BOSTON UNIVERSITY

GRADUATE SCHOOL OF ARTS AND SCIENCES

Dissertation

SINGULAR PERTURBATIONS OF COMPLEX POLYNOMIALS AND CIRCLE INVERSION MAPS

by

DANIEL M. LOOK

M.S., University of Maine, Orono, Maine, 1998
B.S., University of Maine, Orono, Maine, 1996

Submitted in partial fulfillment of the

requirements for the degree of

Doctor of Philosophy

2005

Approved by

First Reader ____
Robert L. Devaney, Ph.D.
Professor of Mathematics

Second Reader ____
Paul Blanchard, Ph.D.
Associate Professor of Mathematics

Third Reader ____
Tasso Kaper, Ph.D.
Professor of Mathematics

Acknowledgments

I would like to begin by thanking my advisor, Bob Devaney. Without Bob's help and inspiration I do not believe that I would have obtained my Ph.D. Bob is a fantastic educator and researcher and I would be happy if I had half of Bob's mathematical abilities. I often find myself seeking Bob's advice on many different aspects of education and he seems to always have an answer. (Maybe he is just quick at making them up?) A great "Thank You" goes out to Paul Blanchard for all of his help with my research and my understanding of Complex Dynamics. Further, I thank Tasso Kaper for agreeing to read my dissertation as well as helping me with the necessary paper work and hoop-jumping required to graduate. I also want to acknowledge Gene Wayne for chairing my dissertation committee.

I am going to avoid thanking friends because I fear I would forget someone. However, you know who you are and thank you so much for standing by me during this time.

There are several people in my life that have served special roles. First and foremost in this category are my parents. My parents never doubted me and have always allowed me to make my own decisions in life. Without their love and support this work would never have happened. I love you Mom and Dad. I would also like to acknowlege Grampy and Grammie Morse. They have never let me forget how proud they are of me. I love them both dearly and I want my grandfather to know that I will never drive by his house and say "I used to know someone that lived there". My other grandparents, Grammie and Grampy Look, are not with us in body any longer. However, I know how proud they would have been and I want them to know that they inspired me in many ways.

In my life there have been several influential people in academia (both as friends and instructors). I would like to thank Dr. Phil Wass for helping me stay on course by reminding me that "the professor is the last of the truly free professionals". I thank Dr. Dale Lick for taking an interest in my mathematical education early in my life. Also, Dr. Robert Franzosa, my advisor during my masters education at the University of Maine. Bob taught me much about how I would like to be as a college professor. I thank him for his fantastic educational skills and for all of his guidance. I can not thank Bob without also thanking Dr. Charlie Slavin. Through the 9 years that Charlie and I have worked together for the University of Maine Regional Math/Science Center in Orono I have grown to respect Charlie greatly. Charlie has an understanding of people that I envy and the breadth of his knowledge never ceases to amaze me.

Finally, this work would never have been possible without the help of the following coffee shops: Curious Liquids, 1369 and Herrell's Opium Den and Lounge.

SINGULAR PERTURBATIONS OF COMPLEX POLYNOMIALS AND CIRCLE INVERSION MAPS

(Order No.)

DANIEL M. LOOK

Boston University Graduate School of Arts and Sciences, 2005

Major Professor: Robert L. Devaney, Professor of Mathematics

Abstract

The dynamics of the family of complex functions $F_\lambda(z) = z^n + \lambda/z^d$ is explored. For $n = d = 2$ conditions are given on λ ensuring that the Julia set of F_λ is a Sierpinski curve. Further, a complete description of the symbolic dynamics of F_λ restricted to its Julia set (in the cases where the Julia set is a Sierpinski curve) is given. For $n = 2$ and $d = 1$ the existence of a sequence of $\lambda_n \in \mathbb{R}^-$ is shown where F_{λ_n} has a superattracting n-cycle and $J(F_{\lambda_n})$ is a Sierpinski curve.

The dynamics of a function derived from multiple circle inversions is also investigated. A description of the dynamics of this function restricted to its Julia set via symbolic dynamics is given. Finally, a description of the (topological and dynamical) bifurcations that occur as this system is perturbed is presented.

Contents

List of Figures

Chapter 1

Preliminaries for the family F_λ

The first three parts of this work deal with the family of rational maps of the complex plane given by

$$F_\lambda(z) = z^n + \frac{\lambda}{z^d}.$$

When $\lambda = 0$, these maps reduce to $z \mapsto z^n$ and the dynamical behavior in this case is well understood: the Julia set of F_λ is just the unit circle and all other orbits tend either to ∞ or to the superattracting fixed point at 0.

When $\lambda \neq 0$, several things happen. First of all, the map F_λ now has degree $n+d$ rather than n. Secondly, the origin is a pole rather than a fixed point. And, finally, there are $n+d$ free critical points, in addition to the critical points at ∞ and 0. As we show below, the orbits of all of these free critical points behave symmetrically, so we essentially have only one free critical orbit for each of these maps. As is well known in complex dynamics, the behavior of this critical orbit determines much of the structure of the Julia sets of these maps. In this paper we shall describe a trichotomy in the structure of the Julia sets that arises when the critical points all tend to ∞.

For comparison, we first recall the dichotomy that occurs for the well-studied family of quadratic polynomials, $Q_c(z) = z^2 + c$. As in our family, there is only one critical orbit for Q_c, namely the orbit of the critical point at 0. The following facts are well known (see [Mil99]):

1. If the critical orbit for Q_c tends to ∞, then the Julia set of Q_c is a Cantor set and Q_c is conjugate on the Julia set to the one-sided shift of two symbols.

2. If the critical orbit does not tend to ∞, then the Julia set of F_λ is a connected set.

In the quadratic polynomial case, ∞ is a superattracting fixed point and so ∞ is surrounded by an immediate basin of attraction. If the critical orbit tends to ∞, then it is known that the critical point must lie in this basin and, consequently, the entire critical orbit lies in this basin.

For the family F_λ, the point at ∞ is still a superattracting fixed point and so we still have an immediate basin of attraction which we denote by B_λ. However, unlike the quadratic polynomial case, the full basin of attraction may be larger: there may be infinitely many disjoint preimages of the immediate basin of ∞. This is why we find three different topological types of Julia sets when the critical points escape to ∞. In particular, the component of the basin that contains 0 may be disjoint from B_λ. If this is the case, then we denote this component by T_λ and note that the only two preimages of B_λ are B_λ and T_λ. Then T_λ must have disjoint preimages under F_λ^n for $n = 1, 2, 3 \ldots$, and so the basin of attraction of ∞ has infinitely many disjoint components.

We consider the maps

$$F_\lambda(z) = z^n + \frac{\lambda}{z^d}$$

where $n, d \in \mathbb{Z}^+$ and $n \geq 2$. The *Julia set* of F_λ, $J(F_\lambda)$, is defined to be the set of points at which the family of iterates of F_λ fails to be a normal family in the sense of Montel. Equivalently, the Julia set is the closure of the set of repelling periodic points for F_λ as well as the set of points on which F_λ behaves chaotically. The complement of the Julia set is called the *Fatou set.*

There are $n + d$ critical points for F_λ, and all are of the form $\omega^k c_\lambda$ where c_λ is one of the critical points and $\omega^{n+d} = 1$. Similarly, the critical values are arranged symmetrically with respect to $z \mapsto \omega z$, though there need not be $n + d$ of them. There are $n + d$ prepoles at the points $(-\lambda)^{1/(n+d)}$.

Note that $F_\lambda(\omega z) = \omega^n F_\lambda(z)$. Hence the orbits of points of the form $\omega^j z$ all behave "symmetrically" under iteration of F_λ. For example, if $F_\lambda^i(z) \to \infty$, then $F_\lambda^i(\omega^k z)$ also tends to ∞ for each k. If $F_\lambda^i(z)$ tends to an attracting cycle, then so does $F_\lambda^i(\omega^k z)$. Note, however, that the cycles involved may be different depending on k and, indeed, they may even have different periods. Nonetheless, all points on these attracting cycles are of the form $\omega^j z_0$ for some $z_0 \in \overline{\mathbb{C}}$.

Proposition ($(n+d)$-fold Symmetry). *Both B_λ and T_λ have $n+d$-fold symmetry, i.e., if $z \in B_\lambda$, then $\omega z \in B_\lambda$ as well, where $\omega^{n+d} = 1$.*

Proof: Let $U \subset B_\lambda$ be the set of points z in B_λ that have the property that the point ωz also lies in B_λ. U is an open, nonempty set since B_λ contains an open neighborhood around ∞. If $U \neq B_\lambda$, let $z_0 \in \partial U$. Then $z_0 \in B_\lambda$ but $\omega z_0 \notin B_\lambda$. Hence $\omega z_0 \in \partial B_\lambda$. Therefore $F_\lambda^i(z_0) \to \infty$ whereas $F_\lambda^i(\omega z_0) \not\to \infty$. But

$$F_\lambda^i(\omega z_0) = \omega^{ni} F_\lambda^i(z_0) \to \infty.$$

This gives a contradiction.

The case of T_λ is similar.

□

In particular, since the critical points are arranged symmetrically about the origin it follows that if one of the critical points lies in B_λ (resp. T_λ) then all of the critical points lie in B_λ (resp. T_λ).

For other components of the Fatou set, the symmetry situation is somewhat different: either a component contains $\omega^j z_0$ for a given z_0 in the Fatou set and each $j \in \mathbb{Z}$, or else such a component contains none of the $\omega^j z_0$ with $j \neq 0 \bmod n+d$:

Symmetry Lemma. *Suppose U is a connected component of the Fatou set of F_λ. Suppose also that both z_0 and $\omega^j z_0$ belong to U, where $\omega^j \neq 1$. Then in fact $\omega^i z_0$ belongs to U for all i and, as a consequence, U has $n+d$-fold symmetry and surrounds the origin.*

Proof: Suppose that z_0 and $\omega^j z_0$ lie in U but $\omega^i z_0$ does not lie in U. Let α_1 be a continuous curve in U that connects z_0 to $\omega^j z_0$. Define a second curve α_2 by $\omega^j \alpha_1$. By symmetry, α_2

also lies in a component of the Fatou set, but since $\omega^j z_0$ lies on α_2, it follows that α_2 also lies in U and so U also contains $\omega^{2j} z_0$. Continuing in this fashion, we see that U contains $\omega^{\ell j} z_0$ for all ℓ and that the analogous curve α_ℓ also lies in U.

Now suppose $\omega^{kj} = 1$. Then the union of the curves $\alpha_1, \ldots, \alpha_k$ forms a closed curve that lies in U and surrounds the origin. Call this curve α. By assumption, $\omega^i z_0$ does not lie on α. If we set $\omega^i \alpha_l = \beta_l$ for each l, we get another closed curve, call it β, that surrounds the origin and is contained in $\omega^i U$. Since $\omega^i z_0 \in \omega^i U$ but $\omega^i z_0 \notin U$ we know that $\omega^i U \neq U$. In fact, since U is a Fatou component and $F_\lambda(\omega z) = \omega^n F_\lambda(z)$ we get that $\omega^i U$ is also a Fatou component and hence $\omega^i U \cap U = \emptyset$. Since $\beta \subset \omega^i U$ and $\alpha \subset U$ we see that $\alpha \cap \beta = \emptyset$. However, α and β are both curves that surround the origin and $\beta = \omega^i \alpha$, implying that α and β must cross. This implies that α and β lie in the same Fatou component, yielding a contradiction.

Chapter 2

Criterion for Sierpinski Curve Julia sets

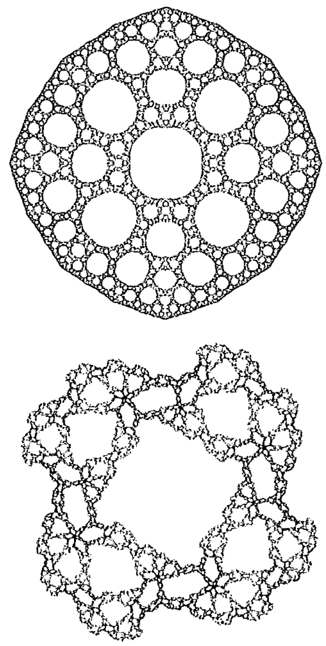
In this section we restrict to the case $n = d = 2$. Our goal is to give a criterion for the Julia set of such a map to be a Sierpinski curve. A Sierpinski curve is a rather interesting topological space that is homeomorphic to the well known Sierpinski carpet fractal. The interesting topology arises from the fact that a Sierpinski curve contains a homeomorphic copy of any one-dimensional plane continuum. Hence any such set is a universal planar continuum. In [BDL$^+$03a] it is shown that, in every neighborhood of $\lambda = 0$ in the parameter plane, there are infinitely many disjoint open sets of parameters for which the Julia set is a Sierpinski curve. This result should be contrasted with the situation that occurs for the related family $G_\lambda(z) = z^n + \lambda/z^d$ with $1/n + 1/d < 1$. McMullen [McM88] has shown that, provided λ is sufficiently small, the Julia set of G_λ is always a Cantor set of circles. A dynamical criterion for this is given in [DLU03]. On the other hand, Hawkins [Haw03] has shown that very different phenomena arise in the family $H_\lambda(z) = z + \lambda/z$.

Our goal is to investigate the dynamics of the family F_λ for all λ-values, not just those close to the origin. Our main result is a criterion for the Julia set of F_λ to be a Sierpinski curve:

Theorem. *Suppose that the critical orbit of F_λ tends to ∞ but the critical points of F_λ do not lie in the immediate basin of ∞. Then the Julia set of F_λ is a Sierpinski curve. In particular, any two Julia sets corresponding to an eventually escaping critical orbit are homeomorphic.*

In Figure 2·1, we display two Sierpinski curve Julia sets drawn from the family F_λ.

This is reminiscent of the Fundamental Dichotomy mentioned in Chapter 1. For the



$\lambda = -1/16$ $\lambda = 0.132 + 0.097i$

Figure 2·1: The Sierpinski curve Julia sets for two values of λ.

family F_λ, we shall prove that there is a similar fundamental dichotomy, but there is a subtle yet extremely important difference. Our result is:

Theorem. *If the entire critical orbit of F_λ lies in the immediate basin of attraction of ∞, then the Julia set of F_λ is a Cantor set. On the other hand, if the entire critical orbit does not lie in the immediate basin, then the Julia set is connected.*

The subtle difference here lies in our assumption that the entire critical orbit lies in the *immediate basin* of ∞. For quadratic polynomials, if the critical orbit escapes to ∞, then its entire orbit must lie in the immediate basin of ∞. However, for F_λ, it is possible that the critical orbit escapes to ∞ but that the entire orbit does not lie in the immediate basin. That is, the critical points may lie in one of the (disjoint) preimages of the immediate basin, or, said another way, the critical orbit may jump around before entering the immediate basin of B_λ. This is the case in which we find Sierpinski curve Julia sets.

There are other significant differences between the Julia sets of the family of rational maps and those of the quadratic polynomials. For example, in the case of connected quadratic Julia sets, it is often the case that the boundary of the basin at ∞ has infinitely many pinch points. That is, the complement of the closure of the immediate basin of ∞ consists of infinitely many disjoint open sets. For example, if Q_c admits an attracting periodic point of period $n \geq 2$, then the complement of the closure of the immediate basin of ∞ always consists of infinitely many disjoint components made up of the various basins of attraction and their preimages. These are the Fatou components for the map.

For F_λ, a very different situation occurs. Let B_λ denote the immediate basin of ∞ for F_λ. Then we shall prove:

Theorem. *Suppose $J(F_\lambda)$ is connected. Then $\overline{\mathbb{C}} - \overline{B_\lambda}$ is an open, connected, simply connected set.*

For "nice" simply connected open sets, the boundary of such sets is a simple closed curve, but as is well known, this need not be the case. For example, the topologists' sine curve and other, non-locally connected sets may bound a simply connected open set in the plane. In our case, however, we often have simple closed curves bounding the basin of ∞. We shall also show:

Theorem. *The boundary of the immediate basin of ∞ is a simple closed curve in each of the following cases:*

1. $|\lambda| < 1/16$*;*

2. *The critical orbits lie on the boundary of the basin of ∞ but are preperiodic (the Misiurewicz case);*

3. *The critical points do not accumulate on the boundary of the basin of ∞, as in the special case where they eventually tend to ∞ and we have a Sierpinski curve Julia set.*

2.1 Preliminaries for F_λ when $n = d = 2$

In this section we describe some of the basic properties of the family $F_\lambda(z) = z^2 + \lambda/z^2$ where, as always, we assume that $\lambda \neq 0$. Observe that $F_\lambda(-z) = F_\lambda(z)$ and $F_\lambda(iz) = -F_\lambda(z)$ so that $F_\lambda^2(iz) = F_\lambda^2(z)$ for all $z \in \overline{\mathbb{C}}$. Also recall that 0 is the only pole for each function in this family. The points $(-\lambda)^{1/4}$ are prepoles for F_λ since they are mapped directly to 0. The four critical points for F_λ occur at $\lambda^{1/4}$. Note that $F_\lambda(\lambda^{1/4}) = \pm 2\lambda^{1/2}$ and $F_\lambda^2(\lambda^{1/4}) = 1/4 + 4\lambda$, so each of the four critical points lies on the same forward orbit after two iterations. We call the union of these orbits the *critical orbit* of F_λ.

Let B_λ be the immediate basin of attraction of ∞ and denote by ∂B the boundary of B_λ. The map F_λ is conjugate to $z \mapsto z^2$ on B_λ, at least in a neighborhood of ∞. The basin B_λ is a (forward) invariant set for F_λ in the sense that, if $z \in B_\lambda$, then $F_\lambda^n(z) \in B_\lambda$ for all $n \geq 0$. The same is true for ∂B_λ.

We denote by $K_\lambda = K(F_\lambda)$ the set of points whose orbit under F_λ is bounded. K_λ is the *filled Julia set* of F_λ. K_λ is given by $\overline{\mathbb{C}} - \cup F^{-n}(B_\lambda)$. Recall that $J_\lambda = J(F_\lambda)$ is the Julia set of F_λ and $J)\lambda = \partial K_\lambda$ (see [Mil99]). Both J_λ and K_λ are completely invariant subsets in the sense that if $z \in J_\lambda$ (resp. K_λ), then $F_\lambda^n(z) \in J_\lambda$ (resp. K_λ) for all $n \in \mathbb{Z}$.

Proposition. (Fourfold Symmetry) *The sets B_λ, ∂B_λ, J_λ, and K_λ are all invariant under $z \mapsto iz$.*

Proof: We prove this for B_λ; the other cases are similar. Let $U = \{z \in B_\lambda \,|\, iz \in B_\lambda\}$. U is an open subset of B_λ. If $U \neq B_\lambda$, there exists $z_0 \in \partial U \cap B_\lambda$, where ∂U denotes the boundary of U. Hence $z_0 \in B_\lambda$ but $iz_0 \in \partial B_\lambda$. It follows that $F_\lambda^n(iz_0) \in \partial B_\lambda$ for all n. But since $F_\lambda^2(z_0) = F^2(iz_0)$, it follows that $z_0 \notin B_\lambda$ as well. This contradiction establishes the result.

□

There is a second symmetry present in this family. Consider the map $H_\lambda(z) = \sqrt{\lambda}/z$. Note that we have two such maps depending upon which square root of λ we choose. H_λ

is an involution and we have $F_\lambda(H_\lambda(z)) = F_\lambda(z)$. As a consequence, H_λ preserves both J_λ and K_λ. The involution H_λ also preserves the circle of radius $\lambda^{1/4}$ and interchanges the interior and exterior of this circle. Hence both J_λ and K_λ are symmetric about this circle with respect to the action of H_λ.

2.2 The Fundamental Dichotomy

Our goal in this section is to prove the fundamental dichotomy for the family F_λ:

Theorem. *If one and hence all of the critical points of F_λ lie in B_λ, then $J(F_\lambda)$ is a Cantor set; if the critical points of F_λ do not lie in B_λ, then both $J(F_\lambda)$ and $K(F_\lambda)$ are compact, connected sets.*

Proof: Suppose first that no critical point lies in B_λ. Then we may extend the conjugacy between F_λ and z^2 to all of B_λ and so B_λ is a simply connected open set in $\overline{\mathbb{C}}$. Let $U_0 = \overline{\mathbb{C}} - B_\lambda$. U_0 is compact and connected with boundary ∂B_λ. Let $U_1 = U_0 - F_\lambda^{-1}(B_\lambda)$. $F_\lambda^{-1}(B_\lambda) - B_\lambda$ is a simply connected open set containing 0 which is mapped two-to-one onto B_λ. Hence $F_\lambda^{-1}(B_\lambda) - B_\lambda$ lies in U_0 and is disjoint from ∂B_λ since orbits in ∂B_λ remain bounded. Therefore U_1 is compact and connected. Inductively, U_k is given by $U_{k-1} - F_\lambda^{-k}(B_\lambda)$. Since $F_\lambda^{-k}(B_\lambda)$ is a collection of disjoint, simply connected, open sets which do not intersect the boundary of U_{k-1}, it follows that U_k is also compact and connected. Then $K(F_\lambda) = \cap U_k$ is compact and connected. Since J_λ is the boundary of K_λ, J_λ too is compact and connected.

The proof that $J(F_\lambda)$ is a Cantor set when all critical points lie in B_λ is standard. See, for example, [Mil99].

□

We emphasize again that the critical points for F_λ may eventually escape but not lie in B_λ. In this case we still have a connected Julia set. In fact, we shall show in Section 5 that $J(F_\lambda)$ is a Sierpinski curve in this case.

We denote the set of parameter values for which $J(F_\lambda)$ is connected by $\mathcal{M}$; $\mathcal{M}$ is called the *connectedness locus* for this family. This set is the analogue of the Mandelbrot set for quadratic polynomials.

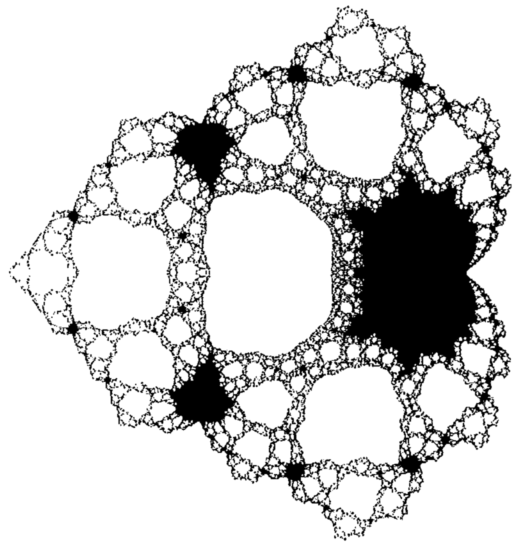


Figure 2·2: The parameter plane for the family $z^2 + \lambda/z^2$. White regions correspond to λ-values for which the critical orbit escapes to ∞.

