SINGULAR PERTURBATIONS OF z^n

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1. Introduction

Our goal in this paper is to describe the topology of and dynamics on certain Julia sets of functions drawn from the family of rational maps of the complex plane given by

$$F_\lambda(z) = z^n + \frac{\lambda}{z^d}$$

where $n, d \in \mathbb{Z}^+$. When $\lambda = 0$, these maps reduce to $z \mapsto z^n$ and the dynamical behavior in this case is well understood: the Julia set of F_λ is just the unit circle and all other orbits tend either to ∞ or to the superattracting fixed point at 0.

When $\lambda \neq 0$, several things happen. First of all, the map F_λ now has degree $n + d$ rather than n. Secondly, the origin is a pole rather than a fixed point. And, finally, there are $n + d$ new critical points in addition to the original critical points at 0 and ∞. As we discuss below, the orbits of all of these new critical points behave symmetrically, so we essentially have only one additional "free" critical orbit for each of these maps. As is well known in complex dynamics, the behavior of this critical orbit determines much of the structure of the Julia sets of these maps.

One of our main goals in this paper is to describe what happens to the Julia set when the parameter λ is nonzero but small. In this case, the map F_λ is called a *singular perturbation* of z^n. The reason for the interest in such a perturbation arises from Newton's method. Suppose we are applying Newton's method to find the roots of a family of polynomials P_λ which has a multiple root at, say, the parameter $\lambda = 0$. For example, consider the especially simple case of $P_\lambda(z) = z^2 + \lambda$. When $\lambda = 0$ this polynomial has a multiple root at 0 and the Newton iteration function is simply $N_\lambda(z) = z/2$. However, when $\lambda \neq 0$, the Newton iteration function becomes

$$N_\lambda(z) = \frac{z^2 - \lambda}{2z}$$

and we see that, as in the family F_λ, the degree jumps as we move away from $\lambda = 0$. In addition, instead of a fixed point at the origin, after the perturbation, there is a pole at the origin.

For the families F_λ, there are a number of different cases to consider depending on the values of n and d. When $n \geq 2$, the point at ∞ is a superattracting fixed point whereas when $n = 1$ this point is a parabolic fixed point. Since much of the interesting dynamical behavior occurs when the free critical points tend to or land at ∞, the singular perturbations therefore behave very differently in these two cases.

One of the main results that we describe below is the following. When $n \geq 2$, we have an immediate basin of attraction B_λ of the superattracting fixed point at ∞. Note that F_λ is n to 1 on a neighborhood of ∞ in B_λ. Since 0 is a pole of order d, the only preimages of points in this neighborhood lie in a neighborhood of the origin. We let T_λ be the preimage of B_λ surrounding the origin. (The sets B_λ and T_λ may or may not be disjoint.) As we shall show, for λ small, it is possible that the critical orbits eventually land in B_λ and hence tend to ∞. In this case, we have the following result described in Section 3.

Theorem (The Escape Trichotomy). *Suppose $n \geq 2$ and that the orbits of the free critical points of F_λ tend to ∞. Then*

