# Homework Set 15: The Plane Pendulum

**Due: Wednesday, October 25, 2023**

## Problem From AOD

**1)** Write out in **explicit detail** the mathematical steps between TM5's equations 4.28 and 4.29. In particular

**a)** Expand expression 4.28c to show the three elliptical integrals and show the steps in determing their solutions using TM5's Appendix E.3, pp. 615 & 616 (the $\Gamma$ functions).

**b)** To go from expression 4.28e to 4.29, expand k = sin($\theta$/2) as a series using TM5's Appendix D.3 on p. 610. Keep terms with powers up to $\theta^4$ in k, $k^2$ and $k^4$. Substitute into 4.28c and show the arithmetic needed to derive 4.29.

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from which

$$\tau = 4\sqrt{\frac{l}{g}}\int_0^1 [(1-z^2)(1-k^2z^2)]^{-1/2}dz \tag{4.28}$$

Numerical values for integrals of this type can be found in various tables.

For oscillatory motion to result, $|\theta_0| < \pi$, or, equivalently, $\sin(\theta_0/2) = k$, where $-1 < k < +1$. For this case, we can evaluate the integral in Equation 4.28 by expanding $(1-k^2z^2)^{-1/2}$ in a power series:

$$(1-k^2z^2)^{-1/2} = 1 + \frac{k^2z^2}{2} + \frac{3k^4z^4}{8} + \cdots \tag{4.28b}$$

Then, the expression for the period becomes

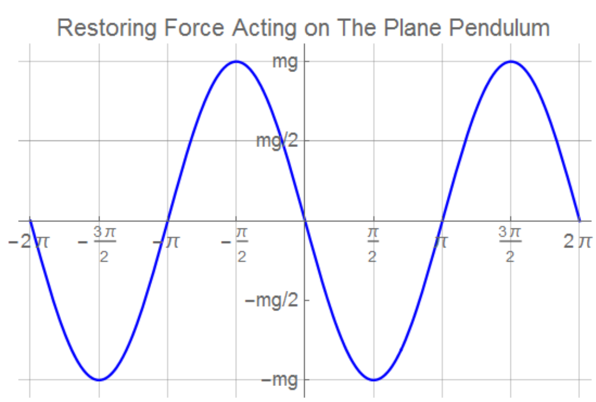
$$\tau = 4\sqrt{\frac{l}{g}}\int_0^1 \frac{dz}{(1-z^2)^{1/2}}\left[1 + \frac{k^2z^2}{2} + \frac{3k^4z^4}{8} + \cdots\right] \tag{4.28c}$$

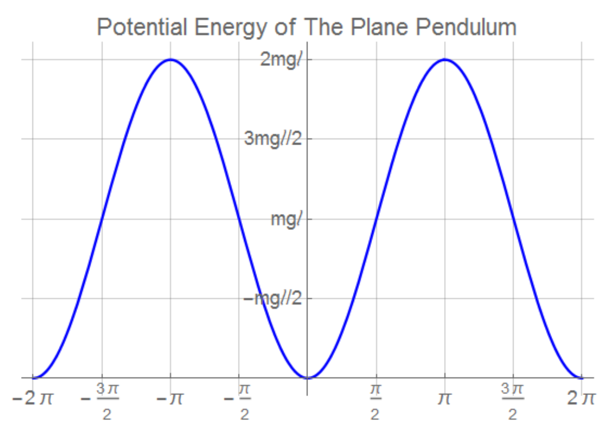
$$= 4\sqrt{\frac{l}{g}}\left[\frac{\pi}{2} + \frac{k^2}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{3k^4}{8}\cdot\frac{3}{8}\cdot\frac{\pi}{2} + \cdots\right] \tag{4.28d}$$

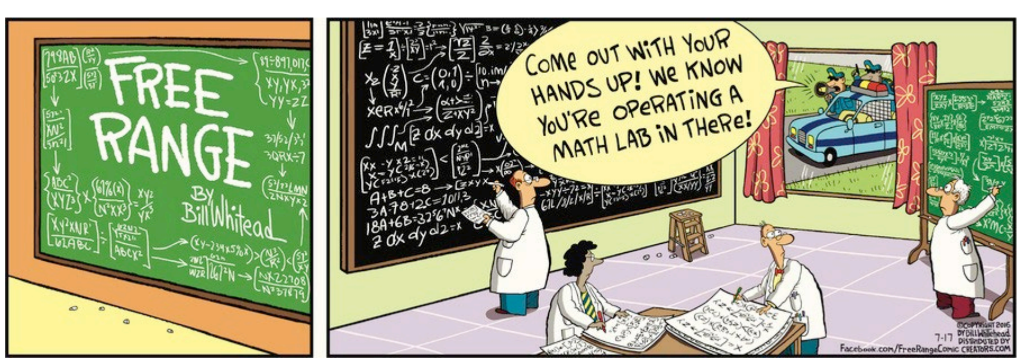
$$= 2\pi\sqrt{\frac{l}{g}}\left[1 + \frac{k^2}{4} + \frac{9k^4}{64} + \cdots\right] \tag{4.28e}$$

If $|k|$ is large (i.e., near 1), then we need many terms to produce a reasonably accurate result. But for small $k$, the expansion converges rapidly. And because $k = \sin(\theta_0/2)$, then $k \cong (\theta_0/2) - (\theta_0^3/48)$; the result, correct to the fourth order, is

$$\tau \cong 2\pi\sqrt{\frac{l}{g}}\left[1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4\right] \tag{4.29}$$



Integrals and Gamma Functions (after TM5[1] Appendix E.3, pp. 615-616)

Derivation of the period of oscillation for a plane pendulum requires the solution of some standard integrals given in the form:

$$\int_0^1 \frac{dx}{\sqrt{\left(1-x^n\right)}} = \frac{\sqrt{\pi}}{n}\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)} \tag{E.26}$$

$$\int_0^1 x^m\left(1-x^2\right)^n dx = \frac{\Gamma\left(n+1\right)\Gamma\left(\frac{m+1}{2}\right)}{2\Gamma\left(n+\frac{m+3}{2}\right)} \tag{E.27a}$$

where n and m can take on various integer values and the Gamma Functions must be determined.

The Gamma Function is an extension of the factorial and defined for complex numbers as a convergent, improper integral:

$$\Gamma\left(n\right) = \int_0^\infty x^{n-1}e^{-x}dx. \tag{E.19a}$$

Some particular values of the Gamma Function are

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{E.21}$$

and

$$\Gamma\left(1\right) = 1. \tag{E.22}$$

Using the recurrence relation allows us to find other values of other Gamma Functions from these two:

$$n\Gamma\left(n\right) = \Gamma\left(n+1\right) \tag{E.20}$$

[1] Thornton, S.T. & Marion, J.B. (2004), *Classical Dynamics of Particles and System* (5th Ed.), Belmont, CA: Brooks/Cole – Thomson Learning