2.3 The case $|\lambda| < 1/16$

In this section we deal with the very special case where $|\lambda| < 1/16$. We first prove:

Theorem. *Suppose that $|\lambda| < 1/16$. Then the boundary of B_λ is a simple closed curve.*

Proof: Consider the *critical circle* S_λ given by $r = |\lambda|^{1/4}$. Note that S_λ contains all four critical points as well as the four prepoles. Write $\lambda = \rho \exp(i\psi)$ and $z = \rho^{1/4}\exp(i\theta) \in S_\lambda$. Then we compute

$$\begin{aligned} F_\lambda(z) &= \rho^{1/2}(e^{2i\theta} + e^{i(\psi - 2\theta)}) \\ &= \rho^{1/2}\left((\cos(2\theta) + \cos(\psi - 2\theta)) + i(\sin(2\theta) + \sin(\psi - 2\theta))\right) \end{aligned}$$

If we set $x = \cos(2\theta) + \cos(\psi - 2\theta))$ and $y = \sin(2\theta) + \sin(\psi - 2\theta)$, then a computation shows that

$$\frac{d}{d\theta}\left(\frac{y}{x}\right) = 0.$$

Hence the image of S_λ under F_λ is a line interval passing through the origin. F_λ maps S_λ onto this line in four-to-one fashion, except at the two endpoints, which are the critical values $\pm 2\sqrt{\lambda}$. Note that these two critical values lie inside S_λ provided we have

$$2|\sqrt{\lambda}| < |\lambda|^{1/4},$$

which occurs when $|\lambda| < 1/16$. Hence the condition $|\lambda| < 1/16$ guarantees that the image of S_λ lies strictly inside S_λ.

Now if V_λ is another circle surrounding the origin whose radius is slightly larger than $|\lambda|^{1/4}$, then the image of V_λ also lies inside S_λ and hence inside V_λ. Moreover, the image of V_λ is a simple closed curve since there are no critical points or prepoles on V_λ. The involution H_λ maps V_λ to a second circle W_λ that lies strictly inside the critical circle and we have $F_\lambda(V_\lambda) = F_\lambda(W_\lambda)$. The annular region between V_λ and W_λ is mapped in four-to-one fashion onto the disk surrounding the origin and bounded by $F_\lambda(V_\lambda)$. In particular, the image of this annulus is disjoint from the annulus provided that V_λ is sufficiently close to the critical circle.

We claim that the preimage of V_λ consists of a pair of disjoint simple closed curves, one lying inside the critical circle and one lying outside V_λ. This follows from the fact that F_λ maps the exterior of V_λ in two-to-one fashion onto the exterior of the curve $F_\lambda(V_\lambda)$. The interior of the circle W_λ is mapped in similar fashion onto the exterior of $F_\lambda(V_\lambda)$. Let U_λ denote the preimage of V_λ lying outside V_λ, and let A_λ denote the annular region bounded by V_λ and U_λ. Note that A_λ is mapped in two-to-one fashion onto the annulus bounded by V_λ and $F_\lambda(V_\lambda)$.

We now use quasiconformal surgery to modify F_λ to a new map G_λ which agrees with

F_λ on the exterior of A_λ but which is conjugate to $z \mapsto z^2$ in the interior of U_λ with a fixed point at the origin. To obtain G_λ, we first replace F_λ in the disk bounded by V_λ by a map which is a quasiconformal deformation of $z \mapsto z^2$ on $|z| < 1/2$. Then we extend G_λ to A_λ so that the new map is quasiconformally conjugate to z^2 on and inside A_λ and agrees with F_λ on the outer boundary U_λ of A_λ. The new map G_λ is continuous and has degree 2 with two superattracting fixed points, one at 0 and one at ∞. Hence G_λ is everywhere conjugate to z^2. Therefore the boundary of the basin of attraction of ∞ for G_λ is a simple closed curve. Since G_λ agrees with F_λ in the exterior of A_λ, the same is true for F_λ. This proves that ∂B_λ is a simple closed curve when $|\lambda| < 1/16$.

□

We now use this result to prove:

Theorem. *Suppose that $|\lambda| < 1/16$ and that the critical points of F_λ tend to ∞ but do not lie in the the immediate basin B_λ of ∞. Then $J(F_\lambda) = K(F_\lambda)$ is a Sierpinski curve.*

Proof: It is known [Why58] that any planar set that is compact, connected, locally connected, nowhere dense, and has the property that any two complementary domains are bounded by simple closed curves that are disjoint is homeomorphic to the Sierpinski carpet and is therefore a Sierpinski curve. In our case, the fact that both J_λ and K_λ are compact and connected was shown in the previous section. Since all of the critical orbits tend to ∞, it follows that $J_\lambda = K_\lambda$ and hence, using standard properties of the Julia set, J_λ is nowhere dense. Also, since no critical points accumulate on J_λ, it is known [Mil99] that J_λ is locally connected.

It therefore suffices to show that the complementary domains are all bounded by disjoint simple closed curves. By the previous result, ∂B_λ is bounded by a simple closed curve lying strictly outside the critical circle. Using the involution H_λ, the boundary of the trap door is given by $H_\lambda(\partial B_\lambda)$, and so this region is bounded by a simple closed curve lying inside the critical circle and therefore disjoint from ∂B_λ.

Now consider the preimage of the trap door. This preimage is an open set. It cannot consist of a single component, for if this were the case, this component would necessarily surround the origin (by fourfold symmetry) and thereby disconnect the Julia set. Hence each of the components of the preimage of T_λ is an open set that is mapped in either one-to-one or two-to-one fashion onto T_λ depending upon whether or not a critical point lies in the preimage. (In fact, the critical points cannot lie in the first preimage of T_λ, but we do not need this fact here.)

It follows that each component of the preimage of T_λ is a simply connected open set whose boundary is a simple closed curve that is mapped onto ∂T_λ. The boundaries of these components are disjoint from ∂B_λ, since this curve is invariant under F_λ and hence cannot be mapped to ∂T_λ. They are also disjoint from ∂T_λ since the boundary of the trap door is mapped to ∂B_λ whereas the boundary of the components are mapped to ∂T_λ, and we know that $\partial T_\lambda \cap \partial B_\lambda = \emptyset$. Finally, the boundary of each component is disjoint from any other such boundary for a point in the intersection would necessarily be a critical point. If this were the case, then the critical orbit would eventually map to ∂B_λ, contradicting our assumption that the critical orbit tends to ∞. Hence the first preimages of T_λ are all bounded by simple closed curves that are disjoint from each other as well as the boundaries of B_λ and T_λ. Continuing in this fashion, we see that the preimages $F_\lambda^{-n}(T_\lambda)$ are similarly bounded by simple closed curves that are disjoint from all earlier preimages of ∂B_λ. This gives the result.

□

2.4 The Boundary of $\mathbf{B}_\lambda$

In this section we consider any λ-value for which the Julia set of F_λ is connected, not just those that satisfy $|\lambda| < 1/16$. Our aim is to show that the open set $\overline{\mathbb{C}} - \overline{B_\lambda}$ is a connected and simply connected set. This implies that the interior of the set containing the origin and bounded by the boundary of B_λ has just one connected component. Moreover, we show below that if z lies in the intersection of the boundaries of both B_λ and T_λ, then z must be a critical point of F_λ. Hence there are at most four points in the intersection of these two boundaries.

Proposition. *The open set $\overline{\mathbb{C}} - \overline{B_\lambda}$ is connected and simply connected whenever $B_\lambda \cap T_\lambda$ is empty.*

Proof: Let W_0 denote the open connected component of $\overline{\mathbb{C}} - \overline{B_\lambda}$ that contains 0. Note that W_0 contains all of T_λ since the boundary of B_λ does not meet T_λ. Hence the closure of W_0 also contains ∂T_λ.

Lemma. *W_0 is symmetric under $z \mapsto iz$ and hence has fourfold symmetry.*

Proof: Let X denote the set of points z in W_0 for which iz also lies in W_0. Note that X is an open subset of W_0. Note also that $X \supset T_\lambda$ since T_λ possesses fourfold symmetry and lies in W_0. Hence X is nonempty. Now suppose that $X \neq W_0$. Then there must be a point $z_1 \in \partial X \cap W_0$. So $z_1 \in W_0$ but $iz_1 \notin W_0$. Therefore iz_1 lies in ∂W_0, which is contained in ∂B_λ. Since $iz_1 \in \partial B_\lambda$ and it was earlier shown that ∂B_λ has fourfold symmetry we know that $z_1 \in \partial B_\lambda$, contradicting our assumption that $z_1 \in W_0$. This proves the lemma.

□

Lemma. *All four preimages of any point in W_0 lie in W_0.*

Proof: Since $H_\lambda(B_\lambda) = T_\lambda$ and $T_\lambda \subset W_0$, we have $H_\lambda(\partial B_\lambda) \subset \overline{W}_0$. Therefore $H_\lambda(\partial W_0) \subset \overline{W}_0$ and so H_λ maps $\overline{\mathbb{C}} - \overline{W}_0$ into W_0.

Now H_λ maps prepoles to prepoles. If one of the prepoles lies in $\overline{\mathbb{C}} - \overline{W}_0$, then its image under H_λ lies in W_0. This cannot occur since, by the previous lemma, W_0 has fourfold symmetry. Hence each prepole lies in $\overline{W}_0$. In fact, each prepole must lie in W_0 since ∂W_0 is mapped to ∂B_λ.

It follows that all four preimages of 0 lie in W_0. Therefore the entire set $F_\lambda^{-1}(W_0)$ is contained in W_0 for, otherwise, there would be points in $\partial W_0 \subset \partial B_\lambda$ that are mapped into W_0. This cannot happen since ∂B_λ is invariant.

□

Remark. By the lemma, $F_\lambda : F_\lambda^{-1}(W_0) \to W_0$ is a proper map of degree four. By the Riemann-Hurwitz Theorem, either $F_\lambda^{-1}(W_0)$ is an annulus containing all four critical points, or else $F_\lambda^{-1}(W_0)$ is a union of four disjoint disks, each of which is mapped homeomorphically onto W_0. This latter case occurs when the critical points all lie in ∂B_λ, as we discuss below.

We now complete the proof that $\overline{\mathbb{C}} - \overline{B_\lambda}$ is connected and simply connected. It suffices to show that W_0 is the only component of $\overline{\mathbb{C}} - \overline{B_\lambda}$.

Assume that there is an additional component of $\overline{\mathbb{C}} - \overline{B_\lambda}$ that is disjoint from W_0. Call this component W_1. Note that $-W_1$ is also a component of $\overline{\mathbb{C}} - \overline{B_\lambda}$ and that $\pm W_1$ are disjoint, since otherwise this component would surround W_0. We have that $F_\lambda(W_1)$ does not meet W_0 since all preimages of points in W_0 lie in W_0. Also, as above, we have that $H_\lambda(\pm W_1)$ lies in W_0.

We claim that there are no critical points in W_1. For, if $c_\lambda \in W_1$, then we must have $-c_\lambda \in -W_1$ and so F_λ maps both $\pm W_1$ onto an open set Q in two-to-one fashion. Now Q lies in $\overline{\mathbb{C}} - \overline{B_\lambda}$ and hence Q must be some connected component of this set, say $Q = W_k$. Then we have $k \neq 0$ and all four preimages of any point in W_k lie in $\pm W_1$. But, since $F_\lambda(H_\lambda(z)) = F_\lambda(z)$, there must also be preimages of these points in $H_\lambda(\pm W_1) \subset W_0$, as we saw above. Thus we have more than four preimages for these points, so this cannot happen. We conclude that there can be no critical points in W_1.

Thus we have that any additional component of $\overline{\mathbb{C}} - \overline{B_\lambda}$ cannot contain either a critical point or a prepole of F_λ. Now we know that the set of components $\cup W_j$ excluding W_0 is mapped onto itself by F_λ. But then either one of these domains must be periodic under F_λ or else we have no periodic domains in $\cup W_j$. The former is impossible, since such a periodic domain would necessarily have a critical point belonging to it, while the latter is impossible by the Sullivan No Wandering Domains Theorem. See [Mil99].

We conclude that there are no other W_j to start with in $\overline{\mathbb{C}} - \overline{B_\lambda}$, and so $\overline{\mathbb{C}} - \overline{B_\lambda} = W_0$, an open, connected, simply connected set as claimed.

□

As a remark, the fact that there is only one component to the complement of $\overline{B_\lambda}$ does not preclude the existence of quadratic-like filled Julia sets with infinitely many pinch points. These often reside as subsets of W_0 as depicted in Figure 2·3.

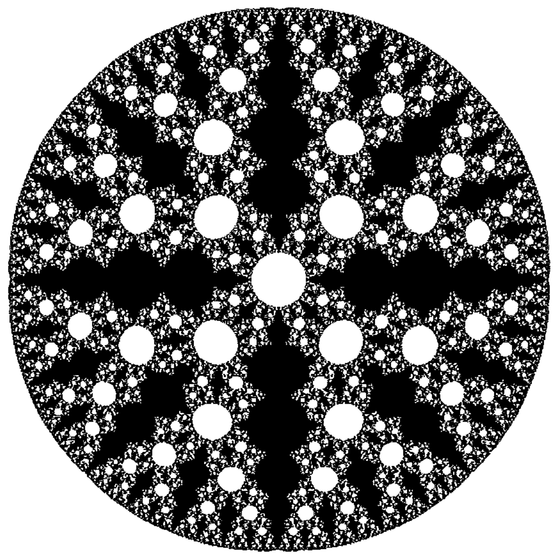


Figure 2·3: The Julia set of F_λ when $\lambda = 0.01$. For this λ-value, F_λ admits an attracting cycle of period 2. Note the black regions lying inside W_0 that resemble the Julia set of $z^2 - 1$; these are the basins of attraction of the two-cycle for F_λ.

Corollary. *Suppose $z_0 \in \partial B_\lambda \cap \partial T_\lambda$. Then z_0 is a critical point of F_λ.*

Proof: Suppose that z_0 is not a critical point of F_λ. Then $F_\lambda(z_0) = w_0$ has four distinct preimages: $\pm z_0$ and $\pm z_1$ with $z_0 \neq \pm z_1$. Let $\pm U_i$ be open neighborhoods of $\pm z_i$ and suppose that the $\pm U_i$ are disjoint and that $F_\lambda(\pm U_i) = W$ where W is an open neighborhood of w_0.

Since $z_0 \in \partial B_\lambda$, we may find a pair of external rays α_0 and β_0 that land at distinct points in $U_0 \cap \partial B_\lambda$. Let $\gamma(\alpha_0, \beta_0)$ denote the union of the external rays contained between α_0 and β_0 (where we assume that the angle between these two rays is smaller than $\pi/2$). We may choose α_0 and β_0 so that the closure of $\gamma(\alpha_0, \beta_0)$ contains z_0, i.e., that these external rays land on either "side" of z_0. The set $-\gamma(\alpha_0, \beta_0)$ lies in B_λ and has similar properties near $-z_0$.

Now z_0 lies in $\partial T_\lambda \cap \partial B_\lambda$. Hence $\pm z_1$ also lies in $\partial T_\lambda \cap \partial B_\lambda$ since $H_\lambda(\pm z_0) = \pm z_1$ and H_λ maps ∂T_λ to ∂B_λ and ∂B_λ to ∂T_λ. Thus we may find two other external rays α_1 and β_1 in B_λ such that α_1 and β_1 terminate in U_1 on either side of z_1 and, moreover, have the property that $\gamma(\alpha_1, \beta_1)$ is disjoint from $\pm\gamma(\alpha_0, \beta_0)$. As above, $-\gamma(\alpha_1, \beta_1)$ lies in B_λ and terminates in $-U_1$.

Since $\pm\gamma(\alpha_0, \beta_0)$ and $\pm\gamma(\alpha_1, \beta_1)$ are disjoint, it follows that the images of these sets are distinct. (Since F_λ is even we know that $F_\lambda(\gamma(\alpha_i, \beta_i)) = F_\lambda(-\gamma(\alpha_i, \beta_i))$ and hence if the images of these four sets were not distinct then there would exist points in B_λ with four preimages in B_λ and this cannot happen since F_λ is $2-1$ on B_λ.) Also, these images accumulate near w_0. Thus we have two disjoint intervals of rays in B_λ that accumulate on ∂B_λ, and both contain w_0 in their closure. It follows that there must be a subset R of ∂B_λ that is disjoint from the boundary of W_0 and R is separated from W_0 by these sets of rays. (See Figure 2·4.)

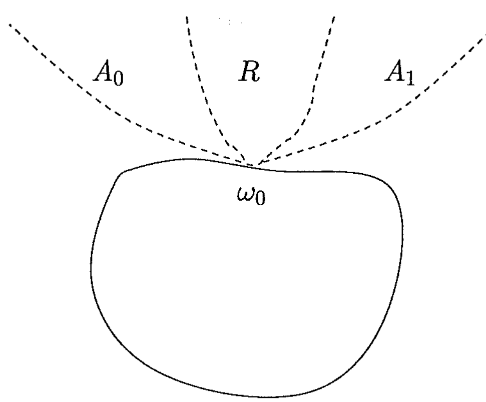


Figure 2·4: The region R where $F_\lambda(\pm\gamma(\alpha_i, \beta_i)) = A_i$

The set R cannot bound an open component of $\overline{\mathbb{C}} - \overline{B_\lambda}$, as we saw above. Hence R, which is separate from W_0, must have empty interior. But there must be preimages of this region on the boundary of W_0, and so there must be points in W_0 that map arbitrarily close to these points. This is impossible.

We conclude that w_0 could not have had four preimages and so z_0 must have been a critical point. □

2.5 Proof of the Sierpinski Curve Criterion

In this section we investigate the general case where the critical orbit escapes through the trap door into B_λ. Here we complete the proof that, when this occurs, the Julia set is a Sierpinski curve.

In this case F_λ is a rational map of degree $d \geq 2$ whose postcritical closure is disjoint from its Julia set. Hence F_λ is dynamically hyperbolic. Since B_λ is a simply connected Fatou component for F_λ, we have that ∂B_λ is locally connected and that the Julia set of F_λ is connected and locally connected [Mil99]. Since B_λ is simply connected we know by the Riemann mapping theorem that there is a conformal isomorphism $\psi : \mathbb{D} \to B_\lambda$ where $\mathbb{D}$ is the open unit disk. The following result is well known. See [Mil99].

Theorem (Caratheodory). *A conformal isomorphism $\psi : \mathbb{D} \to U \subset \overline{\mathbb{C}}$ extends to a continuous map from the closed disk $\overline{\mathbb{D}}$ onto $\overline{U}$ if and only if the boundary ∂U is locally connected.*

This tells us that the Riemann map $\psi : \mathbb{D} \to B_\lambda$ extends to a continuous map $\hat{\psi} : \overline{\mathbb{D}} \to \overline{B_\lambda}$. In particular, we have a continuous map from $\partial\mathbb{D}$ to ∂B_λ. Therefore we know that all external rays R_t (with $t \in \mathbb{R}/\mathbb{Z}$) in B_λ land on a single point in ∂B_λ.

This allows us to prove:

Theorem. *∂B_λ is a simple closed curve.*

Combining this result with the techniques described in Section 3 allows us to conclude that $J(F_\lambda)$ is a Sierpinski curve when the critical orbit escapes through the trap door.

Proof: ∂B_λ is a simple closed curve if and only if exactly one external ray lands at each point in ∂B_λ. Assume this is not the case. Suppose that there exists $p \in \partial B_\lambda$ such that two external rays R_{t_1} and R_{t_2} land on p. Since these rays together with the point p form a Jordan curve and W_0 is connected and simply connected, we have that W_0 lies entirely within one of the two open components created by this Jordan curve. Without loss of generality, assume that W_0 is such that $W_0 \cap \gamma(t_1, t_2) = \emptyset$ (so W_0 is "outside" the sector $\gamma(t_1, t_2)$ between R_{t_1} and R_{t_2}).

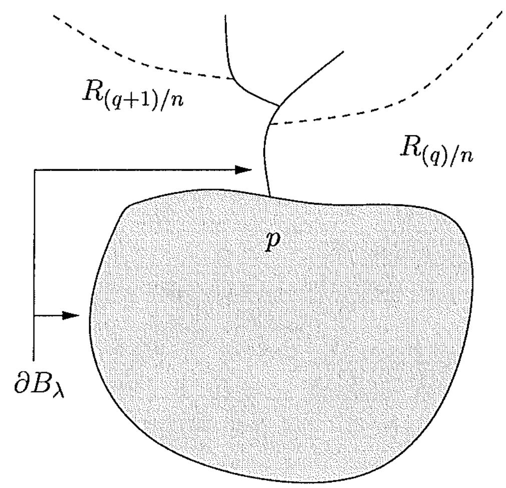


Figure 2·5: A possible landing pattern.

We claim that there exist positive integers q and n such that the region

$$\gamma\left(\frac{q}{n}, \frac{q+1}{n}\right) \subset \gamma(t_1, t_2)$$

and neither of $R_{q/n}$ nor $R_{(q+1)/n}$ land on ∂W_0. If this is not possible then all rays with angle s such that $R_s \subset \gamma(t_1, t_2)$ land at p, This gives a contradiction because the set of $s \in \mathbb{R}/\mathbb{Z}$ such that $\gamma(s) = p$ has measure zero [Mil99]. Therefore, if we have two rays landing at p, then we can find q and n such that

$$\gamma\left(\frac{q}{n}, \frac{q+1}{n}\right) \subset \gamma(t_1, t_2)$$

14 and neither of $R_{q/n}$ nor $R_{(q+1)/n}$ land on ∂W_0. (See Figure 2·5.)

Assume that we have such q and n. As above, let $\gamma(q/n, (q+1)/n)$ denote the union of the external rays contained between q/n and $(q+1)/n$. After n iterations $\gamma(q/n, (q+1)/n)$ is mapped over all of ∂B_λ. In particular, if R_θ is an external ray landing on ∂W_0 we know that there is a ray $R_\phi \in \gamma(q/n, (q+1)/n)$ such that $F_\lambda^n(R_\phi) = R_\theta$. Since $R_\phi \in \gamma(q/n, (q+1)/n)$ we know that the landing point $\gamma(\phi)$ is not on ∂W_0. Hence there exists a neighborhood N_ϕ of $\gamma(\phi)$ such that $N_\phi \cap W_0$ is empty. However, since $F_\lambda^n(\gamma(\phi))$ is on the boundary of W_0 we know that $F_\lambda^n(N_\phi) \cap W_0$ is not empty. This is a contradiction since points not in W_0 never enter W_0. Hence, we can never have two rays landing at the same point on ∂B_λ, implying that ∂B_λ is a simple closed curve.

□

Although the above was written for λ such that the critical points escape, the results also hold for the Misiurewicz case. In this case F_λ is subhyperbolic and all of the theorems above hold with minor adjustments (i.e., all of the proofs depending on hyperbolicity still go through when hyperbolicity is replaced by subhyperbolicity). Therefore we know that ∂B_λ is a simple closed curve for the Misiurewicz case as well.