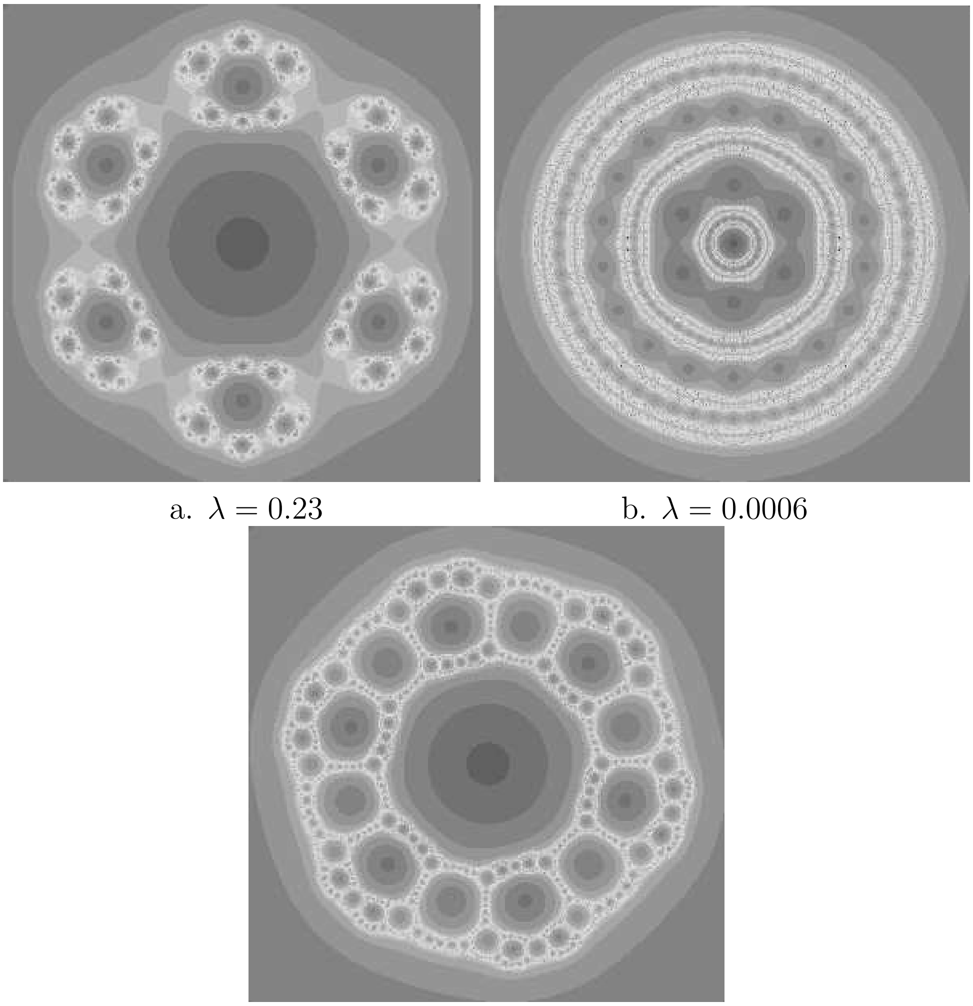
(1) *If one of the critical values lies in B_λ, then $J(F_\lambda)$ is a Cantor set and $F_\lambda \mid J(F_\lambda)$ is a one-sided shift on $n + d$ symbols. Otherwise, the preimage T_λ is disjoint from B_λ.*
(2) *If one of the critical values lies in $T_\lambda \neq B_\lambda$, then $J(F_\lambda)$ is a Cantor set of simple closed curves (quasicircles).*
(3) *If one of the critical values lies in a preimage of B_λ different from T_λ, then $J(F_\lambda)$ is a Sierpinski curve.*

Several Julia sets illustrating this trichotomy and drawn from the family where $n = d = 3$ are included in Figure 1.

A *Sierpinski curve* is a very interesting topological space. By definition, a Sierpinski curve is a planar set that is homeomorphic to the well-known Sierpinski carpet fractal. But a Sierpinski curve has an alternative topological characterization: any planar set that is compact, connected, locally connected, nowhere dense, and has the property that any two complementary domains are bounded by disjoint simple closed curves is known to be homeomorphic to the Sierpinski carpet [24]. Moreover, such a set is a universal planar set in the sense that it contains a homeomorphic copy of any compact, connected, one-dimensional subset of the plane.

When $n \geq 2$, there are certain cases of this Theorem that may or may not hold, depending on the value of d. For example, if n and d satisfy

$$\frac{1}{n} + \frac{1}{d} < 1,$$



a. $\lambda = 0.23$ b. $\lambda = 0.0006$

c. $\lambda = 0.125i$

FIGURE 1. Some Julia sets for $z^3 + \lambda/z^3$: if $\lambda = 0.23$, $J(F_\lambda)$ is a Cantor set; if $\lambda = 0.0006$, $J(F_\lambda)$ is a Cantor set of circles; and if $\lambda = 0.125i$, $J(F_\lambda)$ is a Sierpinski curve.

then there is a neighborhood of $\lambda = 0$ for which the critical values all lie in $T_\lambda \neq B_\lambda$ and so the Julia set is a Cantor set of simple closed curves. This phenomenon was first observed by McMullen for small λ (see [14]) and so we call the regime in the λ-plane where this occurs the *McMullen domain.* There is no McMullen domain if this inequality does not hold, i.e., if n or d is equal to 1 or if $n = d = 2$. Instead, in the special cases where $n = d = 2$ or $n > 1, d = 1$, we have the following result which is described in Section 4:

Theorem. *Suppose $n = d = 2$ or $n > 1$, $d = 1$. Then, in every neighborhood of the origin in the parameter plane, there are infinitely many disjoint open sets $\mathcal{O}_j$ with $j = 1, 2, 3. \ldots$ of parameters having the following properties:*

(1) *If* $\lambda \in \mathcal{O}_j$*, then the Julia set of* F_λ *is a Sierpinski curve, so that if* $\lambda \in \mathcal{O}_j$ *and* $\mu \in \mathcal{O}_k$*, the Julia sets of* F_λ *and* F_μ *are homeomorphic;*
(2) *But if* $k \neq j$*, the maps* F_λ *and* F_μ *are not topologically conjugate on their respective Julia sets.*

The case where $n = 1$ is fundamentally different from the other cases since the function

$$F_\lambda(z) = z + \frac{\lambda}{z^d}$$

has a parabolic fixed point at ∞. Furthermore, any map of this form is linearly conjugate to the case where $\lambda = 1$. So, instead of considering this case, we adjust the family slightly to deal instead with the family

$$F_\lambda(z) = \lambda\left(z + \frac{1}{z^d}\right)$$

when $n = 1$. For this family ∞ is an attracting fixed point when $|\lambda| > 1$ and is repelling when $|\lambda| < 1$. So when $|\lambda| < 1$, ∞ is in the Julia set, and we may have that the critical orbits map onto ∞. In this case, the Julia set is the entire Riemann sphere. Much else occurs near $\lambda = 0$, for, as we show in Section 5, we have:

Theorem. *Let* $F_\lambda(z) = \lambda(z + 1/z)$*. Then, in any neighborhood of* $\lambda = 0$ *in the parameter plane:*

(1) *There are infinitely many parameter values* λ *for which the Julia set of* F_λ *is the entire Riemann sphere;*
(2) *There are also infinitely many parameter values for which the critical orbit is superattracting.*

Unlike the situation that is described in the previous two theorems for λ near 0, in the case where we have a McMullen domain, the dynamical behavior of F_λ is the same for any λ sufficiently close to 0. However, away from this region, F_λ exhibits a rich array of different dynamical behavior. For example, in Section 6 we show that there are many different ways that the Julia sets may be Sierpinski curves. In the previous Sierpinski curve examples, the complement of $J(F_\lambda)$ was simply B_λ together with all of its preimages. However, there are parameter values in these families for which the Julia set is a Sierpinski curve whose complementary domains consist of a variety of different attracting basins (not just B_λ) together with their preimages. Again, while these Julia sets are all homeomorphic to one another, the dynamics on different pairs of these sets is often quite different.

There is another famous Sierpinski "object" in fractal geometry, namely, the Sierpinski gasket or triangle. Objects similar in construction to this shape also occur in these families. In Section 7 we construct infinitely many "Sierpinski gasket-like" Julia sets for F_λ. Unlike the Sierpinski curves, each pair of these Julia sets are topologically as well as dynamically distinct.

This paper is respectfully dedicated to the memory of Professor Noel Baker. Professor Baker's numerous contributions to the field of complex dynamics have been an inspiration to all of us.

2. Preliminaries

We consider the maps

$$F_\lambda(z) = z^n + \frac{\lambda}{z^d}$$

where $n, d \in \mathbb{Z}^+$. The *Julia set* of F_λ, $J(F_\lambda)$, is defined to be the set of points at which the family of iterates of F_λ fails to be a normal family in the sense of Montel. Equivalently, the Julia set is the closure of the set of repelling periodic points for F_λ or, alternatively, the set of points on which F_λ behaves chaotically. The complement of the Julia set is called the *Fatou set*.

