8.47) a) Show that $\Theta = \sin\theta$ is a solution for the 2p states ($\ell = 1, m = \pm 1$)

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left(\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right)\Theta = 0 \qquad (8.65)$$

b) Show that the sum of the wave functions

$$\Psi_{2,1,\pm1} = R_{2p}(r)\sin\theta\, e^{\pm i\phi}$$

is the $2p_x$ state, whereas the difference is $2p_y$

a) For $\ell = 1$, $m = \pm 1$, (8.65) becomes $[m^2 = (\pm 1)^2 = 1]$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left(2-\frac{1}{\sin^2\theta}\right)\Theta = 0$$

For $\Theta = \sin\theta$, $\Theta' = \cos\theta$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\cos\theta)+\left(2\sin\theta-\frac{1}{\sin\theta}\right) = 0$$

$$\frac{1}{\sin\theta}\left(\cos^2\theta-\sin^2\theta\right)+\left(2\sin\theta-\frac{1}{\sin\theta}\right) = 0$$

Multiplying through by $\sin\theta$,

$$\cos^2\theta-\sin^2\theta+2\sin^2\theta-1 = 0$$

$$\cos^2\theta+\sin^2\theta-1 = 0$$

$$\boxed{0 = 0}$$ QED! $\Theta = \sin\theta$ is a solution!

b) Take the sum of $\Psi_{2,1,\pm1} = R_{2p}(r)\sin\theta\, e^{\pm i\phi}$

$$\Psi_{2,1,1}+\Psi_{2,1,-1} = R_{2p}(r)\sin\theta\left(e^{i\phi}+e^{-i\phi}\right)$$

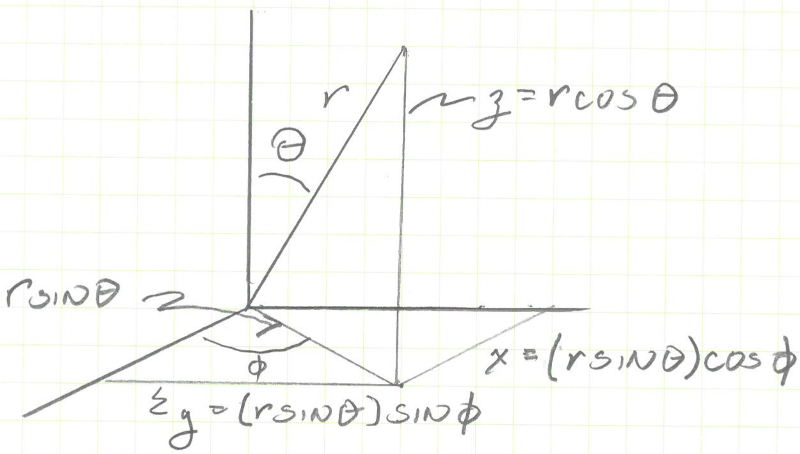
Since $\cos\phi = \frac{1}{2}\left(e^{i\phi}+e^{-i\phi}\right)$ [TZD II Pe. 7.16]

$$\Psi_{2,1,1}+\Psi_{2,1,-1} = 2R_{2p}(r)\sin\theta\cos\phi$$

$\longrightarrow$

8.47) CONTINUED

RECALLING SPHERICAL - CARTESIAN TRANSFORMATIONS,



BY INSPECTION

$$\psi_{2,1,1} + \psi_{2,1,-1} = 2R_{2p}\overbrace{(r\sin\theta)\cos\phi}^{x!} = 2xR_{2p}(r)$$

$$\Rightarrow \boxed{\psi_{2,1,1} + \psi_{2,1,-1} = 2p_{2x}}$$

SO THE SUM OF THE $m = \pm 1$ WAVE FUNCTIONS GIVES $2p_x$

THE DIFFERENCE GIVES

$$\psi_{2,1,1} - \psi_{2,1,-1} = R_{2p}(r)\sin\theta\left(e^{i\phi} - e^{-i\phi}\right)$$

SINCE $\sin\phi = \frac{1}{2}\left(e^{i\phi} - e^{-i\phi}\right)$ [TZD II Pr 7.16]

$$\psi_{2,1,1} - \psi_{2,1,-1} = 2R_{2p}(r)\sin\theta\sin\phi$$

FROM ABOVE, $y = \sin\theta\sin\phi$

$$\psi_{2,1,1} - \psi_{2,1,-1} = 2yR_{2p}(r)$$

$$\Rightarrow \boxed{\psi_{2,1,1} - \psi_{2,1,-1} = 2p_{2y}}$$

SO THE DIFFERENCE OF THE $m = \pm 1$ WAVE FUNCTIONS GIVES $2p_y$