Chapter 3

Symbolic Dynamics for F_λ when $n = d = 2$ and $J(F_\lambda)$ is a Sierpinski Curve

In this section we discuss the dynamics of F_λ when $n = d = 2$. Of particular interest is the dynamical behavior of these maps on the Julia set of F_λ, $J_\lambda = J(F_\lambda)$. The following Theorem was proved in [DLU03]:

Theorem. *There are infinitely many open sets $\mathcal{O}_j$ in the λ-plane for this family having the following properties:*

1. *For each $\lambda \in \mathcal{O}_j$, the Julia set of F_λ is a Sierpinski curve; as a consequence, any two of these Julia sets are homeomorphic.*

2. *However, if $\lambda \in \mathcal{O}_j$ and $\mu \in \mathcal{O}_k$ with $j \neq k$, then F_λ is not topologically conjugate to F_μ on their respective Julia sets.*

In Figure 2·2 we display the parameter plane (the λ-plane) for the family F_λ. The bounded white regions in this figure contain parameters for which $J(F_\lambda)$ is a Sierpinski curve; we call these regions *Sierpinski holes.*

By the above Theorem, any two Julia sets drawn from Sierpinski holes in parameter space are necessarily homeomorphic and hence these sets are identical from a topological point of view. But there are infinitely many of these holes in which the dynamics are different (i.e., the maps are not conjugate on their Julia sets). The basic reason for the difference in the dynamical behavior is the following. In each Sierpinski hole, all of the critical points of F_λ eventually escape to ∞, which is a superattracting fixed point. That is, the critical orbits eventually enter the immediate basin of attraction of ∞. If it takes a different number of iterations for the critical orbits to land in the immediate basin for different maps, then the corresponding maps cannot be conjugate as shown in [DLU03]. Since there exist Sierpinski holes in which the critical orbits escape after exactly k iterations for each $k \geq 3$, this explains why we have different dynamical behavior in certain of these regions.

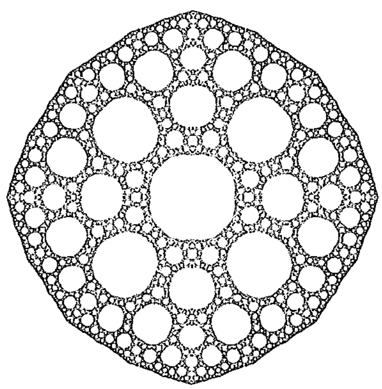


Figure 3·1: The Julia set for $F(z) = z^2 - 1/16z^2$.

However, this fact does not provide a method for understanding the dynamics on the Sierpinski curve Julia sets of these maps. Our goal in this paper is to use symbolic dynamics to provide a complete description of this dynamical behavior on these sets. Rather than deal with the general case, we concentrate on a single example, namely when $\lambda = -1/16$. This λ value is the center of the largest Sierpinski hole visible in the center of Figure 4·4. The Julia set of F_λ when $\lambda = -1/16$ is displayed in Figure 3·1. This parameter value has dynamical behavior that is the simplest to understand, but it should be clear from the analysis below how to extend this analysis to other, more complicated maps drawn from different Sierpinski holes. Our main result is:

Theorem. *There is a quotient space Σ of the space of one-sided sequences on four symbols on which the shift map is conjugate to the dynamics of F_λ on its Julia set.*

3.1 A Special Case

In this section we restrict attention to

$$F_{-1/16}(z) = z^2 - \frac{1}{16z^2},$$

that is, the case where $\lambda = -1/16$. We denote $F_{-1/16}$ by F for the remainder of this paper. In this section we give two especially simple examples of how symbolic dynamics can be used to analyze the dynamics of F on a pair of important invariant subsets of J. Later we use these two subsets as the cornerstone of the more complicated analysis of the dynamics on all of J.

The four critical points of F lie at the points $\omega/2$ where ω is a fourth root of -1. The critical values are given by $\pm v = \pm i/2$ and we have $F(\pm v) = 0$. Thus the second iterate of each of the critical points lands on the pole at the origin; this is what makes the case $\lambda = -1/16$ special. There are prepoles at $\pm 1/2$ as well as at $\pm i/2$.

We first investigate the dynamics of F on $\mathbb{R}$. Note that F preserves the real axis. The graph of F on this axis shows that there is a pair of repelling fixed points in $\mathbb{R}$ (see Figure 3·2). Let p be the fixed point in $\mathbb{R}^+$. The graph of F also shows that the orbit of $x \in \mathbb{R}$ tends directly to ∞ if $|x| > p$, so (p, ∞) and $(-\infty, -p)$ lie in $B = B_{-1/16}$ and $\pm p \in \partial B$.

Let I denote the interval $[-p, p]$ and let Γ be the set of points whose orbits remain in I for all iterations. Let $\pm q \in I$ be the points for which $F(\pm q) = -p$ so that $F^2(\pm q) = p$. If $x \in (-q, q)$ then $F(x) < -p$ and $F^2(x) > p$. Hence $F^n(x) \to \infty$ for all $x \in (-q, q)$. This interval is the *trap door* in $\mathbb{R}$; any orbit in I that enters this open interval falls through the trap door and then tends to ∞.

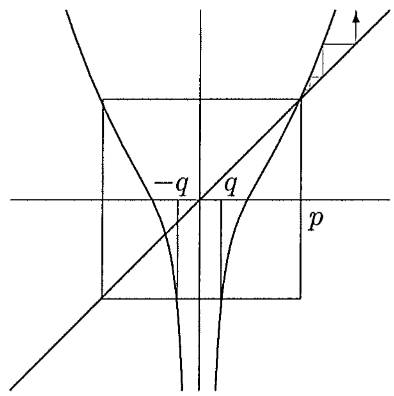


Figure 3·2: The graph of F on the real line.

Proposition. *The set of points Γ whose orbits remain in $[-p,p]$ for all iterations is a Cantor set, and F is conjugate to the one-sided shift map on two symbols on this set. Moreover $\Gamma \subset J(F)$.*

Proof: An easy computation shows that $|F'(x)|$ has a minimum in $\mathbb{R}$ at $\pm 3^{1/4}/2$, and that the minimum value of $|F'(x)|$ is $3^{1/4} + 3^{-3/4} \approx 1.75$. Hence F is expanding on $[-p,p]$. Let $I_a = [q,p]$ and $I_b = [-p,-q]$. Let Σ_2 be the sequence space on the two symbols a and b, and let $S : \Gamma \to \Sigma_2$ be the itinerary map that assigns the sequence $(s_0 s_1 s_2 \dots)$ to $x \in \Gamma$ where $s_j = a$ if $F^j(x) \in I_a$ or $s_j = b$ otherwise. Then standard arguments show that S is a homeomorphism that conjugates $F|\Gamma$ to the shift map on Σ_2. Therefore Γ is a Cantor set.

Since the orbits of points in Γ are bounded, it follows that Γ is contained in the filled Julia set of F. The set of points in Γ whose orbits eventually land on p are dense in Γ, and these points lie in the boundary of the set of escaping points. Hence the closure of this set of points, namely Γ, lies in $J(F)$.

$\square$

The preimage of $\mathbb{R}$ under F consists of the real and imaginary axes; each of these axes is mapped two-to-one over $\mathbb{R}$. Hence there is a Cantor set on the imaginary axis that is mapped in two-to-one fashion onto Γ in $\mathbb{R}$, and all other points on the imaginary axis lie in B.

The preimage of the imaginary axis consists of two sets: the four rays $\theta = \pm\pi/4,\ \pm 3\pi/4$ and the circle of radius $r = 1/2$ centered at the origin. We call this circle the *critical circle.* Note that the four rays meet the critical circle at the four critical points of F. A point on the critical circle given by $r = (1/2)e^{i\theta}$ is mapped to points of the form $(i/2)\ \sin(2\theta)$ on the imaginary axis. Therefore the critical circle is mapped in four-to-one fashion over the interval $[-1/2, 1/2]$ on the imaginary axis (except at the endpoints). Each of the four rays is mapped in two-to-one fashion over either $[1/2, \infty)$ or $(-\infty, -1/2]$ on the imaginary axis.

We now investigate the behavior of F near ∞.

Proposition. *The boundary of the basin of attraction of ∞ is an invariant simple closed curve ∂B on which F is conjugate to the map $z \to z^2$.*

Proof: Let W denote the annulus given by $3/4 \leq |z| \leq 2$. We claim that F is an expanding, two-to-one covering map on W. To see this, first consider the circle $r = (3/4)e^{i\theta}$. For z on this circle, we have

$$|F(z)| = \left|\frac{9}{16}e^{2i\theta} - \frac{1}{9}e^{-2i\theta}\right| \leq \frac{9}{16} + \frac{1}{9} < \frac{3}{4}.$$

Hence F maps this circle strictly inside itself. If $|z| \geq 2$, then we have

$$|F(z)| \geq |z|^2 - \frac{1}{16|z|^2} \geq |z|^2 - \frac{1}{64} > 1.5|z|.$$

Therefore F maps the circle $|z| = 2$ strictly outside itself and we have

$$|F^n(z)| \geq (1.5)^n |z|$$

for all $n \geq 1$ and $|z| \geq 2$. Therefore the entire region $|z| \geq 2$ lies in B.

If $|z| \geq 3/4$, we also have that

$$|F'(z)| \geq 2|z| - \frac{1}{8|z|^3} \geq \frac{3}{2} - \frac{8}{27} > 1,$$

so it follows that F is expanding on the annulus W. Since the critical points lie inside the circle $|z| = 3/4$, it follows that F is a two-to-one covering map on W, and $F(W) \supset W$. Standard arguments then show that the set of points whose orbits remain for all iterations inside W is a simple closed curve, and F is conjugate to z^2 on this curve. Moreover, all points outside this curve lie in B, and so this invariant curve is ∂B.

□

As a consequence of this result, the boundary of the trap door is also a simple closed curve in $\overline{\mathbb{C}}$, as are all other preimages of ∂B. We denote the boundary of the trap door by ∂T. Note that F maps ∂T onto ∂B in two-to-one fashion and that ∂T is disjoint from ∂B. By the Proposition, we also have the fact that $K(F)$ is the set of all points whose orbits remain for all time on or inside ∂B. We also remark that a similar result holds for other λ-values for which the critical orbit eventually enters T; the proof is more complicated as it involves quasiconformal surgery and the Measurable Riemann Mapping Theorem (see [DLU03]).

With an eye toward our discussion of the full symbolic dynamics on the Julia set, we introduce the usual coding of orbits on ∂B. Let Σ_2' be the sequence space on the two symbols 0 and 1 (not a and b, as before). In Σ_2' we identify two sequences that begin with the same finite string of digits and end in either all zeroes or ones. That is, if $s = (s_0 \ldots s_n\overline{0})$ and $t = (s_0 \ldots s_n\overline{1})$, then we identify these two points in Σ_2'. Let $\tilde{\Sigma}_2'$ denote the corresponding quotient space.

There is a natural conjugacy between the dynamics of F on ∂B and the one-sided shift map on $\tilde{\Sigma}_2'$ determined as follows. Let K_0 denote the portion of ∂B lying on or above the real axis, and let K_1 denote that portion lying on or below this axis. Note that, by our previous work, we know that ∂B meets the real axis only at the points $\pm p$. We then associate an itinerary $S(z)$ in $\tilde{\Sigma}_2'$ to each $z \in \partial B$ by recording how the orbit of z visits either K_0 or K_1 exactly as in the case of the dynamics on the real line. Since $K_0 \cap K_1 = \{\pm p\}$, it follows immediately that this assignment respects the identifications in $\tilde{\Sigma}_2'$ and yields a conjugacy.

Proposition. *The itinerary map S gives a conjugacy between $F \,|\, \partial B$ and the one-sided shift map on the quotient space $\tilde{\Sigma}_2'$.*

3.2 Cantor Necklaces

One of the principal objects contained in the Julia set of F is a Cantor necklace. To define this set, let Λ denote the Cantor middle thirds set in the unit interval $[0, 1]$. We regard this interval as a subset of the real axis in the plane. For each open interval of length $1/3^n$ removed from the unit interval in the construction of Λ, we replace this interval by a circle of diameter $1/3^n$ centered at the midpoint of the removed interval. Thus this circle meets the Cantor set at the two endpoints of the removed interval. We call the resulting set the *Cantor middle-thirds necklace* (see Figure 3·3). Any set homeomorphic to the Cantor middle-thirds necklace is called a *Cantor necklace.*

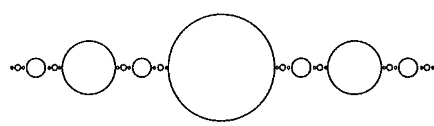


Figure 3·3: The Cantor middle-thirds necklace.

There is a natural invariant Cantor necklace contained in $J(F)$. Recall that ∂T is mapped in two-to-one fashion onto ∂B. Moreover, ∂T meets $\mathbb{R}$ only at the two points $\pm q$. There are four preimages of T in $\overline{\mathbb{C}}$, but only two of them meet $\mathbb{R}$. These two preimages of T meet the real axis in the two open intervals that were removed at the first stage of the construction of the invariant Cantor set Γ described in the previous section. Hence their boundaries are simple closed curves that meet Γ in two points that are the preimages of $\pm q$. Since ∂B and ∂T are disjoint, it follows that these two curves are also disjoint from ∂B

and ∂T. Continuing in this fashion, at the n^{th} stage, we replace the n^{th} intervals removed in the Cantor set construction with the corresponding n^{th} preimage of ∂T. The resulting set is then a Cantor necklace $\mathcal{N}$, and $\mathcal{N} \subset J(F)$ since every point in this set lies in the closure of $\cup F^{-n}(B)$.

Now consider $\partial B \cup \mathcal{N}$. This set is invariant under F. Any point in the Cantor set portion Γ of $\mathcal{N}$ has orbit that remains in Γ, whereas any other point in $\mathcal{N}$ eventually maps to ∂B, where the orbit is then trapped. We could use symbolic dynamics to expand our previous symbolic description to this set, but we will instead take a wider viewpoint and use this set as the skeleton of a larger symbolic description.

3.3 Symbolic Dynamics on the Julia Set

In this section we give a symbolic description of the itinerary of each point in the Julia set of F. Let A be the closed annulus bounded by ∂B and ∂T. All points in $J(F)$ are contained in A. Indeed, $J(F)$ is A minus all of the preimages of T. We divide A into four overlapping closed sets I_0, I_1, I_2, and I_3, each of which is a semi-annulus. I_0 and I_2 lie in the upper half plane while I_1 and I_3 lie in the lower half plane. I_0 and I_1 are bounded by portions of ∂B, the critical circle, and $\mathbb{R}$, while I_2 and I_3 are bounded by portions of ∂T, the critical circle, and $\mathbb{R}$ (see Figure 3·4).

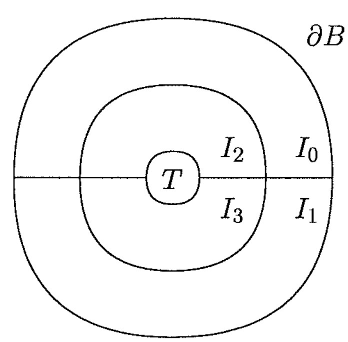


Figure 3·4: The regions I_j for $j = 0, 1, 2, 3$.

Note that F maps each I_j onto all of $\overline{\mathbb{C}} - B$. For example, the boundary of I_0 is mapped by F as follows:

1. $I_0 \cap \partial B$ is mapped one-to-one to all of ∂B (two-to-one to p);
2. $I_0 \cap \mathbb{R}^+$ is mapped one-to-one to $[0, p] \subset \mathbb{R}^+$;
3. $I_0 \cap \mathbb{R}^-$ is mapped one-to-one to $[-p, 0] \subset \mathbb{R}^-$;
4. the portion of the critical circle bounding I_0 is mapped two-to-one to the interval $[-v, v]$ on the imaginary axis.

Let Σ denote the set of one-sided sequences on the four symbols $0, 1, 2,$ and 3. We assign a sequence $S(z) = (s_0 s_1 s_2 \ldots) \in \Sigma$ to each $z \in J(F)$ in the usual fashion so that the entry $s_j = k$ if and only if $F^j(z) \in I_k$. $S(z)$ is the itinerary of z. Since $F(I_j) \supset \overline{\mathbb{C}} - B$ for each j, it follows easily that there is at least one point in $\overline{\mathbb{C}} - B$ associated to each sequence in Σ. These points have bounded orbits and so must lie in J. So S gives a map from $J(F)$ onto Σ.

In order for S to capture the precise dynamics of F on the Julia set, we need to modify this picture in several ways. First, certain points in $J(F)$ correspond to two different sequences since the I_j overlap, so we must first make a number of identifications in Σ, much as we did earlier on ∂B. Second, F will not be conjugate to the shift map on this quotient space, but rather to a "quotient" of the shift map.

We first modify Σ to take into account the identifications necessary along ∂B. As is Section 3, we identify any pair of sequences in Σ of the form $(s_0 s_1 \ldots s_n \overline{0})$ and $(s_0 s_1 \ldots s_n \overline{1})$. Let Σ^1 denote the resulting quotient space. Then we have:

Proposition. *Let $s = (s_0 s_1 s_2 \ldots) \in \Sigma^1$ be a sequence consisting of only 0s or 1s. Then there is a unique point in $J(F)$ whose itinerary is s and this point lies in ∂B.*

Proof: Let $I_{s_0 s_1 \ldots s_n}$ denote the set of points whose itinerary begins $s_0 s_1 \ldots s_n$. If each of the s_j is either 0 or 1, then $I_{s_0 s_1 \ldots s_n}$ is a closed, simply connected set that is properly contained in $I_{s_0s_1\dots s_{n-1}}$ and F^n maps $I_{s_0s_1\dots s_n}$ in one-to-one fashion onto I_{s_n}. Note that each $I_{s_0s_1\dots s_n}$ intersects ∂B in an arc. Using the appropriate branches of the inverse of F, we have that F^{-n} is a contraction in the Poincaré metric on I_{s_n} taking I_{s_n} onto $I_{s_0s_1\dots s_n}$. Hence the intersection of these sets as $n \to \infty$ is a unique point which necessarily lies in ∂B, and this point has the itinerary s.

□

The description of the further identifications that must be made in Σ^1 is more complicated. The reason for this is that the sets $I_{s_0s_1\dots s_n}$ are not always simply connected if the s_j involve 2s or 3s. For example, the interior of the set I_{02} consists of two disjoint components whose boundaries meet at a single critical point.

3.4 Symbolic Dynamics on the Real Line

To describe the further identifications in Σ^1, we begin on the real axis. Note that our previous symbolic description of points in $J(F) \cap \mathbb{R}$ (using the symbols a and b) is no longer valid. Now we have four symbols describing such points and each point in $J(F) \cap \mathbb{R}$ will have two distinct symbolic representations.

Recall that $\mathcal{N}$ denotes the Cantor necklace lying along $\mathbb{R}$ and that Γ denotes the Cantor set given by $\mathcal{N} \cap \mathbb{R}$. We have $F(\mathcal{N}) = \mathcal{N} \cup \partial B$. If we remove the set of points in $\mathcal{N}$ whose images do not lie in $\mathcal{N}$, then we are left with a pair of Cantor subnecklaces, one to the left of the trap door T (extending from $-p$ to $-q$ along $\mathbb{R}^-$) and one to the right of T (extending from q to p along $\mathbb{R}^+$). Indeed, we simply remove the upper and lower open semicircles on the boundary of T (i.e., $\partial T - \{\pm q\}$) from $\mathcal{N}$ to produce these subnecklaces.

The preimage in $\mathbb{R}$ of this pair of subnecklaces then consists of four Cantor subnecklaces, and then the preimage in $\mathbb{R}$ of these subnecklaces consists of eight subnecklaces. Call this set of eight subnecklaces $\mathcal{N}'$.

Points that lie the upper or lower portions of a subnecklace in $\mathcal{N}'$ can be distinguished using a pair of symbols. Recall that $I_{\alpha\beta}$ denotes the set of points in J whose itinerary begins with the string $\alpha\beta$. Each of the eight subnecklaces in $\mathcal{N}'$ is associated to a pair of distinct $I_{\alpha\beta}$; one of the $I_{\alpha\beta}$ contains all points in the upper half of a subnecklace in $\mathcal{N}'$, the other to the bottom half. Using the dynamics of F on $\mathbb{R}$, we see that the upper pieces of the subnecklaces in $\mathcal{N}'$ are contained from left to right in

$$I_{01}, I_{03}, I_{23}, I_{21}, I_{20}, I_{22}, I_{02}, I_{00}.$$

The bottom pieces are contained from left to right in

$$I_{10}, I_{12}, I_{32}, I_{30}, I_{31}, I_{33}, I_{13}, I_{11}.$$

Applying F, we then see that, for example,

$$F(I_{00}) \supset I_{00} \cup I_{02}$$

and

$$F(I_{23}) \supset I_{32} \cup I_{30}.$$

This implies that points in I_{00} have itineraries that begin 000 or 002 while points in I_{23} have itineraries that begin 230 or 232. We write this more succinctly as

$$00 \to 00\{0,2\} \text{ and } 23 \to 23\{0,2\}.$$

A straightforward computation shows that we have, in order,

$$00 \to 00\{0,2\},\ 02 \to 02\{0,2\},\ 22 \to 22\{1,3\},\ 20 \to 20\{1,3\},$$

$$21 \to 21\{0,2\},\ 23 \to 23\{0,2\},\ 03 \to 03\{1,3\},\ 01 \to 01\{1,3\},$$

$$11 \to 11\{1,3\},\ 13 \to 13\{1,3\},\ 33 \to 33\{0,2\},\ 31 \to 31\{0,2\},$$

$$30 \to 30\{1,3\},\ 32 \to 32\{1,3\},\ 12 \to 12\{0,2\},\ 10 \to 10\{0,2\},$$

Note that, in this recipe, if an even digit is preceded by a 0 or 1, then the following digit must also be even. If the even digit is preceded by a 2 or 3, the following digit must be odd. Similarly, an odd digit preceded by 0 or 1 is followed by an odd digit while it is followed by an even digit if preceded by a 2 or 3. More succinctly, we have:

Proposition (The Criterion for Real Itineraries.) *Suppose* $s = (s_0 s_1 s_2 \dots)$ *is a sequence that corresponds to a point in* $J \cap \mathbb{R}$. *Then for each* j, s_j *satisfies*

1. *If* $s_j = 0$ *or* 1, *then* s_{j+1} *and* s_{j+2} *have the same parity;*

2. *If* $s_j = 2$ *or* 3, *then* s_{j+1} *and* s_{j+2} *have different parities.*

Conversely, any sequence that satisfies this rule for each s_j *corresponds to a unique point in* $J \cap \mathbb{R}$.

We say that a sequence in Σ^1 corresponds to a real itinerary if it obeys this criterion for all j. Using the criterion for real itineraries, we can now "identify" all of the sequences corresponding to points in $J \cap \mathbb{R}$. Let $R : \Sigma^1 \to \Sigma^1$ be the map that exchanges 0's and 1's or 2's and 3's in each sequence in Σ^1. Then we have that, if $s \in \Sigma^1$ corresponds to a point in $J \cap \mathbb{R}$, then $R(s)$ also corresponds to such a point and, moreover, these two points are the same. Therefore we identify any two such sequences in Σ^1. Note that, under this identification, the sequences $(\overline{0})$ and $(\overline{1})$ are identified, as are $(0\overline{1})$ and $(1\overline{0})$. These represent the two points where this new identification coincides with the previous identification.