There are $n+d$ finite and nonzero critical points for F_λ and all are of the form $\omega^k c_\lambda$ where c_λ is one of the critical points and $\omega^{n+d} = 1$. Similarly, the critical values are arranged symmetrically with respect to $z \mapsto \omega z$, though there need not be $n+d$ of them. For example, if $n = d$, the $n+d$ critical points are given by $\lambda^{1/2n}$, while there are only two critical values given by $\pm 2\sqrt{\lambda}$. There are $n+d$ prepoles at the points $(-\lambda)^{1/(n+d)}$.

Note that $F_\lambda(\omega z) = \omega^n F_\lambda(z)$. Hence the orbits of points of the form $\omega^j z$ all behave "symmetrically" under iteration of F_λ. For example, if $F_\lambda^i(z) \to \infty$, then $F_\lambda^i(\omega^k z)$ also tends to ∞ for each k. If $F_\lambda^i(z)$ tends to an attracting cycle, then so does $F_\lambda^i(\omega^k z)$. Note, however, that the cycles involved may be different depending on k and, indeed, they may even have different periods. Nonetheless, all points lying on this set of attracting cycles are of the form $\omega^j z_0$ for some $z_0 \in \mathbb{C}$. In particular, all $n+d$ critical points have orbits that behave symmetrically, so this is why there is only one free critical orbit for F_λ.

We now restrict attention to the case $n \geq 2$; the case $n = 1$ will be dealt with in Section 5. The point at ∞ is a superattracting fixed point for F_λ and it is well known that F_λ is conjugate to $z \mapsto z^n$ in a neighborhood of ∞, so we have an immediate basin of attraction B_λ at ∞. Since F_λ has a pole of order d at 0, there is an open neighborhood of 0 that is mapped d to 1 onto a neighborhood of ∞ in B_λ. If B_λ does not contain this neighborhood, then there is a disjoint open set about 0 that is mapped d to 1 onto B. This set is called the *trap door* and we denote it by T_λ. Since the degree of F_λ is $n+d$, all points in the preimage of B_λ lie either in B_λ or in T_λ.

Using the symmetry $F_\lambda(\omega z) = \omega^n F_\lambda(z)$, it is straightforward to check that all of B_λ, T_λ, and $J(F_\lambda)$ are symmetric under $z \mapsto \omega z$. We say that these sets possess $n+d$-fold symmetry. In particular, since the critical points are arranged symmetrically about the origin, it follows that if one of the critical points lies in B_λ (resp.,T_λ), then all of the critical points lie in B_λ (resp., T_λ).

For other components of the Fatou set, the symmetry situation is somewhat different: either a component contains $\omega^j z_0$ for a given z_0 in the Fatou set and all $j \in \mathbb{Z}$, or else such a component contains none of the $\omega^j z_0$ with $j \neq 0$ mod $n+d$:

Symmetry Lemma. *Suppose U is a connected component of the Fatou set of F_λ. Suppose also that both z_0 and $\omega^j z_0$ belong to U, where $\omega^j \neq 1$. Then in fact, $\omega^i z_0$ belongs to U for all i and, as a consequence, U has $n+d$-fold symmetry and surrounds the origin.*

See [8] for a proof of this fact.

3. The Escape Trichotomy

For the well-studied family of quadratic maps $Q_c(z) = z^2 + c$ with c a complex parameter there is the well known Fundamental Dichotomy:

(1) If the orbit of the one free critical point at 0 tends to ∞, then the Julia set of Q_c is a Cantor set;
(2) If the orbit of 0 does not tend to ∞, then the Julia set is a connected set.

In this section we discuss a similar result for F_λ that we call the Escape Trichotomy. Unlike the family of quadratic maps Q_c, there exist three different "ways" that the critical orbit for F_λ can tend to infinity. If the critical orbit tends to infinity, then all of the critical values must lie in B_λ or one of its preimages. These three different scenarios lead to three distinct classes of Julia sets for F_λ that comprise the Escape Trichotomy.

3.1. **Critical Values in B_λ.** We first assume that one of the critical values of F_λ lies in B_λ. In this case, $J(F_\lambda)$ is a Cantor set. We sketch a proof of this fact here (for more details, see [8]).

By symmetry, if one of the critical values lies in B_λ, then all of the critical values do so as well. Let v be a critical value of F_λ and let c be a critical point such that $F_\lambda(c) = v$. Let U be an open disk in B_λ containing both v and ∞ with $F_\lambda(U) \subset U$. We may assume that U has $(n+d)$-fold symmetry. Let V be the preimage of $F_\lambda(U)$ containing the origin. We may also assume that U and V are disjoint.

