8.29) FOR $\ell = m = 0$, USE

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0 \qquad (8.65)$$

a) TO VERIFY THAT Θ = CONSTANT = A IS A SOLUTION

b) TO VERIFY THAT

$$\Theta = \ln\left[\frac{1+\cos\theta}{1-\cos\theta}\right]$$

IS A SOLUTION BUT IT BLOWS UP AT 0 AND π.

c) SINCE ANY SOLUTION IS A LINEAR COMBINATIONS OF THESE SOLUTIONS, WRITE DOWN THE GENERAL SOLUTION & SHOW THAT ONLY Θ = CONSTANT IS ACCEPTABLE.

FOR $\ell = m = 0$, THE DE BECOMES

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0$$

a) FOR Θ = CONSTANT = A,

$$\frac{dA}{d\theta} = 0 \Rightarrow \frac{1}{\sin\theta}\cancel{\frac{d}{d\theta}\left(\sin\theta(0)\right)}^{\to 0} = 0$$

$\Rightarrow$ YES, $\Theta = A$ IS A SOLUTION

- NOT A VERY INTERESTING SOLUTION, BUT A SOLUTION!

b) FOR $\Theta = \ln\left|\frac{1+\cos\theta}{1-\cos\theta}\right|$ AND $\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)}f'(x)$

$$\frac{d\Theta}{d\theta} = \frac{d}{d\theta}\left[\ln(1+\cos\theta) - \ln(1-\cos\theta)\right]$$

$$= \frac{-\sin\theta}{1+\cos\theta} - \frac{\sin\theta}{1-\cos\theta} = \frac{-\sin\theta(1-\cos\theta) - \sin\theta(1+\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}$$

$$= \frac{2\sin\theta}{1-\cos^2\theta}$$

AND

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = \frac{1}{\sin\theta}\frac{d}{d\theta}\left[\frac{2\sin^2\theta}{1-\cos^2\theta}\right]$$

$$= \frac{1}{\sin\theta}\frac{d}{d\theta}\left[\frac{2\cancel{(1-\cos^2\theta)}}{\cancel{1-\cos^2\theta}}\right]$$

$$= \frac{1}{\sin\theta}\cancel{\frac{d}{d\theta}\left[2\right]}^{\to 0} = 0!$$

$\Rightarrow$ YES $\Theta = \ln\left[\frac{1+\cos\theta}{1-\cos\theta}\right]$ IS A SOLUTION!

$\longrightarrow$

8.29) CONTINUED

CHECK THE LIMITS OF $\Theta(\theta) = \ln\left[\frac{1+\cos\theta}{1-\cos\theta}\right]$ WHERE $0 \le \theta \le \pi$

$$\left.\begin{aligned}\Theta(\theta=0) &= \ln\left[\frac{1+1}{1-1}\right] = \ln\left[\frac{2}{0}\right]\\ \Theta(\theta=\pi) &= \ln\left[\frac{1-1}{1-(-1)}\right] = \ln\left[\frac{0}{2}\right]\end{aligned}\right\}$$

BOOM!

$\Rightarrow$ THIS IS NOT AN ACCEPTABLE SOLUTION!

c) A LINEAR COMBINATION OF THE PREVIOUS SOLUTIONS IS ALSO A SOLUTION:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = \frac{1}{\sin\theta}\frac{d}{d\theta}\left\{\sin\theta\frac{d}{d\theta}\left[aA + b\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right)\right]\right\}$$

$$= \frac{1}{\sin\theta}\frac{d}{d\theta}\left\{\sin\theta\left[0 + \frac{db}{d\theta}\ln(\;) - \frac{2b\sin\theta}{1-\cos^2\theta}\right]\right\}$$

$$= \frac{1}{\sin\theta}\frac{d}{d\theta}\left\{\frac{2b\sin^2\theta}{1-\cos^2\theta}\right\}$$

$$= \frac{1}{\sin\theta}\frac{d}{d\theta}\left\{\frac{2b\cancel{\sin^2\theta}}{\cancel{\sin^2\theta}}\right\}$$

$$= \frac{1}{\sin\theta}\cancel{\frac{d}{d\theta}(2b)}^{\,0} = 0 \quad \text{QED!}$$

$\Rightarrow$ THE LINEAR COMBINATION IS A SOLUTION!

BUT,

$$\left.\begin{aligned}\Theta(\theta=0) &= aA + b\ln\left[\frac{2}{0}\right] \to \infty\\ \Theta(\theta=\pi) &= aA + b\ln\left[\frac{0}{2}\right] \to \infty\end{aligned}\right\}$$

BOOM!

$\Rightarrow$ SINCE THIS, TOO, BLOWS UP AT THE BOUNDARIES, THE ONLY ACCEPTABLE SOLUTION (OF THESE) IS THAT Θ = CONSTANT.