Now suppose we have a sequence in Σ^1 that does not correspond to a real itinerary. Then there are two possibilities:

1. Either there are only finitely many s_j for which the criterion for a real itinerary fails, or

2. There are infinitely many s_j for which this criterion fails.

In the first case, such a sequence must be of the form

$$(s_0 \dots s_n t_{n+1} t_{n+2} \dots)$$

where n is the largest digit for which the criterion for real itineraries fails. We say that n is the *real itinerary marker.* If the real itinerary marker is 0, then any point with such an itinerary lies along the imaginary axis. Any such point corresponds to two distinct itineraries having the form $(j, (s))$ and $(j, R(s))$ where $j = 0, 1, 2$, or 3 and s is a sequence that corresponds to a real itinerary. Therefore we identify these two sequences in Σ^1 as well.

If the real itinerary marker is 1, then the situation is different. In this case the corresponding points in J lie either on the critical circle or on the four straight rays connecting the origin to ∞ and passing through one of the critical points. The points lying on the critical circle are mapped to the portion of the imaginary axis in $I_2 \cup I_3$ while the other points are mapped to the portion in $I_0 \cup I_1$. The former case presents a problem since points on the critical circle lie on the boundaries between two of the I_j; we deal with this more complicated case in the next section. In the latter case, the itinerary of such a point is either $(s_0, j, (s))$ or $(s_0, j, R(s))$ where $j = 0$ or $j = 1$ and s is a sequence that corresponds to a real itinerary. Hence we identify two such sequences as above.

Finally, if the real itinerary marker is $j \geq 2$ and $s_j = 0$ or 1, we have a similar pair of sequences that correspond to the same point. Again these two sequences are identified. With all of these identifications, we now have a quotient space of Σ^1 that we call Σ^2.

3.5 Symbolic Dynamics on the Critical Circle

The only other points where the I_j intersect lie along the critical circle, so we now describe the identifications that these intersections cause in Σ^2. Sequences that correspond to points on the critical circle have the form $(s_0, j, (s))$ where $j = 2$ or $j = 3$ and s corresponds to a real itinerary.

Let C denote the critical circle. Recall that F maps C four-to-one onto the portion of the imaginary axis lying between the two critical values $\pm v$. One of these critical values is located at the intersection of I_0 and I_2 on the positive imaginary axis; the other is located at the intersection of I_1 and I_3 on the negative imaginary axis.

Recall also that F maps the portion of the imaginary axis lying in I_2 in one-to-one fashion onto the interval $[0, p]$ where we recall that p is the fixed point in $\mathbb{R}^+$ and q is the preimage of $-p$ lying in $\mathbb{R}^+$. So there is a Cantor subnecklace along this portion of the imaginary axis that is mapped onto the portion of the Cantor necklace $\mathcal{N}$ along the interval $[q, p]$. We call this subnecklace U. There is a similar Cantor subnecklace $V = -U$ in I_3 that is also mapped homeomorphically onto the same portion of $\mathcal{N}$ (see Figure 3·5).

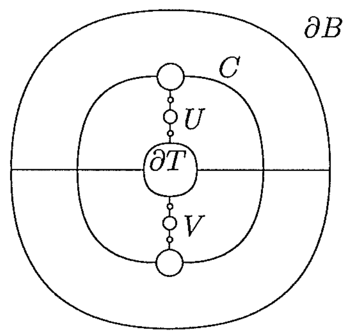


Figure 3·5: The Cantor subnecklaces U and V.

For simplicity, in our figures we will henceforth represent the subnecklaces U and V by lines (see Figure 3·6).

U can be divided into two equal sized subnecklaces which we will denote U_{in} and U_{out} with U_{out} lying above U_{in} on $i\mathbb{R}$. We create this division by removing the set in U that is mapped onto $\partial T - \{\pm q\}$ in two iterations. Likewise, let $V_{in} = -U_{in}$ and $V_{out} = -U_{out}$ (see Figure 3·7). Any point in the Cantor set portion of these subnecklaces lying in I_j has itinerary $(js_0s_1\dots)$ where each entry in the sequence $(s_0s_1s_2\dots)$ satisfies the criterion for a real itinerary but the initial digit j does not, so the real itinerary marker is 0.

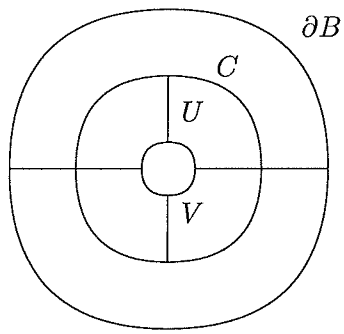


Figure 3·6: The Cantor subnecklaces U and V represented by lines.

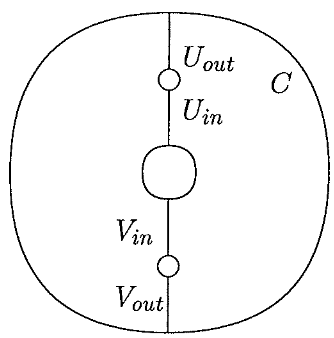


Figure 3·7: Location of U_{in} and U_{out} within C.

For the identifications that occurred along the real line we needed distinct notations for the upper and lower pieces of the Cantor necklace along $\mathbb{R}$. Similarly, we will speak of the left and right pieces of U and V. Our convention will be that points in the right pieces have nonnegative real parts while points in the left pieces have nonpositive real parts. For example, we will denote the left piece of U_{out} by $U_{(out,L)}$ and the right piece by $U_{(out,R)}$. Since $U \subset I_2$ and $V \subset I_3$ it follows that any point in U has itinerary that begins with 2 while any point in V has itinerary that begins with with 3. It is a matter of computation to see that $F(U_{(out,R)}) \subset I_2$ and that $F^{(2)}(U_{(out,R)})$ is contained is I_0 or I_2. Hence, itineraries of points in $U_{(out,R)}$ must begin with 220 or 222. We will write this as

$$U_{(out,R)} \to 22\{0,2\}.$$

Adopting this notation we compute immediately that

$$U_{(out,R)} \to 22\{0,2\}, U_{(out,L)} \to 23\{1,3\}, U_{(in,L)} \to 21\{1,3\}$$

$$U_{(in,R)} \to 20\{0,2\}, V_{(in,R)} \to 31\{1,3\}, V_{(out,R)} \to 33\{1,3\}$$

$$V_{(in,L)} \to 30\{0,2\}, V_{(out,L)} \to 32\{0,2\}.$$

We turn now to the identifications in Σ^2 corresponding to points in C. Recall that C is mapped four-to-one onto the interval $[-v, v]$ on the imaginary axis. Consider the preimage of the Cantor necklace U along the critical circle. This preimage consists of four Cantor necklaces, two in each of the first and third quadrants. Each of these preimages is mapped one-to-one onto U. Similarly the necklace V has four preimage necklaces, two each in the second and fourth quadrants. Let C_1 denote the two preimages of U lying in the first quadrant. Continuing counterclockwise from C_1, we label the other preimages of U or V by C_2, C_3, and C_4 (see Figure 3·8).

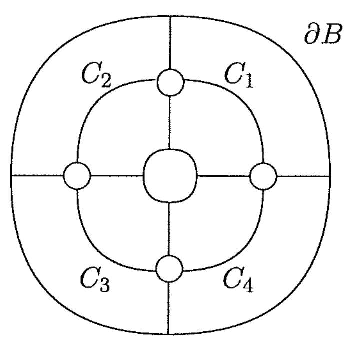


Figure 3·8: The location of the Cantor subnecklaces C_i for $i = 1, 2, 3, 4$.

Note that C_1 and C_2 lie along the critical circle separating I_0 from I_2 while C_3 and C_4 lie on the critical circle separating I_1 from I_3. We next break each of the C_j into four equal sized subnecklaces by removing the middle preimage of ∂T from each. Equivalently, we consider the preimages of the smaller subnecklaces U_{in}, U_{out}, V_{in}, and V_{out}. This yields sixteen Cantor subnecklaces lying along the critical circle, each of which is mapped to the one of the subnecklaces on the imaginary axis. Each of these sixteen subnecklaces has two pieces; one outside C and one inside C. The outside piece of C_1 that lies in I_0 and meets the preimage of ∂T along $\mathbb{R}^+$ is mapped to $U_{(in,R)}$. Starting at this piece and continuing counterclockwise along the critical circle, the outer pieces of the sixteen Cantor subnecklaces lying in $I_0 \cup I_1$ are mapped to

$$U_{(in,R)},\ U_{(out,R)},\ U_{(out,L)},\ U_{(in,L)},\ V_{(in,L)},\ V_{(out,L)},\ V_{(out,R)},\ V_{(in,R)},$$

$$U_{(in,R)},\ U_{(out,R)},\ U_{(out,L)},\ U_{(in,L)},\ V_{(in,L)},\ V_{(out,L)},\ V_{(out,R)},\ V_{(in,R)},$$

respectively. Figure 3·9 shows the pieces of C_1 in I_0 (as well as the pieces in I_2) marked by where they are mapped via F. The first eight of these pieces of necklaces live in I_0 while the second eight live in I_1, giving us itineraries that begin, in the above order, with

$$020\{0,2\},\ 022\{0,2\},\ 023\{1,3\},\ 021\{1,3\},$$

$$030\{0,2\},\ 032\{0,2\},\ 033\{1,3\},\ 031\{1,3\},$$

$$120\{0,2\},\ 122\{0,2\},\ 123\{1,3\},\ 121\{1,3\},$$

$$130\{0,2\},\ 132\{0,2\},\ 133\{1,3\},\ 131\{1,3\}.$$

Now, starting at the inner piece of C_1 that lies in I_2 and intersects the preimage of ∂T on the positive real axis and continuing counterclockwise along the critical circle we see

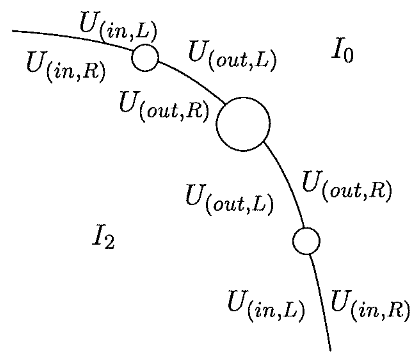


Figure 3·9: The locations of the eight pieces of C_1 marked via their images.

that the inner pieces of the sixteen Cantor subnecklaces lying in $I_2 \cup I_3$ are mapped to

$$U_{(in,L)},\ U_{(out,L)},\ U_{(out,R)},\ U_{(in,R)},\ V_{(in,R)},\ V_{(out,R)},\ V_{(out,L)},\ V_{(in,L)},$$

$$U_{(in,L)},\ U_{(out,L)},\ U_{(out,R)},\ U_{(in,R)},\ V_{(in,R)},\ V_{(out,R)},\ V_{(out,L)},\ V_{(in,L)},$$

respectively. The itineraries of points in these sets begin

$$221\{1,3\},\ 223\{1,3\},\ 222\{0,2\},\ 220\{0,2\},$$

$$231\{1,3\},\ 233\{1,3\},\ 232\{0,2\},\ 230\{0,2\},$$

$$321\{1,3\},\ 323\{1,3\},\ 322\{0,2\},\ 320\{0,2\},$$

$$331\{1,3\},\ 333\{1,3\},\ 332\{0,2\},\ 330\{0,2\}.$$

Note that the second entry for all 32 of these itineraries is either a 2 or a 3. This is because the critical circle is mapped onto the portion of the imaginary axis between the boundary of the trap door and the critical value, and this region is in $I_2 \cup I_3$.

With this information, we can now identify the itineraries of points lying on the critical circle.

Proposition (Criterion for a Critical Circle Itinerary.) *Any point in $J(F) \cap C$ has itinerary of the form $(s_0 s_1 s_2 \dots)$ where the real itinerary marker is 1 and either $s_1 = 2$ or $s_1 = 3$. Conversely, any itinerary that satisfies these conditions corresponds to a point in $J(F) \cap C$.*

Finally, let $s = (s_0 s_1 s_2 \dots)$ be a sequence in Σ^2 corresponding to a point in $J \cap C$ and let $H : \Sigma^2 \to \Sigma^2$ denote the map that changes s_0 and s_j for each $j \geq 2$ as follows:

1. 0s are interchanged with 2s and 1s are interchanged with 3s in s_0,

2. 0s are interchanged with 1s and 2s are interchanged with 3s in s_j if $j \geq 2$.

If $s \in \Sigma^2$ corresponds to a point in $J \cap C$ then $H(s)$ also corresponds to such a point and, moreover, the points corresponding to s and $H(s)$ are the same. Hence we identify these sequences in Σ^2. Similarly any pair of sequences whose first n entries are the same and whose tail is one of these two identified sequences should also be identified in Σ^2. This then gives a new quotient space Σ^3.

We remark that there are two points in C_1 that are mapped to the same point on the imaginary axis. If one of these points corresponds to the identified sequences $(02s)$ and $(22R(s))$, then the other point corresponds to the pair of identified sequences $(22s)$ and $(02R(s))$. So the sequences corresponding to the images of these points are $(2s)$ and $(2R(s))$, both of which have already been identified in Σ^3.

Note also that the sequences that have been identified to form the quotient space Σ^3 correspond to points in $J(F)$ whose orbits eventually land in the invariant Cantor set in $\mathbb{R}$.

3.6 Symbolic Dynamics

Given all of our work constructing the space Σ^3, we can now prove:

Theorem. *The map F on $J(F)$ is conjugate to the shift map on the quotient space Σ^3.*

Proof: Recall that A is the annulus bounded by ∂B and ∂T, i.e., the closed annulus between the basin of ∞ and the trap door. Let $s = (s_0 s_1 s_2 \dots) \in \Sigma^3$ and recall that

$$I_{s_0 s_1 \dots s_n} = \{z \in A \,|\, z \in I_{s_0}, F(z) \in I_{s_1}, \dots, F^n(z) \in I_{s_n}\}.$$

As we have seen, I_j is a closed set that is homeomorphic to a closed disk for each j. We also have

$$I_{s_0 s_1 \dots s_n} \subset I_{s_0 s_1 \dots s_{n-1}} \subset \dots \subset I_{s_0}$$

for each n. If F were one-to-one on each I_j, then we would have a well defined branch of

$$F^{-n} : I_{s_n} \to I_{s_0 s_1 \dots s_n}$$

and this map would then be a contraction in the Poincaré metric on these spaces. Standard arguments would then show that

$$\bigcap_{n=0}^{\infty} I_{s_0 s_1 \dots s_n}$$

is a unique point with the given itinerary, and this would then give the conjugacy.

Unfortunately, F is not one-to-one on I_j. However, the only places where this fails is on the portions of the boundary of I_j that meet the critical circle and the real axis. We remedy this situation as follows.

For clarity, let us restrict to I_0; the other cases are similar. Consider the subsets of I_0 of the form I_{0j}. We now stipulate that I_{0j} corresponds to a closed subset of I_0 that meets the interior of I_0, i.e., this set itself has interior. This eliminates certain intervals from I_{0j} along the real axis that previously made up portions of I_{0j}. In particular, every point in $\mathbb{R} \cap I_0$ now belongs to a unique I_{0j}. So the map $F : I_{0j} \to I_j$ is now one-to-one on $\mathbb{R}$.

Next note that I_{02} and I_{03} now consist of a pair of closed disks that are joined at a single critical point. Preimages of these sets have a similar structure. On the other hand, both I_{00} and I_{01} are now homeomorphic to a closed disk. So we now adopt the convention that I_{02} is one of these two closed disks. That is, two sets bear the name I_{02}. We make the same convention for other sets $I_{s_0s_1\ldots s_n}$ that are similarly joined at isolated points. This presents no ambiguity since F maps one of the sets I_{02} onto the portion of I_2 lying in the right half plane and F maps the other set I_{02} to the left half plane. Hence one of the sets I_{02} contains only I_{022} and I_{020} while the other contains only I_{021} and I_{023}. That is, the subsequent digit in the sequence s determines which disk to consider in the chain $I_{s_0s_1\ldots s_n}$. The set I_{03} has similar properties.

With this specification of $I_{s_0s_1\ldots s_n}$, it now follows that $F^n : I_{s_0s_1\ldots s_n} \to I_{s_0}$ is one-to-one for each n. This completes the proof.

Chapter 4

Buried Sierpinski Curve Julia Sets

In all of the previous cases where the Julia set is a Sierpinski curve, the complementary domains (or the Fatou components) are always preimages of the immediate basin of attraction of ∞, which is a superattracting fixed point for these maps (provided $n \geq 2$). In this chapter, we exhibit a similar infinite collection of dynamically distinct Julia sets, but now the Fatou components are quite different. Instead of being preimages of a single superattracting basin at ∞, we give examples where the complementary domains consist of a collection of a number of different attracting basins together with the basin at ∞ and all of the preimages of these basins. As before, we prove that the dynamics on these Julia sets are all distinct from one another as well as from those mentioned above, but again, all of these Julia sets are homeomorphic.

For simplicity, we restrict attention in this chapter to the special family $F_\lambda(z) = z^2 + \lambda/z$. At the end of the chapter, we describe generalizations to other higher degree families of the form $z^n + \lambda/z^d$. In Figure 4·1, we display the Julia set of F_λ when $\lambda = -0.327$. For this

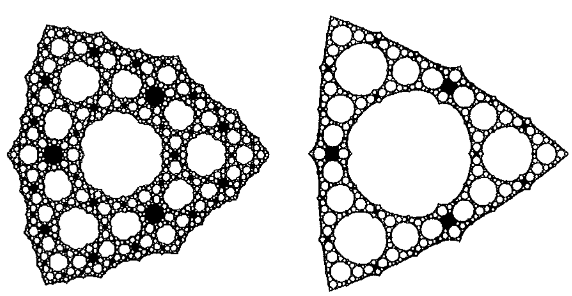


Figure 4·1: The Julia sets for $\lambda = -0.327$ and $\lambda = -0.5066$.

map, there are attracting basins of period 3 and period 6 together with the basin at ∞. We also display the case where $\lambda = -0.5066$ for which there are three different attracting basins of period 4 together with the basin at ∞. In these figures, the black regions represent the basins of the finite attracting cycles while the white regions form the basin of ∞.

4.1 Preliminaries for F_λ when $n = 2, d = 1$.

Consider the degree three family of rational maps of the complex plane given by $F_\lambda(z) = z^2 + \lambda/z$ where λ is a parameter. There are four critical points for F_λ, one at ∞ and the other three of the form $\omega^k c_\lambda$ where $c_\lambda = (\lambda/2)^{1/3}$ is one of the finite critical points and ω is a cube root of unity. So the critical points are arranged with three-fold symmetry about the origin. Similarly, the critical values are arranged symmetrically with respect to ω and are given by $\omega^k v_\lambda$ where

$$v_\lambda = \frac{3}{2^{2/3}}\lambda^{2/3}.$$

There are also three symmetric prepoles given by $(-\lambda)^{1/3}$.

Here, $F_\lambda(\omega z) = \omega^2 F_\lambda(z)$. Therefore, as before, the orbits of points of the form $\omega^j z$ all behave "symmetrically" under iteration of F_λ. Further note that, when λ is real, $F_\lambda(\overline{z}) = \overline{F_\lambda(z)}$, and therefore the orbits of the points z and $\overline{z}$ also behave symmetrically in this case.

In this work we shall restrict attention to the case where $\lambda \in \mathbb{R}^-$. For these λ-values there exists a unique critical point in $\mathbb{R}^-$ which we call $c = c(\lambda)$. Since $\mathbb{R}$ is mapped to itself by F_λ, it follows that $F_\lambda^n(c) \in \mathbb{R}$ for all $n \geq 0$. By symmetry there is a critical point on each of the two lines $\omega\mathbb{R}$ and $\omega^2\mathbb{R}$. Note that $F_\lambda : \omega\mathbb{R} \mapsto \omega^2\mathbb{R}$ and vice versa. We call the three lines $\mathbb{R}$, $\omega\mathbb{R}$, and $\omega^2\mathbb{R}$ the *symmetry axes.* While the orbit of c is trapped in $\mathbb{R}$, the other two critical orbits jump between $\omega\mathbb{R}$ and $\omega^2\mathbb{R}$ at each iteration. Therefore, if there is an attracting n-cycle on $\mathbb{R}$, this cycle attracts only c. By symmetry, there must be attracting cycles on $\omega\mathbb{R} \cup \omega^2\mathbb{R}$ that attract the other two critical points. Since $F_\lambda(\overline{z}) = \overline{F_\lambda(z)}$ and $F_\lambda(\omega z) = \omega^2 F_\lambda(z)$, if there is an attracting n-cycle on $\mathbb{R}$, then $\omega\mathbb{R} \cup \omega^2\mathbb{R}$ has either an attracting $2n$-cycle (when n is odd) or a pair of symmetric n-cycles (when n is even). Since there are only three (finite) critical points, it follows that, if there is an attracting n-cycle on $\mathbb{R}$, these are the only other possibilities for attracting cycles in $\overline{\mathbb{C}}$.

Consider the intervals connecting the critical values to 0 along each of the three symmetry axes. These intervals lie in the rays $\omega^j\mathbb{R}^+$ for $j = 0, 1, 2$. One checks easily that the preimage of the union of these intervals contains a simple closed curve κ that surrounds the origin. All three of the critical points lie in κ as do the three prepoles. See Figure 4·2. We call κ the *critical curve.* Now consider the three rays given by $\mathbb{R}^-$ and its two symmetric images under $z \mapsto \omega z$. These three rays divide the region inside κ into three sectors which we call the *critical sectors.* We denote by S_0 the critical sector that meets the positive real axis. A straightforward computation shows that F_λ maps S_0 onto the sector $2\pi/3 \leq \operatorname{Arg} z \leq 4\pi/3$ in one-to-one fashion.

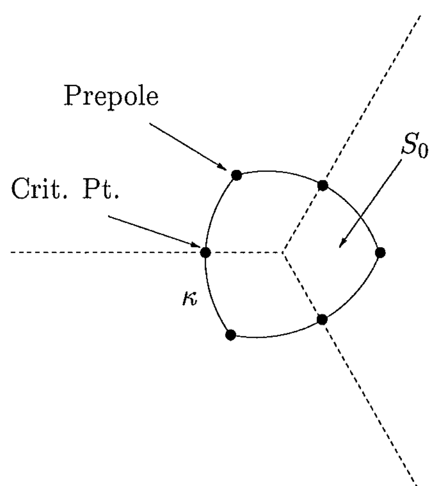


Figure 4·2: The critical curve and critical sectors.

Defining B_λ and T_λ as before we have:

Proposition. *Both B_λ and T_λ have 3-fold symmetry, i.e., if $z \in B_\lambda$, then $\omega z \in B_\lambda$ as well.*

Proof: Let $U \subset B_\lambda$ be the set of points z in B_λ that have the property that the point ωz also lies in B_λ. U is an open, nonempty set since B_λ contains an open neighborhood around ∞. If $U \neq B_\lambda$, let $z_0 \in \partial U$. Then $z_0 \in B_\lambda$ but $\omega z_0 \notin B_\lambda$. Hence $\omega z_0 \in \partial B_\lambda$. Therefore $F^i_\lambda(z_0) \to \infty$ whereas $F^i_\lambda(\omega z_0) \not\to \infty$. But

$$F^i_\lambda(\omega z_0) = \omega^{2^i} F^i_\lambda(z_0) \to \infty.$$

This gives a contradiction.

The case of T_λ is similar.

□

By symmetry of B_λ, if one of the critical points lies in B_λ, then all of the critical points do. The same is true if one of the critical points lies in the i^{th} preimage of T_λ, $F_\lambda^{-i}(T_\lambda)$, with $i > 0$. In this case, it is known that each set $F_\lambda^{-i}(T_\lambda)$ has multiple components and the critical points always lie in different components [DLU03].

We will further restrict attention to the case where F_λ has an attracting cycle on $\mathbb{R}$, and hence all three critical points are attracted to cycles. Therefore we are in case 2 of the Escape Dichotomy and so $J(F_\lambda)$ is connected.

The graph of F_λ on $\mathbb{R}$ shows that B_λ meets $\mathbb{R}$ in the intervals $(p(\lambda), \infty)$ and $(-\infty, q(\lambda))$, where $p(\lambda)$ is the fixed point in $\mathbb{R}^+$ and $q(\lambda)$ is the leftmost preimage of $p(\lambda)$ in $\mathbb{R}^-$. See Figure 4·3.

There are no points in $[q(\lambda), p(\lambda)] \cap B_\lambda$ since, by the $\overline{z}$ symmetry, B_λ would then not be simply connected, or equivalently, $J(F_\lambda)$ would not be connected. That would contradict the Escape Dichotomy Theorem.

4.2 Existence of Superattracing Cycles

In this section we will show that there is a sequence $\lambda_n \in \mathbb{R}^-$ with $n = 3, 4, \ldots$ having the property that F_{λ_n} has a superattracting cycle of period n lying in $\mathbb{R}$. We will later prove

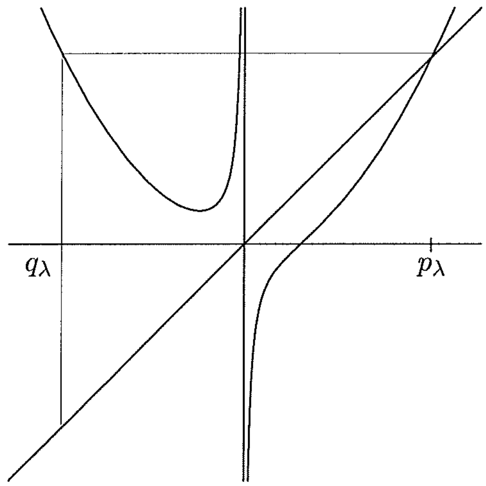


Figure 4·3: The graph of F_λ.

that $J(F_{\lambda_n})$ is a Sierpinski curve.

Let $\lambda^* = -16/27$. A straightforward calculation shows that F_{λ^*} has a repelling fixed point at $4/3$. The critical point on the real axis for this map is given by $-2/3$ and the critical value is $4/3$, so for λ^* the real critical point of F_{λ^*} maps to the fixed point $p(-16/27) = 4/3$.

We consider nearby λ-values. The critical point on the real axis is given by $c(\lambda) = (\lambda/2)^{1/3}$ and the critical value is given by

$$v(\lambda) = \frac{3}{2^{2/3}}\lambda^{2/3}.$$

Recall that $p(\lambda)$ is the real fixed point for F_λ. The graph of F_λ shows that $p(\lambda) > 1$ when $\lambda < 0$. Now $p(\lambda)$ satisfies the equation

$$(p(\lambda))^3 - (p(\lambda))^2 + \lambda = 0.$$

Using this we derive

$$p'(\lambda) = \frac{-1}{3(p(\lambda))^2 - 2p(\lambda)}$$

so that $p'(\lambda^*) = -3/8$. Using the fact that $p(\lambda) > 1$ it follows that $-1 < p'(\lambda) < 0$ for all negative λ. Also, since

$$v(\lambda) = \frac{3}{2^{2/3}}\lambda^{2/3},$$

we have

$$v'(\lambda) = (2/\lambda)^{1/3} < -1$$

as long as $-2 < \lambda < 0$. Therefore it follows that for $\lambda \in (\lambda^*, 0] = (-16/27, 0]$, $v(\lambda)$ decreases faster than $p(\lambda)$, and so $v(\lambda) < p(\lambda)$ for these values of the parameter (since $p(\lambda^*) = v(\lambda^*)$).

Proposition. *There exists a decreasing sequence λ_n for $n \geq 3$ with $\lambda_n \to \lambda^*$ and having the property that F_{λ_n} has a superattracting cycle of period n given by $x_j(\lambda_n) = F_{\lambda_n}(x_{j-1}(\lambda_n))$ where*

1. *$x_0(\lambda_n) = x_n(\lambda_n) = c(\lambda_n)$, and*
2. *$x_0 < 0 < x_{n-1} < x_{n-2} < \cdots < x_1 = v(\lambda_n) < p(\lambda_n)$.*

Proof: If $0 < x < p(\lambda)$, then the iterates $F_\lambda^j(x)$ decrease with j as long as $F_\lambda^{j-1}(x)$ remains positive. In particular, we may choose λ close enough to λ^* so that the forward orbit of $v(\lambda)$ remains in the interval $(0, p(\lambda))$ for as many iterates as we desire.

We claim that there exists a sequence μ_n with $n \geq 2$ satisfying

1. $F_{\mu_n}^n(c(\mu_n)) = 0$, and
2. $0 < F_{\mu_n}^j(c(\mu_n)) < p(\mu_n)$ for $j = 1, \ldots, n-1$.

To see this, note first that μ_2 may be chosen to be $-4/27$. Define $G_n(\lambda) = F_\lambda^n(c(\lambda))$. So $G_2(\mu_2) = 0$ while $G_2(\lambda^*) = 4/3$. Then G_3 maps the interval (λ^*, μ_2) over the entire half line $(-\infty, 4/3)$, so that there exists μ_3 with $\lambda^* < \mu_3 < \mu_2$ and $G_3(\mu_3) = 0$. Continuing inductively yields the sequence μ_n.

Now consider G_n on the interval $(\mu_n, \mu_{n-1}]$ for $n \geq 3$. G_n maps this interval over at least the negative real axis since $G_n(\mu_n) = 0$ and $G_n(\mu_{n-1}) = F_{\mu_{n-1}}(0) = -\infty$, so there exists a λ_n in this interval with $G_n(\lambda_n) = c(\lambda_n)$. This yields the parameters λ_n.

4.3 Buried Basins

In this section we fix a particular parameter value $\lambda = \lambda_n$ for which F_λ has a superattracting periodic point $x_0 = c(\lambda_n)$ lying in $\mathbb{R}^-$ as described in the previous section. Let A_j denote the immediate basin of attraction in $\overline{\mathbb{C}}$ of $x_j = F_\lambda^j(x_0)$. So $F_\lambda^j(A_0) = A_j$. Let $C_j = \omega A_j$ and $C_{j+n} = \omega^2 A_j$. The C_i are the basins of the nonreal superattracting cycle(s), but the indexing here does not necessarily correspond to the iteration, i.e., it is not in general true that $F_\lambda^j(C_0) = C_j$. Also, recall from Section 1 that the C_j surround a pair of attracting n-cycles when n is even and a single attracting $2n$-cycle when n is odd.

We say that a basin of attraction of F_λ is *buried* if the boundary of this basin is disjoint from the boundaries of all other basins of attraction (including B_λ). Note that, if the basin of one point on an attracting cycle is buried, then so too are all forward and backward images of this basin, so the entire basin of the cycle is buried. Our goal is to show that all of the basins of F_λ are buried. To accomplish this, it suffices to show that A_0 and B_λ are buried, for if that is the case, then all forward and backward images of A_0 and B_λ are also buried. By symmetry, the basins of the symmetric cycles are also buried since each C_j has the form $\omega^i A_k$ for some i and k.

We begin by showing that ∂A_0 and ∂B_λ are disjoint. Recall that, in Section 1, we showed that the interval $[q(\lambda), p(\lambda)]$ does not meet B_λ, but that $q(\lambda)$ and $p(\lambda)$ lie in ∂B_λ. By symmetry, the corresponding intervals on the other two symmetry axes also do not meet B_λ. We claim that the endpoints of these three intervals are the only points in the intersection of ∂B_λ and the symmetry axes:

Proposition. *The boundary of B_λ meets each of the symmetry axes in exactly two points, namely $p(\lambda)$ and $q(\lambda)$ or their symmetric images.*

Proof: It suffices to consider the case of $\mathbb{R}$. Recall from Section 1 that $B_\lambda \cap [q(\lambda), p(\lambda)]$ is empty. Suppose $y_0 \in \mathbb{R} \cap \partial B_\lambda$ and $y_0 \neq p(\lambda), q(\lambda)$. Then either y_0 or $y_1 = F_\lambda(y_0)$ lies in the interval $(0, p(\lambda))$ since F_λ maps $\mathbb{R}^-$ to $\mathbb{R}^+$. But then, since F_λ is decreasing on $\mathbb{R}^+$, there is a first point $y_n = F_\lambda^n(y_0)$ such that $y_n \in (0, (-\lambda)^{1/3})$ where we recall that $(-\lambda)^{1/3}$ is the prepole in $\mathbb{R}^+$, i.e., $F_\lambda(-\lambda^{1/3}) = 0$. We have that $y_n \in \partial B_\lambda$ since ∂B_λ is invariant.

Now recall that the critical sector S_0 is the region bounded by the rays $\omega^2 t$, ωt, and a third of the critical curve, where $c(\lambda) \leq t \leq 0$. The vertices of this "triangular" region are given by 0 and the two nonreal critical points of F_λ. We claim that B_λ cannot meet the boundary of S_0. To see this, note that the straight line boundaries of S_0 lie strictly inside the symmetric images of $[q(\lambda), p(\lambda)]$ on the nonreal symmetry axes, so B_λ misses them. Also, the portion of the boundary of S_0 on the critical curve is mapped by F_λ onto the intervals between 0 and the critical value along $\omega\mathbb{R}^+$ and $\omega^2\mathbb{R}^+$. But these intervals are contained inside the symmetric copies of $[0, p(\lambda))$ in these rays. Hence there are no points in B_λ on this part of the boundary of S_0 as well.

Now since y_n lies in the interior of S_0 and also on ∂B_λ, it follows that there are points in B_λ inside the set S. But since B_λ is connected and extends to ∞, it follows that there are points in B_λ that also lie on the boundary of S. This contradiction establishes the result.

□

In particular, note that the proof of this result implies that ∂B_λ does not meet the the critical curve, for otherwise the image of such a point would lie in one of the symmetric copies of $(q(\lambda), p(\lambda))$, in contradiction to the previous Proposition. The same is true for ∂T_λ. Since the critical circle therefore surrounds T_λ, it follows that $\partial B_\lambda \cap \partial T_\lambda$ is empty. It follows immediately that none of the preimages of ∂B_λ meet ∂B_λ.

Now we show that $\partial B_\lambda \cap \partial A_j$ is empty. We first observe that the basins A_j cannot intersect the nonreal symmetry axes. This follows since any point on these two symmetry axes must remain on the union of these axes for all iterations and hence the orbit of this point cannot tend to a (non-zero) cycle in $\mathbb{R}$. The A_j miss 0 since 0 maps to ∞. Now the point x_{n-1} on the real superattracting cycle lies in the interval $(0, (-\lambda)^{1/3})$ since this is the subinterval of $\mathbb{R}^+$ that is mapped to $\mathbb{R}^-$. Consequently A_{n-1} must intersect the critical sector S_0. But the interior of A_{n-1} cannot meet the boundary of this sector for, as in the previous Proposition, this boundary is mapped to the nonreal symmetry axes. Hence ∂A_{n-1} is contained in the closed set S_0 and therefore must be disjoint from ∂B_λ. Therefore all of the basins A_j have this property and we have proved:

Proposition. *The boundaries of B_λ and the A_j are disjoint.*

By symmetry, it follows that the boundaries of B_λ and the C_j are also disjoint. Next we have:

Proposition. *The basins A_j and all of their preimages have disjoint boundaries.*

Proof: This result follows immediately from the fact that ∂A_{n-1} lies in the closed set S_0 and therefore is contained in the half plane $\operatorname{Re} z > 0$ (note that the origin is not in ∂A_{n-1}). At the same time, ∂A_0 is contained in $\operatorname{Re} z < 0$, for otherwise this basin would meet a nonreal symmetry axes. Hence ∂A_0 is disjoint from ∂A_{n-1} and the result follows.

□

To complete the proof that all basins of attraction are buried, we must show that $\partial C_k \cap \partial A_j = \emptyset$ for all k, j. To see this, we first observe that a given C_k cannot intersect both nonreal symmetry axes. If this were to happen, then we would have a pair of points inside C_k whose iterates always lie on different nonreal symmetry axes and so these two orbits could not lie in the same immediate basin of attraction. Now there are $2n-2$ C_k's that lie completely in the "left" sector J_L defined by $\pi/3 < \operatorname{Arg} z < 5\pi/3$ and there are only two C_k's that are completely contained in the "right" sector J_R given by $-2\pi/3 < \operatorname{Arg} z < 2\pi/3$. Recall here that $n \geq 3$, so there are more C_k's in J_L than in J_R. Similarly, there is only one A_j, namely A_0, in J_L, while the remaining $n-1$ A_j's lie in J_R. It follows that if the boundary of some C_k meets ∂A_0, then some subsequent iterate $F_\lambda^i(C_k)$ must lie in J_L whereas $F_\lambda^i(A_0)$ lies in J_R. This uses the fact that $n \geq 3$. But we must have $F_\lambda^i(\partial C_k) \cap F_\lambda^i(\partial A_0) \neq \emptyset$.

Therefore the basin $F_\lambda^i(C_k)$ must intersect both of the nonreal symmetry axes. Since this cannot happen, it follows that ∂C_k must be disjoint from ∂A_0 and hence from each ∂A_j for all j and k. This completes the proof of the fact that all of the attracting basins of F_λ are buried.

4.4 Sierpinski Curves

In this section we complete the proof that $J(F_{\lambda_n})$ is a Sierpinski curve for each n. Again we fix n and write $\lambda = \lambda_n$.

We need to show that $J(F_\lambda)$ is compact, connected, locally connected, nowhere dense, and the boundaries of all the Fatou components are disjoint simple closed curves. We remark that, for topologically constructed Sierpinski curves, the difficulty that usually arises in showing that a set is a Sierpinski curve is proving local connectivity or nowhere density. But complex dynamics makes the proofs of these properties easy.

First, $J(F_\lambda)$ is compact and connected since $J(F_\lambda)$ is the complement of the union of countably many open, simply connected basins of attraction and their preimages. Since $J(F_\lambda)$ omits these basins, it follows that $J(F_\lambda)$ is not the entire Riemann sphere and hence contains no interior points implying that it is nowhere dense. Finally, since all critical points lie on attracting cycles, it follows that F_λ is hyperbolic on $J(F_\lambda)$ and so the Julia set is locally connected. See [Mil99] for details. It remains to prove that the boundaries of the basins are simple closed curves, as the previous section guarantees that they are mutually disjoint. This is straightforward for the bounded basins.

Proposition. *The basins of attraction A_j and C_k have boundaries that are simple closed curves.*

Proof: We prove this for A_0; the other cases follow by symmetry and/or by taking iterates of F_λ. The point $x_0 \in A_0$ is a superattracting fixed point of F_λ^n. Hence there is a conjugacy $\phi_\lambda : \mathbb{D} \to A_0$ satisfying $\phi_\lambda(z^2) = F_\lambda(\phi_\lambda(z))$ where $\mathbb{D}$ is the open unit disk in $\overline{\mathbb{C}}$. The image of a straight ray in $\mathbb{D}$ given by $te^{i\theta}$ with $0 \leq t < 1$ under ϕ_λ is called an internal ray. Since the boundary of A_0 is locally connected, Carathéodory theory (see [Mil99]) guarantees that each internal ray lands on a single point in ∂A_0, i.e.,

$$\lim_{t \to 1} \phi_\lambda(te^{i\theta})$$

exists for each θ. It then suffices to show that no two internal rays land at the same point. But if two rays did land at a given point $p \in \partial A_0$, then the union of these two internal rays together with p forms a simple closed curve γ that lies entirely inside A_0 (except for p). Let Γ denote the interior of this simple closed curve. Then Γ must contain other points in the boundary of A_0, for otherwise an entire interval of rays would land at p, and this is impossible. But then the union of the forward images of Γ cannot meet points on B_λ, for example, since the images of γ all lie in the union of the $\overline{A}_j$. This contradicts Montel's Theorem which says that the union of these images of Γ must cover all of $\overline{\mathbb{C}}$ (except for at most one point).

□

The fact that the boundary of B_λ is a simple closed curve must be handled differently, for in this case the forward images of the analogue of Γ are no longer bounded. Therefore we proceed differently.

Let W denote the open connected component of $\overline{\mathbb{C}} - \overline{B}_\lambda$ that contains the origin. As we showed in Chapter 2, W is the unique component of $\overline{\mathbb{C}} - \overline{B}_\lambda$.

Now we argue as above. Consider the conjugacy ϕ_λ between z^2 and F_λ taking $\mathbb{D}$ to B_λ. Choose the curve γ and the open set Γ as before, where the curve γ now consists of two external (as opposed to internal) rays and the common landing point p. Now we know that the forward images of Γ cannot map onto the interior of W, so just as before, all of the rays associated to ϕ_λ land at unique points and ∂B is a simple closed curve. This completes the proof that $J(F_{\lambda_n})$ is a Sierpinski curve.

By Whyburn's theorem ([Why58]), any two Sierpinski curves are homeomorphic. Hence $J(F_{\lambda_n})$ is topologically equivalent to $J(F_{\lambda_m})$ for any n and m. However, each of these Julia sets is dynamically distinct from the others.

Theorem. *If $n \neq m$, F_{λ_n} is not topologically conjugate to F_{λ_m} on their Julia sets.*

Proof: A conjugacy between F_{λ_n} and F_{λ_m} on their Julia sets must take the boundaries of attracting basins to boundaries of attracting basins. But the three immediate basins that contain critical points are mapped two-to-one onto their images and these are the only basins that have this property (except for B_λ). Since these basin boundaries are dynamically distinct, they must be mapped to each other by the conjugacy. But the periods of these basins are different, and so they cannot be mapped to one another by a conjugacy.

□

In this result we have concentrated on the case where F_{λ_n} has a superattracting cycle. However, the results go over immediately to a neighborhood of each λ_n in the parameter plane. For these nearby parameters, F_λ also has an attracting cycle. While F_λ^n is no longer conjugate to z^2 in the immediate basin of the cycle, quasiconformal surgery allows us to modify these maps so that they have this property and thereby establish the fact that the Julia set is again a Sierpinski curve. See [BDL$^+$03a] for more details on this construction.

4.5 Concluding Remarks

In this section we have concentrated on the family

$$F_\lambda(z) = z^2 + \frac{\lambda}{z}.$$

However, all of the results go over immediately to the higher degree families given by

$$F_\lambda(z) = z^{2n} + \frac{\lambda}{z^{2d+1}}.$$

One checks easily that, for $\lambda \in \mathbb{R}^-$, the real axis is again invariant and we have similar symmetries for this map. The proofs therefore go over more or less unchanged.

In Figure 4·4 we display the parameter plane for the degree three family

$$F_\lambda(z) = z^2 + \frac{\lambda}{z}.$$

together with a magnification of a certain region along the negative real axis.

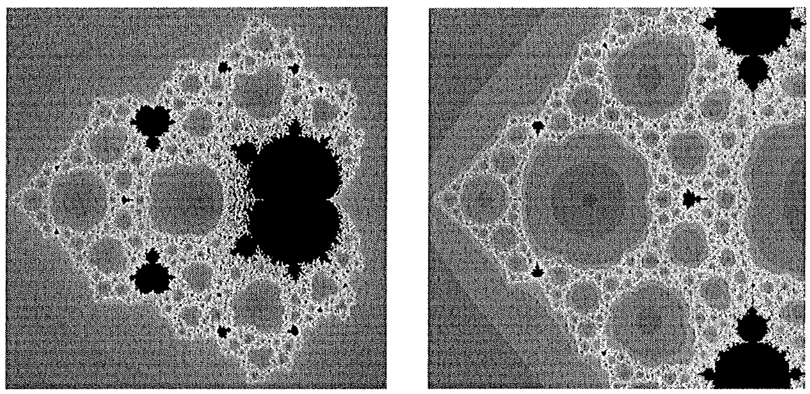


Figure 4·4: The parameter plane for the degree three family of rational maps and a magnification.

The holes in this parameter plane correspond to parameter values for which the Julia set is a Sierpinski curve. See [DLU03] for a complete discussion of these Sierpinski curve Julia sets. Note the existence of a small copy of a Mandelbrot set in this image. The parameters λ_n described in this paper are drawn from the centers of the main cardioids of these Mandelbrot sets.

Note that these Mandlebrot sets are somewhat different in appearance from many of the other baby Mandelbrot sets in this picture. The small copies of the Mandelbrot sets whose cusp meets the outer boundary of the parameter plane also seems to touch many of the other holes in the parameter plane. This is quite different from the Mandelbrot sets from which our parameters are drawn: they do not seem to extend to any of the holes. Indeed, we conjecture that these baby Mandelbrot sets are also "buried" in the sense that there are no parameters in these sets that also lie on the boundaries of one of the Sierpinski curve holes in the parameter plane.

Chapter 5

Singular Perturbations of Circle Inversion Maps

5.1 Introduction

Here we will define a family of maps that arise from three circle inversion and we will investigate the structure of their Julia sets; in particular we obtain either Cantor sets or Apollonian gaskets. We will then introduce a perturbation in the form of a fourth circle with small radius and show that the Julia sets for this family of maps are all homeomorphic to a set sharing many topological characteristics with the Sierpinski curve.

5.1.1 Construction of our Map

We define inversion of a point α about a circle centered at O with radius r to be the point α' on the ray $\overrightarrow{O\alpha}$ such that

$$\frac{\| \overrightarrow{O\alpha} \|}{r} = \frac{r}{\| \overrightarrow{O\alpha'} \|}.$$

Let us say that we have n circles $C_1, C_2, \ldots, C_n$. We will take inversion about these n circles to mean sending the point α to the point β that is the arithmetic mean of $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ where α_i is the inversion of α about circle C_i. (See Figure 5·1.)

We will look at the case of four circles: three with radius r centered at the cube roots of unity and one circle of radius ϵ centered at the origin. If $0 < \epsilon << r$ then our map will be a singular perturbation of the three circle inversion occuring when $\epsilon = 0$. One easily computes our map to be

$$F_{\epsilon,r}(z) = \frac{(3r^2 + \epsilon^2)\overline{z}^3 - \epsilon^2}{\overline{z}^4 - \overline{z}}.$$

In order to use many of the tools of complex dynamics we need our function to be holo-

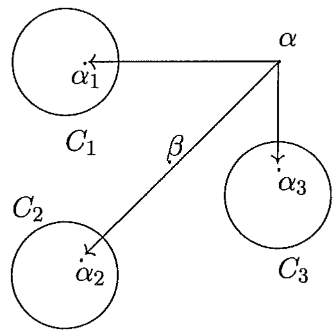


Figure 5·1: β is the inversion of α about the three circles C_1, C_2, C_3.

morphic. Hence, we will define

$$F_{\epsilon,r}(z) = \frac{(3r^2 + \epsilon^2)z^3 - \epsilon^2}{z^4 - z}.$$

This map agrees with the "true" circle inversion map on every second iterate, so dynamically our map is equivalent to the "true" map.

5.1.2 Preliminaries

Throughout this work we will consider ω such that $\omega^3 = 1$ and $\omega \neq 1$. We will only consider ϵ, r such that $0 \leq \epsilon << r$. The map $F_{\epsilon,r}$ has 3-fold symmetry in that $F_{\epsilon,r}(\omega z) = \omega^2 F_{\epsilon,r}(z)$. Hence, we know that if z_0 is attracted to a periodic cycle, then ωz_0 and $\omega^2 z_0$ are also attracted to periodic cycles, although they could be different cycles (and even of different periods). Further, $\mathbb{R} \cup \{\infty\} = \mathbb{R}^\#$ is forward invariant under $F_{\epsilon,r}$ and $F_{\epsilon,r}$ maps the line $\omega\mathbb{R}^\#$ to $\omega^2\mathbb{R}^\#$ and vice versa. Hence, $\omega\mathbb{R}^\# \cup \omega^2\mathbb{R}^\#$ is also a forward invariant set. We will call $\omega\mathbb{R}^\#$ and $\omega^2\mathbb{R}^\#$ the *symmetric axes*. As a final symmetry, note that $\overline{F_r(z)} = F_r(\overline{z})$.

5.2 $\epsilon = 0$

If we set $\epsilon = 0$ our map becomes

$$F_{0,r}(z) = \frac{3r^2z^2}{z^3 - 1},$$

which we will denote F_r throughout this section. This family of maps always has a superattracting fixed point at the origin. Since F_r is degree three there are 4 critical points, located at 0, c, ωc and $\omega^2 c$ where c is a negative real critical point. Since 0 is fixed, we will call the three symmetric critical points $c, \omega c$, and $\omega^2 c$ the *free critical points.* By our symmetries we know that $F_r(\omega c) = \omega^2 F_r(c)$. This implies that F_r has (essentially) only one free critical orbit in that if one of the free critical points is in the basin of attraction for 0 then all of the free critical points are in the basin for 0. Similarly, if one of the free critical points has bounded orbit then so do all of the free critical points. Further, for all values of r, F_r has poles at the cube roots of unity.

5.2.1 Structure of $J(F_r)$ for $r < (1/2)^{1/3}$

When $r < (1/2)^{1/3}$ c is in the immediate basin of attraction for 0. (See Figure 5·2.) Since c is attracted to the origin, our 3-fold symmetry guarantees that ωc and $\omega^2 c$ are also attracted to the origin.

Therefore F_r is a degree three map whose critical points are all attracted to a superattracting fixed point, implying that the Julia set of F_r, $J(F_r)$, is a Cantor set. (See [Bea91].) This result can also be obtained using symbolic dynamics. Let D be a simply connected open set containing the origin such that D is mapped 2 to 1 into itself. We know that such a set exists since 0 is a superattracting fixed point of order 2. Further, we will choose D such that the boundary of D, which we will denote δ, is a simple closed curve containing no critical points or critical values. Now, $F_r(\delta)$ is a simple closed curve contained within D and bounding $F_r(D)$. $F_r(D)$ has two preimages, D (mapped two-to-one over $F_r(D)$)

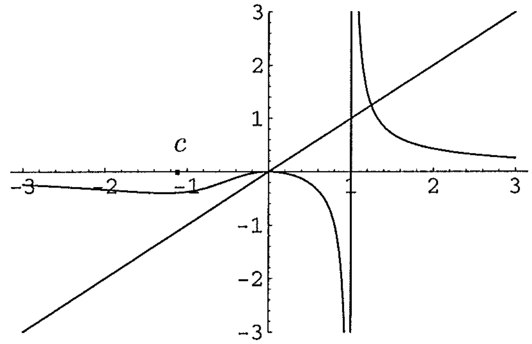


Figure 5·2: F_r with $r = .5$.

and another simply connected open set (mapped one-to-one over $F_r(D)$) that contains ∞. We will denote this second preimage by D_∞. Let δ_∞ denote the boundary of D_∞. Hence, $F_r(D) = F_r(D_\infty)$ and this set is in the basin of attraction for 0. Therefore, the Julia set for F_r must be contained within the closed annular region between δ and δ_∞. We will denote this region by $\mathcal{A}$. (See Figure 5·3.) In Figure 5·3 the shaded regions are all in the basin of attraction of 0.

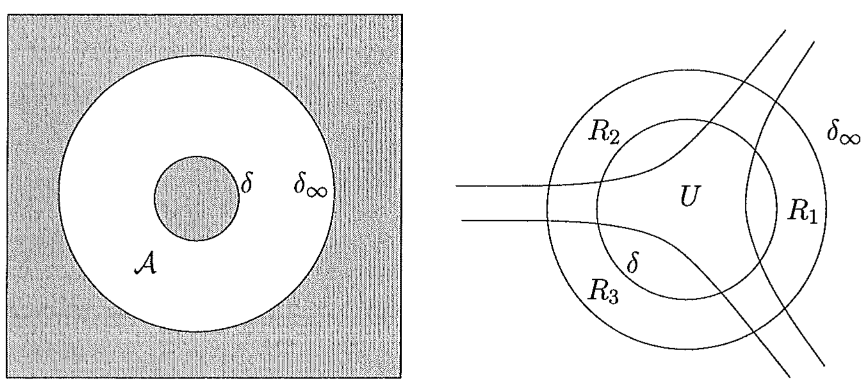


Figure 5·3: The regions R_1, R_2, and R_3, the set U, and the curves δ and δ_∞.

Now note that for all $x \in \mathbb{R}^-$ we have $x < F_r(x) < 0$. (See Figure 5·2.) Hence, we know that all of $\mathbb{R}^-$ is in the basin of attraction for 0. By symmetry, the symmetric axes $\omega\mathbb{R}^-$ and $\omega^2\mathbb{R}^-$ are also in the basin of 0. Since the connected closed set $W = \{0\} \cup \{\mathbb{R}^-\} \cup \{\omega\mathbb{R}^-\} \cup \{\omega^2\mathbb{R}^-\}$ is in the basin of attraction for 0 we can find an open connected set U containing W such that U is in the basin of attraction for 0. (We will view U as a "fattened up" W.) Now, $\mathcal{A} - (\mathcal{A} \cap U)$ consists of three pairwise disjoint closed sets R_1, R_2, and R_3 each of which contains a pole. Further, the R_i can be chosen such $F_r(R_i)$ covers R_1, R_2, and R_3 for any $i = 1, 2, 3$. (See Figure 5·3.) Hence, the Julia set $J(F_r)$ consists of all points in $R_1 \cup R_2 \cup R_3$ that remain in $R_1 \cup R_2 \cup R_3$ for all iterations. Standard arguments show that this set is a Cantor set and that F_r restricted to $J(F_r)$ is conjugate to the shift map on three symbols.

5.2.2 Structure of $J(F_r)$ for $r > (1/2)^{1/3}$

Our map undergoes a bifurcation as r passes through $(1/2)^{1/3}$. (See Figure 5·4.)

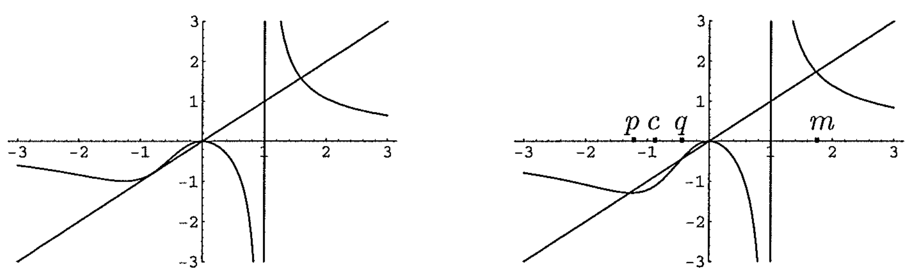


Figure 5·4: The bifurcation as r passes through $r = (1/2)^{1/3}$

The repelling complex conjugate two-cycle coalesces into a parabolic fixed point on the negative real axis and then splits into two real fixed points on the negative real axis. We will denote these p and q, where $|q| < |p|$. For all $r > (1/2)^{1/3}$, q is repelling and p is attracting. There is also a repelling fixed point on the positive real axis which we will denote by m. (See Figure 5·4.)

Since $F_r(\omega z) = \omega^2 F_r(z)$ we know that there are actually three simulataneous bifurcations that occur at $r = (1/2)^{1/3}$; one on each of $\mathbb{R}^-$, $\omega\mathbb{R}^-$, and $\omega^2\mathbb{R}^-$. Note that the existence of an attracting fixed point on the negative real axis, namely p, implies the existence of an attracting two-cycle on the symmetric axes, namely ωp and $\omega^2 p$. From the graph of F_r it is easy to see that c is in the immediate basin of attraction of p, implying that ωc and $\omega^2 c$ are in the immediate basin of attraction for the symmetric two-cycle. Therefore we have all of our critical points accounted for and there can be no other stable domains, for each of these domains necessarily contains a critical point. Let A_p denote the immediate basin of attraction for p and $\mathcal{O}$ denote the immediate basin of attraction for 0. Therefore, ωA_p and $\omega^2 A_p$ form the immediate basin of attraction for the two-cycle formed by ωp and $\omega^2 p$. Since $F_r(\overline{z}) = \overline{F_r(z)}$ we know that $\overline{A_p} = A_p$, $\overline{\mathcal{O}} = \mathcal{O}$, $\overline{\omega A_p} = \omega^2 A_p$, and $\overline{\omega^2 A_p} = \omega A_p$. Note A_p cannot meet either of the symmetry axes, $\omega\mathbb{R}$ and $\omega^2\mathbb{R}$, because all points of A_p are attracted to $p \in \mathbb{R}^-$ while the union of the symmetry axes are forward invariant.

This bifurcation corresponds to the geometric bifurcation that occurs when the three circles go from being mutually disjoint to intersecting. (See Figure 5·5.)

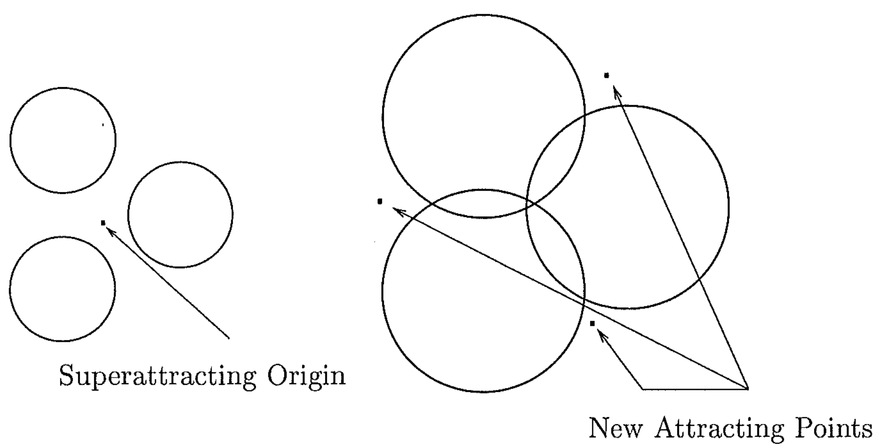


Figure 5·5: The topological relation of the three generating circles pre- and post-bifurcation.

For all $r > (1/2)^{1/3}$ $J(F_r)$ is the Riemann sphere with a countable number of simply connected regions removed, namely

$$J(F_r) = \overline{\mathbb{C}} / \left(\left[\bigcup_{n \leq 0} F_r^n(\mathcal{O}) \right] \bigcup \left[\bigcup_{i=0,1,2} \left(\bigcup_{n \leq 0} F_r^n(\omega^i A_p) \right) \right] \right).$$

Hence $J(F_r)$ is connected. Since the critical points are all attracted to finite attracting cycles we know that F_r is dynamically hyperbolic. It is known that if the Julia set of a dynamically hyperbolic map is connected then it is locally connected. (See [Mil99]). Therefore, $J(F_r)$ is locally connected as well as connected. The Julia set for one of these maps is shown in Figure 5·6.

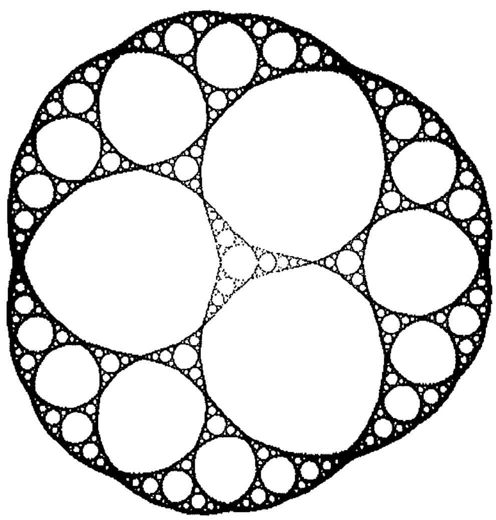


Figure 5·6: The Julia set of F_r with $r = .8$.

We will now show that the boundaries of A_p and $\mathcal{O}$, denoted ∂A_p and $\partial \mathcal{O}$, respectively, are simple closed curves.

Proposition: *The boundary of A_p is a simple closed curve.*

Proof: Since A_p is simply connected we know that the boundary of A_p, ∂A_p, is a closed curve. (See [MNTU00]). Further, since our Julia set is locally connected we know that ∂A_p

is locally connected and Carathéodory theory tells us that ϕ, the inverse of the Riemann map mapping A_p to the unit disk $\mathbb{D}$, extends continuously to $\overline{\phi} : \overline{\mathbb{D}} \to \overline{A_p}$. We will call the images in A_p of straight rays from 0 to $\partial\mathbb{D}$ *internal rays.* Since ϕ extends continuously to $\overline{\phi}$ we know that all of these rays land at a point on ∂A_p. To show that ∂A_p is a simple closed curve we need only show that no two internal rays land at the same point in ∂A_p. Let us assume that two rays land at a point $w \in \partial A_p$. Now, let γ be the Jordan curve consisting of these two rays along with the landing point w and let Γ denote the bounded complement of the curve γ. (See Figure 5·7.) There must be other points of ∂A in Γ, else we would have an entire sector of rays all landing at w, which is not possible.

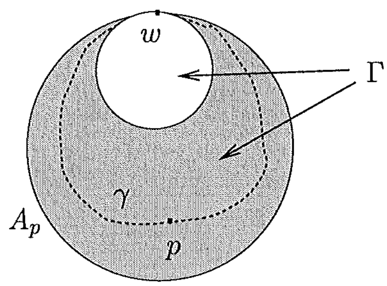


Figure 5·7: The region Γ.

Since Γ is bounded by γ we know by the maximum modulus theorem that $F_r(\Gamma)$ is bounded by $F_r(\gamma)$. (Note that $F_r(\gamma)$ is a simple closed curve.) Because γ lies inside A_p, the basin of attraction for p, (with the exception of the point w which lies on the boundary of this basin) we know that $F_r(\gamma - \{w\})$ lies inside A_p. Therefore, the boundary of $F_r(\Gamma)$ (with the exception of one point) lies inside A_p. Hence, $F_r(\Gamma)$ is either mapped to the unbounded complement of $F_r(\gamma)$ or to the bounded complement. It is known that any neighborhood of the Julia set for a function is eventually mapped by iterates of the function onto the entire Riemann sphere minus at most two points. Since $\Gamma \cap J(F_r) \neq \emptyset$ it cannot be the case that $F_r(\Gamma)$ is mapped to the bounded complement of $F_r(\gamma)$. However, if $F_r(\Gamma)$ is mapped to the unbounded complement then Γ must contain a pole. The poles lie on $\mathbb{R}^+$ and on the symmetric rays $te^{2\pi/3}$ and $te^{4\pi/3}$ with $t > 0$. Since $p \in \mathbb{R}^-$ we know that A_p does not intersect $\mathbb{R}^+$ or the symmetric rays. Hence, the pole must lie in the boundary of A_p, which is not possible. One way to see this is to note that if a pole did lie on the boundary of A_p we could connect p to the pole with a simple curve μ such that μ lies entirely within A_p. The image of μ would then be a simple curve connecting ∞ to p contained entirely within A_p. This is not possible because there exist neighborhoods of ∞ consisting entirely of points in the basin of attraction for 0. This contradiction establishes the result.

□

The same proof, with minor modifications, allows us to show that $\partial\mathcal{O}$ is also a simple closed curve. Symmetry allows us to extend this result to ωA_p and $\omega^2 A_p$. Since all of our critical points tend to attracting cycles, we can generalize this result to all preimages of attracting basins and hence to any component of the Fatou set. We have therefore shown:

Proposition: *If C is the boundary of a Fatou component then C is a simple closed curve.*

There are also restrictions placed on the number of intersection points occuring between boundaries of pairwise disjoint complementary regions of the Julia set.

Theorem: *If $\partial(\omega^i A_p)$ meets $\partial(\omega^j A_p)$, then it does so at exactly one point.*

Proof: Note that A_p is trapped in the region $S = \{z : \text{Re(z)} < 0, 2\pi/3 < \text{Arg}(z) < 4\pi/3\}$. To see this note that A_p cannot meet either of the symmetric axes since the symmetric axes are forward invariant and hence points on these axes cannot lie in the basin of attraction of a real non-zero fixed point. An analagous statement is true for the symmetric basins ωA_p and $\omega^2 A_p$. We know that ωA_p must be trapped in ωS and $\omega^2 A_p$ must be trapped in $\omega^2 S$. Without loss of generality we will show that if $\partial(\omega A_p)$ meets $\partial(\omega^2 A_p)$ then it does so at exactly one point. Since ωA_p is trapped in ωS and $\omega^2 A_p$ is trapped in $\omega^2 S$ we know that $\partial(\omega A_p) \cap \partial(\omega^2 A_p)$ must lie in $\partial(\omega S) \cap \partial(\omega^2 S) = \mathbb{R}^+$. Since ωA_p is mapped onto $\omega^2 A_p$ and vice versa, we know that $\partial(\omega A_p) \cap \partial(\omega^2 A_p) \subset \mathbb{R}^+$ is mapped into $\mathbb{R}^+$. Hence, the set $\partial(\omega A_p) \cap \partial(\omega^2 A_p)$ remains in $\mathbb{R}^+$ for all iterations. There are no two-cycles on the positive real axis and therefore it can be seen from the graph of the function that there is only one point in $\mathbb{R}^+$ that remains in $\mathbb{R}^+$ for all iterations and does not tend to ∞, the repelling fixed point. (See Figure 5·4.) Hence, $\partial(\omega A_p) \cap \partial(\omega^2 A_p)$ is either empty or contains only the positive repelling fixed point.

Symmetry yields the result.

□

Let $S^1 = \{z : |z| = 1\}$ and $h(z) = z^k$. We will now make use of the following Theorem from [MNTU00]:

Theorem: *Assume that the Fatou set of a hyperbolic rational function f contains a simply connected invariant component D whose local degree of f is k. Then there is a continuous map ϕ from S^1 onto ∂D such that*

$$\phi(h(z)) = f(\phi(z)).$$

Moreover, h is injective on $\phi^{-1}(\zeta)$ for all $\zeta \in \partial D$.

Note that F_r is a hyperbolic rational map that maps A_p onto A_p in two-to-one fashion. Hence, by the Theorem, there exists a continuous map ϕ such that $\phi : S^1 \mapsto \partial A_p$ and

$$\phi(z^2) = F_r(\phi(z)).$$

Therefore, $\phi(e^{4\pi i/3})$ and $\phi(e^{2\pi i/3})$ form a two-cycle on ∂A_p.

Since F_r is degree 3 we know that there are at most three distinct two-cycles. Since q is a repelling fixed point on the boundary of A_p we know that ωq and $\omega^2 q$ form a repelling two-cycle with $\omega q \in \omega(\partial A_p) = \partial(\omega A_p)$ and $\omega^2 q \in \omega^2(\partial A_p) = \partial(\omega^2 A_p)$. Hence, the attracting two-cycle and the repelling two-cycle just mentioned both have positive real components. However, ∂A_p is entirely contained in the left half-plane $\{z : \mathrm{Re}(z) < 0\}$. Therefore the two-cycle on ∂A_p must be distinct from the previous two-cycles, giving us our 3 two-cycles.

Recall that we denote the repelling fixed point on the positive real axis by m. Therefore, ωm and $\omega^2 m$ form a two-cycle with negative real component. Since we have all three of our two-cycles accounted for it must be that ωm and $\omega^2 m$ corresponds to the two-cycle in ∂A_p. We have $\omega m \in \partial A_p$ and therefore, by the symmetries, we know that $\omega^2 m \in \omega(\partial A_p) = \partial(\omega A_p)$. However, $\omega^2 m$ is also in ∂A_p. Hence, $\omega^2 m \in \partial A_p \cap \omega(\partial A_p) = \partial A_p \cap \partial(\omega A_p)$. This implies that $\omega m \in \partial A_p \cap \omega^2(\partial A_p) = \partial A_p \cap \partial(\omega^2 A_p)$ and, finally, that $m \in \omega(\partial A_p) \cap \omega^2(\partial A_p) = \partial(\omega A_p) \cap \partial(\omega^2 A_p)$.

We have just shown that $\partial(\omega^i A_p) \cap \partial(\omega^j A_p) \neq \emptyset$ for $i \neq j$. Combining this with the previous Theorem yields:

Corollary: *$\partial(\omega^i A_p)$ meets $\partial(\omega^j A_p)$ at exactly one point for $i, j = 0, 1, 2$ and $i \neq j$.*

We will now show that a similar result holds for $\mathcal{O}$.

Theorem: *$\partial\mathcal{O}$ meets $\partial(\omega^i A_p)$ at exactly one point for $i = 0, 1, 2$.*

Proof: From the graph of F_r we see that $\mathcal{O} \cap \mathbb{R}^-$ is the interval $(q, 0)$, where q is the repelling fixed point on $\mathbb{R}^-$. (See Figure 5·4.) Further, q is in ∂A_p. So $q \in \partial A_p \cap \partial\mathcal{O}$. We claim that $\partial A_p \cap \partial\mathcal{O} = q$. Assume that there exists another point, $\alpha \in \partial A_p \cap \partial\mathcal{O}$. Since $\overline{A_p} = A_p$ and $\overline{\mathcal{O}} = \mathcal{O}$ we know that $\overline{\alpha} \in \partial A_p \cap \partial\mathcal{O}$, too. Let γ_A represent the portion of ∂A_p between α and $\overline{\alpha}$ containing q. Now let $(\alpha, q)_A$ be the portion of γ_A connecting α to q and $(q, \overline{\alpha})_A$ the portion connecting q to $\overline{\alpha}$. In a similar fashion we can define $\gamma_{\mathcal{O}}$ and its subsets $(\alpha, q)_{\mathcal{O}}$ and $(q, \overline{\alpha})_{\mathcal{O}}$.

Denote by Γ_{upper} the bounded complement of the closed curve $\{\alpha\} \cup \{q\} \cup (\alpha, q)_A \cup (\alpha, q)_{\mathcal{O}}$ and Γ_{lower} the bounded complement of the closed curve $\{q\} \cup \{\overline{\alpha}\} \cup (q, \overline{\alpha})_A \cup (q, \overline{\alpha})_{\mathcal{O}}$. Let $\Gamma = \Gamma_{upper} \cup \Gamma_{lower}$. (See Figure 5·8.)

Since ∂A_p and $\partial\mathcal{O}$ are forward invariant $F_r(\partial\Gamma_{upper}) \subset \partial A_p \cup \partial\mathcal{O}$ and the same is true for $\partial\Gamma_{lower}$. Hence, $F_r(\Gamma_{upper})$ and $F_r(\Gamma_{lower})$ are either contained within $\overline{A_p \cup \Gamma \cup \mathcal{O}}$ or are unbounded.

The first is impossible for it would imply that $F_r^n(\Gamma_{lower}) \subset \overline{A_p \cup \Gamma \cup \mathcal{O}}$ for all $n > 0$.

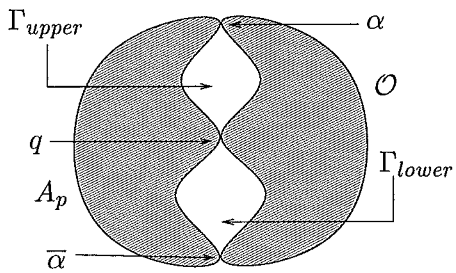


Figure 5·8: ∂A_p meets $\partial\mathcal{O}$ at 3 points.

Hence, we can take a neighborhood N of $\partial\Gamma_{lower}$ that contains only points from $\overline{A_p \cup \Gamma \cup \mathcal{O}}$. This neighborhood is contained in $\overline{A_p \cup \Gamma \cup \mathcal{O}}$ for all iterations. This is a contradiction because N is a neighborhood containing points in the Julia set of F_r (namely $\partial\Gamma_{lower} \cap N$) and therefore its iterates must eventually cover the entire Riemann sphere minus at most two points. The same argument holds for Γ_{upper}.

Hence, it must be the case the $F_r(\Gamma_{upper})$ and $F_r(\Gamma_{lower})$ are unbounded. Therefore, both Γ_{upper} and Γ_{lower} contain poles and both of these regions are incident to A_p and $\mathcal{O}$ only. Hence, for $i = 0, 1, 2$ we have two sets, $\omega^i(\Gamma_{upper})$ and $\omega^i(\Gamma_{lower})$, that are distinct and contain poles. This implies that our degree 3 map has at least 6 poles. This contradiction establishes the result.

5.2.3 Dynamics of F_r on $J(F_r)$

We shall now describe the dynamics of F_r in $\overline{\mathbb{C}}$ completely by means of a geometric construction. For $r < (1/2)^{1/3}$ we know that F_r on $J(F_r)$ is conjugate to the shift map on three symbols. Now let us consider $r > (1/2)^{1/3}$. Consider the closed curve $\partial\mathcal{O}$ 17 encircling zero. Its preimage consists of two closed curves; $\partial\mathcal{O}$ itself and a closed curve surrounding ∞ that we will denote $\partial\mathcal{O}_\infty^{-1}$. Note that $\partial\mathcal{O}$ is mapped in two-to-one fashion over itself making two counterclockwise twists around the origin while $\partial\mathcal{O}_\infty^{-1}$ is mapped in one-to-one fashion over $\partial\mathcal{O}$ making one clockwise twist around the origin. All points outside $\partial\mathcal{O}_\infty^{-1}$

and inside $\partial\mathcal{O}$ are attracted to the origin. Therefore, all of our interesting dynamics occur in the annular region between $\partial\mathcal{O}$ and $\partial\mathcal{O}_\infty^{-1}$. We will denote this region by R.

Let the preimage of q lying on the negative real axis be denoted q^{-1}. Note that $q^{-1} < q$ and the interval $(-\infty, q^{-1}]$ is mapped over the interval $[q, 0)$. (See Figure 5·4.) However, $[q, 0) \subset \mathcal{O}$. Hence, $q^{-1} \in \partial\mathcal{O}_\infty^{-1}$. Further, the interval $[q^{-1}, p]$ is mapped onto $[p, q]$ and $[p, q) \subset A_p$. Therefore, $q^{-1} \in \partial A_p$. Hence, ∂A_p meets $\partial\mathcal{O}_\infty^{-1}$ at exactly one point along the negative real axis. By symmetry, similar results hold for $\partial(\omega A_p)$ and $\partial(\omega^2 A_p)$.

Let us consider the region obtained from R by removing the open sets A_p, ωA_p and $\omega^2 A_p$. We will cut this region into three parts with the rays $\arg z = \pi/3$, $\arg z = \pi$ and $\arg z = 5\pi/3$. We will call these regions R_1, R_2, and R_3; with R_1 between $\arg z = 5\pi/3$ and $\arg z = \pi/3$, R_2 between $\arg z = \pi/3$ and $\arg z = \pi$ and with R_3 between $\arg z = \pi$ and $\arg z = 5\pi/3$. (See Figure 5·9.)

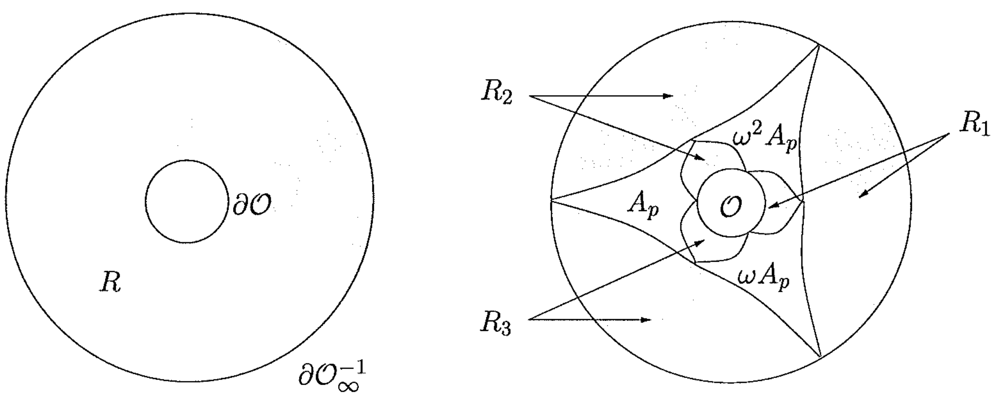


Figure 5·9: The Region R and its subregions R_1, R_2, R_3.

Note that only six points lie in multiple regions. From the graph of F_r on $\mathbb{R}^-$ it is easy to see that the repelling fixed point q and its preimage on $\mathbb{R}^-$, q^{-1}, are the only points in $R_2 \cap R_3$. The symmetries of F_r then imply that we have exactly two points in $R_1 \cap R_2$, namely $\omega^2 q$ and $\omega^2 q^{-1}$, and exactly two points in $R_3 \cap R_1$, namely ωq and ωq^{-1}.

Proposition: *F_r maps R_i over R_1, R_2 and R_3 for $i = 1, 2, 3$ in one-to-one fashion.*

Proof: Let P_1 be the closure of the set of points $z \in R$ such that $-\pi/3 \leq \arg z \leq \pi/3$. (See Figure 5·10.) Note that $R_1 \subset P_1$ and $P_1 - R_1$ is contained in the Fatou set of F_r. (The only points in P_1 that are not in R_1 are those points belonging to ωA_p and $\omega^2 A_p$.)

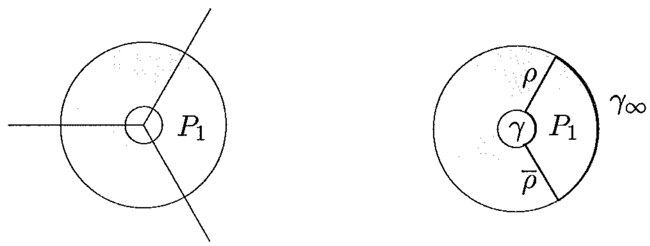


Figure 5·10: The Region P_1 and its boundary.

Let $\rho = \partial P_1 \cap \omega^2\mathbb{R}^-$, $\gamma = \partial P_1 \cap \partial\mathcal{O}$, and $\gamma_\infty = \partial P_1 \cap \partial\mathcal{O}_\infty^{-1}$. Notice that boundary of P_1 is given by $\rho \cup \overline{\rho} \cup \gamma \cup \gamma_\infty$. (See Figure 5·10.)

By the definition of $\partial\mathcal{O}$, note that γ is mapped onto the points of $\partial\mathcal{O}$ with argument between $\pi/3$ and $5\pi/3$ while γ_∞ is mapped onto the points of $\partial\mathcal{O}$ with argument between -$\pi/3$ and $\pi/3$. So $\gamma \cup \gamma_\infty$ is mapped over $\partial\mathcal{O}$ in one-to-one fashion except for the two points $\partial\mathcal{O} \cap \rho$ and $\partial\mathcal{O} \cap \overline{\rho}$. Further, ρ is mapped two-to-one (except at the critical value) over the portion of $\overline{\rho}$ between $\overline{\rho} \cap \partial\mathcal{O}$ and the critical value on $\overline{\rho}$. Hence, by symmetry, we know that $\overline{\rho}$ is mapped two-to-one (except at the critical value) over the portion of ρ between $\rho \cap \partial\mathcal{O}$ and the critical value on ρ. (See Figure 5·11.)

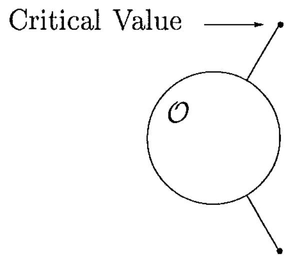


Figure 5·11: The image of the boundary of P_1.

By the maximum modulus principle we know that $\partial F_r(P_1)$ is given by $F_r(\partial P_1)$. We know that there are no zeros in P_1 (0 and ∞ are the only zeros) so the interior of P_1 must be mapped over the entire unbounded complement of $F_r(\partial P_1)$. Hence, P_1 itself is mapped over $\overline{\mathbb{C}} - \mathcal{O}$ in essentially one-to-one fashion (the map is one-to-one except along the rays ρ and $\overline{\rho}$). Hence, R_i is contained in $F_r(P_1)$ for $i = 1, 2, 3$. Since we have a degree three map and all three preimages of the R_i are accounted for we know F_r maps R_1 over R_1, R_2, and R_3 in one-to-one fashion.

Symmetry gives the desired result.

□

We are now in a position to describe the dynamics of F_r on $J(F_r)$ using symbolic dynamics on the regions R_1, R_2, and R_3.

Theorem: *There is a semiconjugacy between the mapping F_r on its Julia set $J(F_r)$ and the full shift on the space of sequences of three symbols. The semiconjugacy is given by associating to each point $z \in J(F_r)$ a sequence $\{s_0, s_1, \ldots\}$ where $s_n = 1, 2, 3$ and is given by $F_r^n(z) \in R_{s_n}$.*

Proof: We need to show that each sequence corresponds to at least one point in the Julia set and that no sequence corresponds to multiple points. Since each region R_i is mapped over all three regions we are guaranteed that each sequence will correspond to at least one point in the Julia set.

The Julia set is contained in $\bigcup R_i$ but $\bigcup R_i$ contains no critical points. Hence we know that the critical points are disjoint from the Julia set implying that F_r is dynamically hyperbolic on its Julia set. This expansion guarantees that no sequence can correspond to multiple points.

□

We only have semi-conjugacy and not a conjugacy due to the six points that lie in multiple regions. Any sequence ending with $\{111111\ldots\}$ is equivalent to the same sequence ending with $\{232323\ldots\}$ or $\{323232\ldots\}$ and any sequence ending with $\{222222\ldots\}$ is equivalent to the same sequence ending in $\{333333\ldots\}$. The first of these identifications corresponds to the two cycle $\{\omega q, \omega^2 q\}$ and its preimages while the latter identification corresponds to the fixed point q and its preimages. If we exclude these cases our semiconjugacy becomes a conjugacy.

5.2.4 The Bifurcation as r passes through $(1/2)^{1/3}$

As r passes through the bifurcation value $(1/2)^{1/3}$ a saddle node bifurcation occurs on the negative real axis. (See Figure 5·4.) When this occurs we have the appearance of a new attracting fixed point $p \in \mathbb{R}^-$ (and therefore a new attracting two-cycle $\{\omega p, \omega^2 p\}$ on the symmetric axes). The basin of 0 is now bounded and no longer extends to ∞. The basins of attraction for these attracting points, $A_p, \omega A_p$, and $\omega^2 A_p$ extend to ∂O and ∂O_∞^{-1}. The regions R_2 and R_3 intersect at $\partial O \cap \mathbb{R}^-$ and $\partial O_\infty^{-1} \cap \mathbb{R}^-$. Further, A_p meets $\omega^2 A_p$ at one point causing R_2 to be pinched as shown in figure 5·12. There are now points of intersection among the R_i causing the identifications described last section.

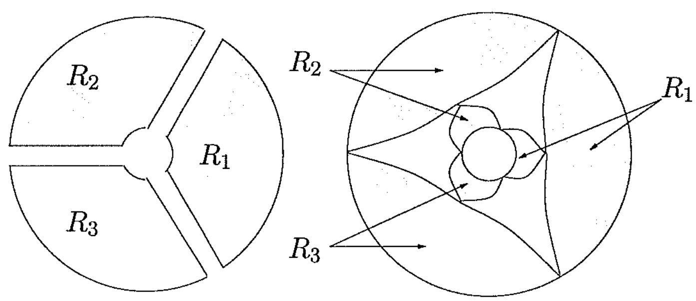


Figure 5·12: The regions R_i pre- and post-bifurcation.

5.3 $\epsilon \neq 0$

When $\epsilon \neq 0$ our map has degree 4 so there are 6 critical points. When $\epsilon = 0$ we had four critical points; one at each of the cube roots of -2 and one at 0. When ϵ becomes nonzero each of the three non-zero critical points persists while the critical point at 0 splits into three separate critical points, giving us our six total critical points. The critical points for this map are all of the form $\omega^i c_1$ or $\omega^i c_2$ with ω a cube root of unity and $c_2 < c_1$ the two critical points on the negative real axis.

While 0 was a superattracting fixed point when $\epsilon = 0$, for $\epsilon \neq 0$ we have 0 and ∞ lying on a repelling 2-cycle. For $r < r^*$, where the value of r^* depends on ϵ, there is one attracting fixed point on the negative real axis. (See Figure 5·13.) We will call this fixed point p.

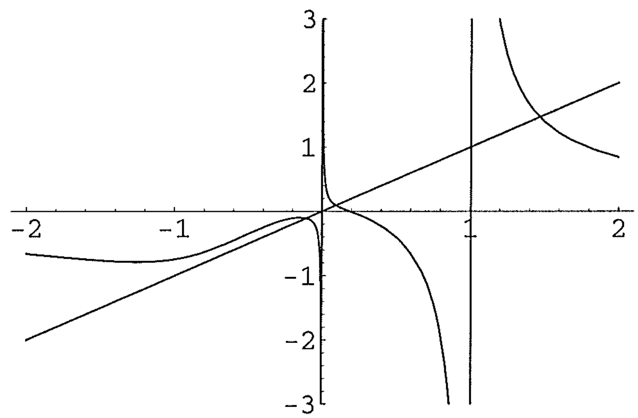


Figure 5·13: The graph of $F_{\epsilon,r}$ for $r < r^*$.

Note that since we still have the symmetry $F_{\epsilon,r}(\omega z) = \omega^2 F_{\epsilon,r}(z)$ the existence of the attracting fixed point p implies that ωp and $\omega^2 p$ form an attracting two-cycle. It is easy to see that c_i is in the immediate basin of attraction of p for $i = 1, 2$. Hence, we have all six of our critical points accounted for since $\omega^j c_i$ will be attracted to the two-cycle $\{\omega p, \omega^2 p\}$ for $i, j = 1, 2$. The Julia set for one of these maps is shown in Figure 5·14, together with a magnification of the same image.

One notices from the graph of $F_{\epsilon,r}$ on the real line that the entire negative real axis is contained in the basin of attraction for p. The preimage of $\mathbb{R}$ consists of $\mathbb{R}$ itself and a

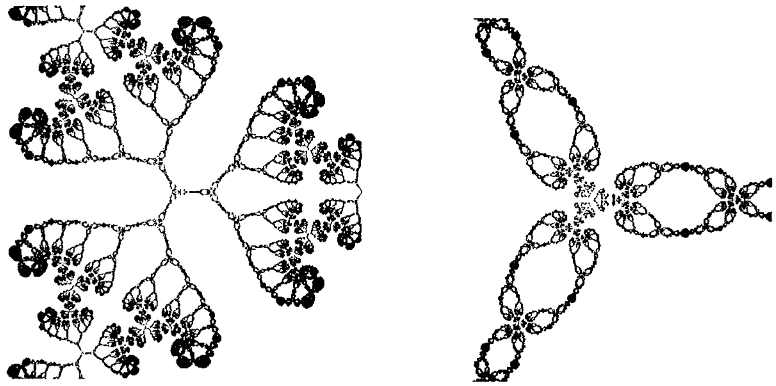


Figure 5·14: The Julia set of $F_{\epsilon,r}$ with $\epsilon = .1$ and $r = .8$ and a magnification of the same set.

curve with negative real parts. (See Figure 5·15.)

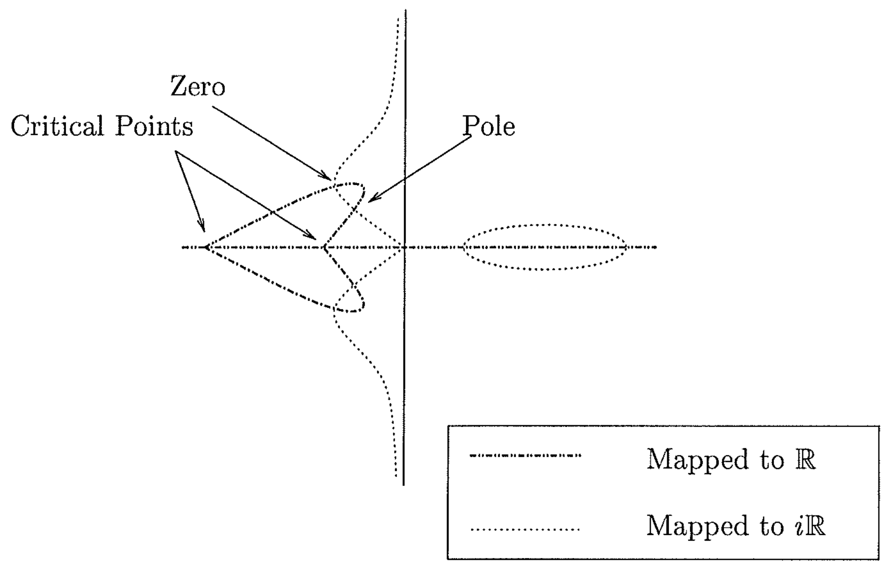


Figure 5·15: The preimages of $\mathbb{R}$ and $i\mathbb{R}$

In particular, there is a curve running from pole to pole through c_2 (entirely contained between $\omega^2\mathbb{R}^+$ and $\omega\mathbb{R}^+$) that is mapped into $\mathbb{R}^-$ and, therefore, is attracted to p. Call this curve K_1. Hence there is an open set surrounding $\mathbb{R}^- \cup K_1$ that is contained in the basin of attraction for p and is mapped inside of itself. Call this set W_1. (See Figure 5·16.)

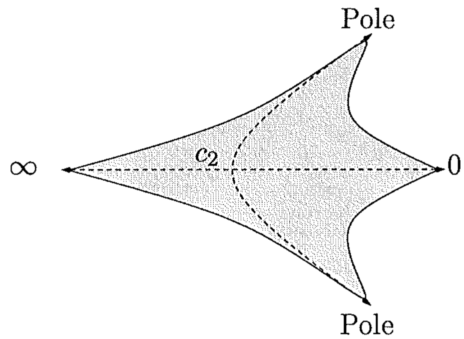


Figure 5·16: The curve K (dashed) and the region W_1.

Via symmetry we know that there exists sets W_2 and W_3, where $W_i = \omega^{i-1}W_1$, such that $W_2 \cup W_3$ is attracted to the two-cycle $\{\omega p, \omega^2 p\}$. We show these regions in Figure 5·17. (Since the regions W_i all extend to ∞ we represent ∞ as a cirlce in Figure 5·17.)

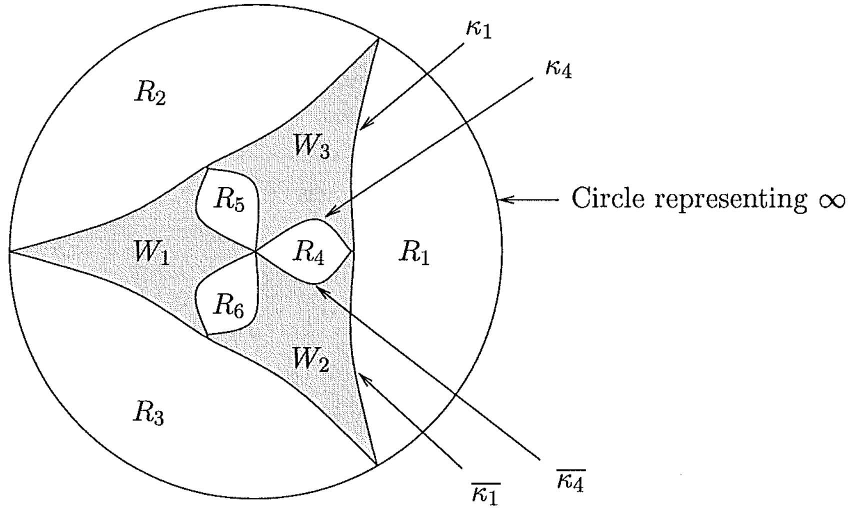


Figure 5·17: The regions W_i and R_i.

Now, W_1, W_2, and W_3 all intersect at ∞ and 0. Further, W_1 and W_2 (resp. W_3) also intersect at the pole on $\omega^2\mathbb{R}^+$ (resp. $\omega\mathbb{R}^+$) while W_2 and W_3 intersect at the pole in $\mathbb{R}^+$. These are all of the interesections among the W_i. Hence, $\overline{\mathbb{C}} - (W_1 \cup W_2 \cup W_3)$ consists of six regions. We will denote these regions R_i for $i = 1, 2, \ldots, 6$. Specifically, let R_1 be the region of $\overline{\mathbb{C}} - (W_1 \cup W_2 \cup W_3)$ containing the unbounded portion of $\mathbb{R}^+$ and R_4 the region containing the bounded portion of $\mathbb{R}^+$. Continuing, let $R_2 = \omega R_1$, $R_3 = \omega^2 R_1$, $R_5 = \omega R_4$ and $R_6 = \omega^2 R_4$. We will use these six regions to describe the dynamics of $F_{\epsilon,r}$ on its Julia set, which is entirely contained in the R_i. (See Figure 5·17.)

Theorem: *The regions R_4, R_5, R_6 are each mapped over all of the R_i. R_1 is mapped over R_1 and R_4, R_2 is mapped over R_3 and R_6, and R_3 is mapped over R_2 and R_5. All of these mappings are one-to-one.*

Proof: We will show that R_4 covers all of the R_i in one-to-one fasion and R_1 covers only R_1 and R_4 in one-to-one fashion. The result will then follow via symmetry.

To begin, note that $\overline{W_2} = W_3$. The boundary of R_4 consists of a curve in ∂W_2 and another in ∂W_3. Let κ_4 denote the portion of ∂W_3 connecting 0 to the pole such that κ_4 is in ∂R_4. Note that $\kappa_4 \cup \overline{\kappa_4} = \partial R_4$ and $\kappa_4 \cap \overline{\kappa_4}$ consists of the origin and the pole on the positive real axis. (See Figure 5·17.) Since $F_{\epsilon,r}(\overline{z}) = \overline{F_{\epsilon,r}(z)}$ we know $\overline{F_{\epsilon,r}(\kappa_4)} = F_{\epsilon,r}(\overline{\kappa_4})$. The two endpoints of κ_4 are both mapped to ∞ and κ_4 consists of no critical points, so κ_4 is mapped to a closed loop containing ∞. Since W_3 is mapped inside W_2, $F_{\epsilon,r}(\kappa_4)$ is contained in W_2. Hence, $F_{\epsilon,r}(\overline{\kappa_4})$ is contained in $W_3 = \overline{W_2}$. Therefore, the boundary of R_4 is mapped inside $W_2 \cup W_3$. Since R_4 is not mapped strictly inside $W_2 \cup W_3$, R_4 must be mapped to the compliment of $F_{\epsilon,r}(\partial R_4)$ that is not entirely contained in $W_2 \cup W_3$. Therefore, R_4 is mapped over all of the R_i. Therefore, by symmetry R_5 and R_6 are mapped over all of the R_i.

We will now show that R_1 is mapped only over R_1 and R_4. The boundary of R_1 consists of a curve in ∂W_2 and another in ∂W_3. Let κ_1 denote the portion of ∂W_3 connecting the pole in $\mathbb{R}^+$ to ∞ such that κ_1 is in ∂R_1. Note that $\kappa_1 \cup \overline{\kappa_1} = \partial R_1$ and $\kappa_1 \cap \overline{\kappa_1}$ consists of the pole on the positive real axis and ∞. (See Figure 5·17.) One of the endpoints of κ_1 is mapped to ∞ and the other is mapped to 0. There are no critical points on κ_1 and κ_1 is mapped strictly inside W_2, hence $F_{\epsilon,r}(\kappa_1)$ is a curve connecting 0 and ∞ and completely contained in W_2. By the symmetries of the map we know that $F_{\epsilon,r}(\overline{\kappa_1}) = \overline{F_{\epsilon,r}(\kappa_1)}$, and hence $F_{\epsilon,r}(\partial R_1)$ is a closed curve including 0 and ∞ and lying inside $W_2 \cup W_3$. Therefore $F_{\epsilon,r}(\partial R_1)$ separates $\overline{\mathbb{C}}$ into two regions: one containing R_1 and R_4 and another containing R_2, R_3, R_5, and R_6. We want to show that $F_{\epsilon,r}(R_1)$ is contained within the region containing R_1 and R_4. The portion of $\mathbb{R}$ that is contained in R_1 is mapped over all of $\mathbb{R}^+$ yet misses $\mathbb{R}^-$. Hence, R_1 is not mapped over the entire Riemann sphere and therefore must be contained in one of the two regions bounded by $F_{\epsilon,r}(\partial R_1)$. Since $\mathbb{R} \cap R_1$ is mapped over all of $\mathbb{R}^+ = \mathbb{R} \cap (R_1 \cup R_4)$ it must be that $F_{\epsilon,r}(R_1)$ is the region of $\overline{\mathbb{C}} - F_{\epsilon,r}(\partial R_1)$ containing R_1 and R_4 yet missing the other R_i.

Now, we know that each R_i has four preimages since $F_{\epsilon,r}$ is degree 4. We now know where all of these preimages lie. For example, R_1 has a preimage in each of R_4, R_5, R_6, and R_1. Hence we have all of the preimages of R_1. This implies that each of these mappings is one-to-one.

The result follows from symmetry.

□

We assign each point in the Julia set a sequence $s(z) \in \Sigma_6$, the set of sequences of six symbols. We associate to $z \in J(F_{\epsilon,r})$ the sequence $s(z) = \{s_0 s_1 s_2 \dots\}$ where $s_n = 1, 2, 3, 4, 5$ or 6 via the rule $F^n_{\epsilon,r}(z) \in R_{s_n}$.

Since not all of the R_i's cover all of the other regions not all sequences represent actual points in the Julia set. However, we can define the set of admissable sequences, $\Sigma'_6 \subset \Sigma_6$, as follows: A sequence $s = s_0 s_1 s_2 \dots$ will be in Σ'_6 if $s_n = 1$ implies that $s_{n+1} = 1$ or 4, $s_n = 2$ implies that $s_{n+1} = 2$ or 5, and $s_n = 3$ implies that $s_{n+1} = 3$ or 6.

Theorem: *There is a semiconjugacy between the mapping $F_{\epsilon,r}$ on its Julia set, $J(F_{\epsilon,r})$, and the shift map on Σ'_6. The semiconjugacy is given by associating to each point $z \in J(F_{\epsilon,r})$ a sequence $\{s_0 s_1 s_2 \dots\}$ where $s_n = 1, 2, 3, 4, 5, 6$ and is given by $F^n_{\epsilon,r}(z) \in R_{s_n}$.*

Proof: We need to show that each sequence in Σ'_6 corresponds to at least one point in the Julia set and that no sequence corresponds to multiple points. Due to the definition of Σ'_6

we are guaranteed that each sequence will correspond to at least on point in the Julia set.

The Julia set is contained in $\bigcup R_i$ and $\bigcup R_i$ contains no critical points. Hence we know that the critical points are bounded from the Julia set implying that $F_{\epsilon,r}$ is dynamically hyperbolic on its Julia set. This expansion guarantees that no sequence can correspond to multiple points.

□

We only have a semi-conjugacy and not a conjugacy due to the five points lying in multiple regions. Any sequence ending with $\{1414\ldots\}$ is equivalent to the same sequence ending with $\{2626\ldots\}$ or $\{3535\ldots\}$. We will denote this by $\{1414\overline{14}\} \to \{2626\overline{26}\} \to \{3535\overline{35}\}$. These identifications corresponds to the origin and its preimages. Using the same notation ∞ and its preimages yield the following identifications: $\{4141\overline{41}\} \to \{5353\overline{53}\} \to \{6262\overline{62}\}$. Finally , the three poles on $\mathbb{R}^+$, $\omega\mathbb{R}^+$, and $\omega^2\mathbb{R}^+$ and their preimages yield the following identifications:

$$\{4262\overline{626}\} \to \{4353\overline{535}\} \to \{1141\overline{414}\},$$

$$\{5141\overline{414}\} \to \{5262\overline{626}\} \to \{2353\overline{535}\},$$

$$\{6353\overline{535}\} \to \{6141\overline{414}\} \to \{3262\overline{626}\}.$$

If we exclude these cases our semiconjugacy becomes a conjugacy.

5.3.1 The Perturbation as ϵ becomes nonzero

In this section we will attempt to better understand what happens as ϵ becomes nonzero. While 0 was a superattracting point when $\epsilon = 0$, when ϵ becomes non-zero three critical points split from the origin, and the origin is no longer a critical point and now forms a repelling two-cycle with ∞. Therefore, 0 and ∞ are now in the Julia set. The attracting fixed point p (and the symmetric two-cycle $\{\omega p, \omega^2 p\}$) are born from 0 as ϵ becomes nonzero.

The basin of attraction for zero is split into basins of attraction for the fixed point p and the two-cycle $\{\omega p, \omega^2 p\}$. Since these basins intersect (as previously described) they pinch the regions R_i in a manner similar to the pinching that occurs when $\epsilon = 0$ and r passes through $(1/2)^{1/3}$. (See Figure 5·12.) However, the origin is now a pole and these regions will no longer be mapped in one-to-one fashion. It is therefore necessary to rename the regions, splitting R_i into R_i and R_{i+3} for $i = 1, 2, 3$. (See Figure 5·17.) Notice that in Figure 5·12 the large circle is ∂O_∞^{-1}, while in Figure 5·17 the large circle actually represents ∞ since the basins W_i stretch to ∞.

5.3.2 $r > r^*$

There are two attracting fixed points in $\mathbb{R}^-$ which we will denote p_1 and p_2 where we assume $p_2 < p_1$. The existence of the attracting fixed points p_1 and p_2 on $\mathbb{R}^-$ implies that ωp_i and $\omega^2 p_i$ form attracting two-cycles. Hence we have two attracting cycles of period two lying on the symmetric axes.

It is easy to see that c_i is in the immediate basin of attraction of p_i for $i = 1, 2$. Hence, by symmetry we know that ωc_i and $\omega^2 c_i$ are in the immediate basin of attraction for the two-cycle formed by ωp_i and $\omega^2 p_i$ for $i = 1, 2$. This accounts for all 6 of our critical points. Hence, we know that there are no other attracting periodic points because all attracting periodic points must attract a critical point.

Let A_i denote the immediate basin of attraction for p_i. Note that A_i is trapped in the region $S = \{z : \text{Re(z)} < 0, 2\pi/3 < \text{Arg}(z) < 4\pi/3\}$. To see this note that A_i cannot meet either of the symmetric axes since the symmetric axes are forward invariant and hence points on these axes cannot lie in the basin of attraction of a real fixed point. Since $\mathbb{R}$ is foward invariant we see that none of the immediate basins on $\omega\mathbb{R}$ or $\omega^2\mathbb{R}$ meet $\mathbb{R}$. Further, $\mathbb{R}$ and $\omega\mathbb{R} \cup \omega^2\mathbb{R}$ forward invariant imply that these results hold for all preimages of these basins, too. Using similar techniques as before we can show that $\partial A_1 \cap \partial A_2$ is a single point on the negative real axis. Further, $\partial \omega^j A_i \cap \partial \omega^k A_i$ is a single point for $j, k = 0, 1, 2$ and

$i = 1, 2$. The basins are show in Figure 5·18.

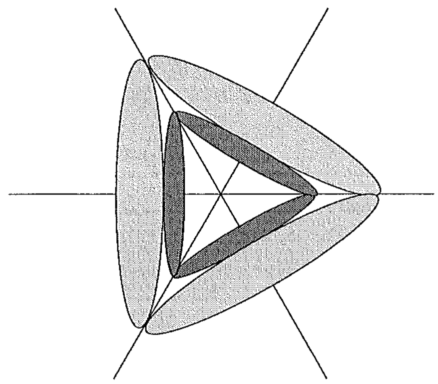


Figure 5·18: The immediate basins of attraction for $\epsilon > 0$ and $r > r^*$.

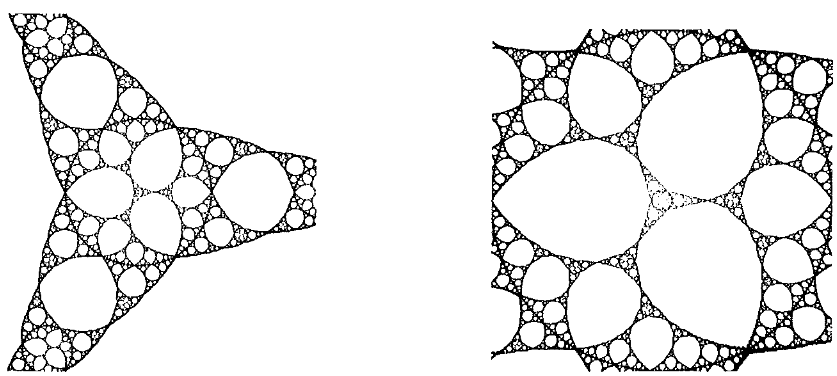


Figure 5·19: The Julia set for $\epsilon > 0$ and $r > r^*$ and a magnification of the same set.

It can be shown that these basin boundaries have the same properties as the prior basins in that they are all simple closed curves. All of our critical points are accounted for, so the remainder of the Julia set consists of preimages of these basins, all of which will have simple closed curve boundaries. The Julia set is shown in Figure 5·19. Since ∞ lies on a repelling two-cycle with 0 we know, as in the case for $r < r^*$, that the Julia set extends to ∞.

Notice that much of the structure from $\epsilon = 0, r > r^*$ is still present in this Julia set, as can be seen in Figure 5·19.

Bibliography

[BDL$^+$03a] P. Blanchard, R. L. Devaney, D. M. Look, P. Seal, and Y. Shapiro, *Sierpinski curve Julia sets and singular perturbations of complex polynomials*, To appear in *Ergodic Theory and Dynamical Systems*, 2003.

[BDL$^+$03b] P. Blanchard, R.L. Devaney, D. M. Look, M. Moreno Rocha, P. Seal, S. Siegmund, and D. Uminsky, *Sierpinski carpets and gaskets as Julia sets of rational maps*, Preprint, Boston University, 2003.

[Bea91] A.F. Beardon, *Iteration of rational functions*, first ed., Springer, New York, 1991.

[Bla84] P. Blanchard, *Complex analytic dynamics on the Riemann sphere*, Bulletin of the American Mathematical Society **2** (1984), no. 1, 85–141.

[Dev04] R. L. Devaney, *Cantor and Sierpinski, Julia and Fatou: Complex topology meets complex dynamics*, Notices of the American Mathematics Society (2004), no. 51, 9–15.

[DH82] A. Douady and J. Hubbard, *Itération des polynômes quadratiques complexes*, Comptes Rendus de l'Académie des Sciences, Paris (1982), no. 29, 123–126.

[DH85] A. Douady and J. Hubbard, *On the dynamics of polynomial-like mappings*, Annales Scientifiques des Ecoles Normales (1985), no. 18, 287–343.

[DJS04] R.L. Devaney, K. Josic, and Y. Shapiro, *Singular perturbations of quadratic maps*, International Journal of Bifurcation and Chaos (2004), no. 14, 161–169.

[DLU03] R.L. Devaney, D.M. Look, and D. Uminsky, *The escape trichotomy for singularly perturbed rational maps*, 2003.

[DMRS03] R.L. Devaney, M. Moreno Rocha, and S. Siegmund, *Rational maps with generalized Sierpinski gasket Julia sets*, 2003.

[Haw03] J. Hawkins, *Lebesgue ergodic rational maps in parameter space*, International Journal of Bifurcation and Chaos (2003), no. 13, 1423–1447.

[McM88] C. McMullen, *Automorphisms of rational maps*, Holomorphic Functions and Moduli **1** (1988).

[McM95] C. McMullen, *The classification of conformal dynamical systems*, Current Developments in Mathematics (1995), 323–360.

[Mil99] J. Milnor, *Dynamics in one complex variable*, first ed., Vieweg, New York, 1999.

[ML93] J. Milnor and Tan Lei, *A "Sierpinski carpet" as Julia set*, Experimental Mathematics (1993), no. 2, 37–83.

[MNTU00] S. Morosawa, Y. Nighimura, M. Taniuchi, and T. Ueda, *Holomorphic dynamics*, first ed., Cambridge University Press, Cambridge, UK, 2000.

[Sul85] D. Sullivan, *Quasiconformal maps and dynamical systems I, solutions of the Fatou-Julia problem on wandering domains*, Annals of Mathematics (1985), no. 122, 401–418.

[Why58] G.T. Whyburn, *Topological characterization of the Sierpinski curve*, Fundaments of Mathematics (1958), no. 45, 320–324.

[Wit88] B. Wittner, *On the bifurcation loci of rational maps of degree two*, 1988.

Curriculum Vitae

Daniel M. Look

Education

09/1999–05/2005 Ph.D. degree in Mathematics at Boston University, Boston MA. Dissertation title: *Singular Perturbations of Complex Polynomials and Circle Inversion Maps* Advisor: Prof. R.L. Devaney

09/1996–12/1998 M.A. Mathematics at the University of Maine, Orono ME.

09/1992–05/1996 B.A. in Mathematics at the University of Maine, Orono ME.

Publications

Sierpinski Curve Julia Sets and Singular Perturbations of Complex Polynomials, P. Blanchard, R. L. Devaney, D. M. Look, P. Seal, and Y. Shapiro, to appear in Ergodic Theory and Dynamical Systems.

Symbolic Dynamics for a Sierpinski Curve Julia Set, R.L. Devaney and D. M. Look, to appear in Journal of Difference Equations and Applications.

Sierpinski Carpets and Gaskets as Julia Sets of Rational Maps, P. Blanchard, R. L. Devaney, D. M. Look, M. Moreno Rocha, P. Seal, and D. Uminsky, submitted to Proceedings of the Bodilfest.

Singular Perturbations of z^n, R. L. Devaney, M. Holzer, D. M. Look, M. Moreno Rocha, and D. Uminsky, submitted to Noel Baker Proceedings.

A Criterion for Sierpinski Curve Julia Sets for Rational Maps, R. L. Devaney and D. M. Look, submitted to Topology Proceedings.

Buried Sierpinski Curve Julia Sets, R. L. Devaney and D. M. Look, submitted to Discrete and Continuous Dynamical Systems.

The Escape Trichotomy for Singularly Perturbed Rational Maps, R. L. Devaney, D. M. Look, and D. Uminsky, submitted to Indiana University Mathematics Journal.

Locally Sierpinski Julia Sets of Weierstrass Elliptic $\wp$ Functions, J. Hawkins and D. M. Look, to appear in International Journal of Bifurcation and Chaos.

Classifying Simple Closed Curve Pairs on the 2-Sphere: A Generalization of the Schoenflies Theorem, R. Franzosa, I. Gotchev, and D. M. Look, submitted to Topology and Its Applications.

Talks

03/30/2005 *Mathematics Seminar,* Colby College, Waterville, Maine.

03/21/2005 *Mathematics Seminar,* Indiana University of Pennsylvania, Indiana, Pennsylvania.

03/19/2005 *Spring Topology and Dynamics Conference,* Berry College, Mount Berry, Georgia.

03/16/2005 *Topology and Dynamics Seminar,* University of Alabama, Birmingham, Alabama.

03/14/2005 *Dynamics Seminar,* Boston University, Boston, Massachusetts.

12/09/2004 *Workship on Holomorphic Dynamics,* University of Warwick, UK.

11/20/2004 *Northeastern Section Meeting for the Mathematical Association of America,* Worcester Polytechnic Institute, Worcestor, Massachusetts.

11/11/2004 *Mathematics Colloquium,* University of Maine, Orono, Maine.

10/23/2004 *Midwest Dynamical Systems Seminar,* University of Cincinnati, Cincinnati, Ohio.

04/30/2004 *Mathematics Seminar,* Tufts University, Medford, Massachusetts.

04/30/2004 *Complex Analysis course meeting,* Tufts University, Medford, Massachusetts.

03/25/2004 *Spring Topology and Dynamics Conference,* University of Alabama, Birmingham, Alabama.

03/15/2004 *Dynamics Seminar,* Boston University, Boston, Massachusetts.

03/15/2004 *Graduate Student Dynamics Seminar,* Boston University, Boston, Massachusetts.

Teaching Experience

09/2004–06/2005 *Calculus Instructor,* Boston University Academy, Boston, Massachusetts.

09/2001–05/2005 *Assistant Instructor for Differential Equations,* Boston University, Boston, Massachusetts.

09/2004–12/2004 *Teaching Fellow for Multivariable Calculus,* Boston University, Boston, Massachusetts.

01/2004–05/2004 *Instructor for Calculus II,* Boston University, Boston, Massachusetts.

09/2003–12/2003 *Instructor for Calculus II,* Boston University, Boston, Massachusetts.

09/2003–12/2003 *Teaching Fellow for Introduction to Statistics,* Boston University, Boston, Massachusetts.

05/2003–06/2003 *Instructor for Calculus II,* Boston University, Boston, Massachusetts.

01/2003–05/2003 *Instructor for Calculus II,* Boston University, Boston, Massachusetts.

09/2002–12/2002 *Teaching Fellow for Introduction to Chaotic Dynamical Systems,* Boston University, Boston, Massachusetts.

05/2002–06/2002 *Instructor for Discrete Mathematics for Engineers,* Boston University, Boston, Massachusetts.

09/2000–12/2000 *Teaching Fellow for Calulus I/II Review,* Boston University, Boston, Massachusetts.

09/2000–12/2000 *Teaching Fellow for Linear Algebra,* Boston University, Boston, Massachusetts.

09/1999-12/1999 *Teaching Fellow for Multivariable Calculus,* Boston University, Boston, Massachusetts.

01/1999–05/1999 *Instructor for College Alegebra,* University of Maine, Orono, Maine.

01/1999–05/1999 *Instructor for Geometry for High School Teachers,* University of Maine, Orono, Maine.

09/1996–12/1998 *Teaching Assistant for College Algebra,* University of Maine, Orono, Maine.

Related Work

09/2002–05/2003 *Helped proofread second edition of Differential Equations, Dynamical Systems and An Introduction to Chaos by M. W. Hirsch, S. Smale and R. L. Devaney.*

Outreach Activities

Summers 1997-Present *Mathematics Instructor for Upward Bound Regional Math/Science Program,* University of Maine, Orono, Maine.

11/06/2004 *Helped coordinate Mathmeet 2004,* Colby College, Waterville, Maine.

10/29/2003 *Spoke at 12th Annual Mathematics Field Day,* Boston University, Boston, Massachusetts.

10/29/2002 *Spoke at 11th Annual Mathematics Field Day,* Boston University, Boston, Massachusetts.

Awards and Honors

05/2000 *Teaching Fellow Award,* Awarded for outstanding performance as a Teaching Fellow. Boston University, Boston, Massachusetts.

05/1997 *Teaching Assistant Award,* Awarded for outstanding performance as a Teaching Assistant. University of Maine, Orono, Maine.