Let γ be an arc in U connecting v to ∞. The preimage of γ is an arc γ' that contains c and is mapped two-to-one onto γ. One portion of γ' connects c to ∞. The curve γ' must therefore also lie in B_λ, and so we see that c and hence all of the critical points must lie in B_λ.

Since c is a critical point, it follows that γ' contains a second preimage of ∞. One checks easily that this second preimage of ∞ is 0, not ∞, and so γ' extends all the way from 0 to ∞. In particular, γ' meets both U and V, and so both of these sets lie in B_λ. Therefore B_λ and T_λ are not disjoint sets. Let W be the preimage of U. It follows that W contains U, V, and a neighborhood of γ'.

Since v was an arbitrary critical value of F_λ we can repeat this process and obtain $n+d$ arcs connecting 0 and ∞ such that each arc contains a distinct critical point. Furthermore, these arcs may be chosen so that they do not intersect and are symmetric under $z \to \omega z$ where $\omega^{n+d} = 1$. Each of these arcs also lies in W and so W consists of the Riemann sphere with $n+d$ disjoint and symmetric disks A_j for $j = 1, \ldots, n+d$ removed. Finally, it is easy to check that each A_j in the complement of W is mapped univalently over the complement of U and hence over all of the other A_i. Therefore, each of the $n+d$ sets A_j contain preimages of all of the other A_i, and the Julia set is contained in the union of these $(n+d)^2$ sets. See Figure 2. Standard arguments then show that the Julia set is a Cantor set and F_λ is a one-sided shift on $n+d$ symbols on this set. Figure 1a displays an example of a Julia set for which the critical values lie in B_λ.

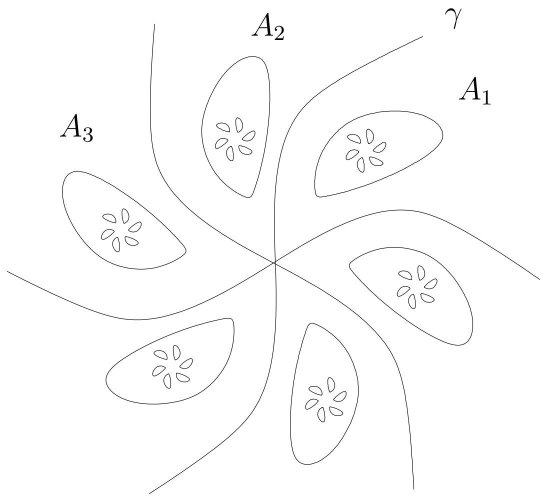


FIGURE 2. The sets A_j and their preimages.

3.2. **Critical Values in T_λ.** Assume now that B_λ and T_λ are disjoint and that one, and hence all, of the critical values of F_λ now lie in T_λ. In this case, $J(F_\lambda)$ is a Cantor set of simple closed curves. To see this, note first that, since B_λ and T_λ are both open disks, the Riemann-Hurwitz formula shows that the preimage of T_λ is an open annulus surrounding the origin and located between $\overline{T}_\lambda$ and $\overline{B}_\lambda$. We denote this preimage of T_λ by T_λ^{-1} and the n^{th} preimage of T_λ by T_λ^{-n}. The annulus T_λ^{-1} contains all of the critical points and its closure divides the region between $\overline{T}_\lambda$ and $\overline{B}_\lambda$ into two open subannuli that are mapped onto $\mathbb{C} - (\overline{B}_\lambda \cup \overline{T}_\lambda)$. We call these subannuli A_{in} and A_{out}, with A_{in} the subannulus bordering T_λ and A_{out} the subannulus bordering B_λ. Note that since the boundary of T_λ^{-1}, ∂T_λ^{-1}, is mapped onto ∂T_λ, whereas both ∂T_λ and ∂B_λ are both mapped onto ∂B_λ, it must be the case that ∂T_λ^{-1}

is disjoint from ∂T_λ and ∂B_λ. See Figure 3. Let A denote the union of the three annuli A_{in}, A_{out}, and T_λ^{-1}.

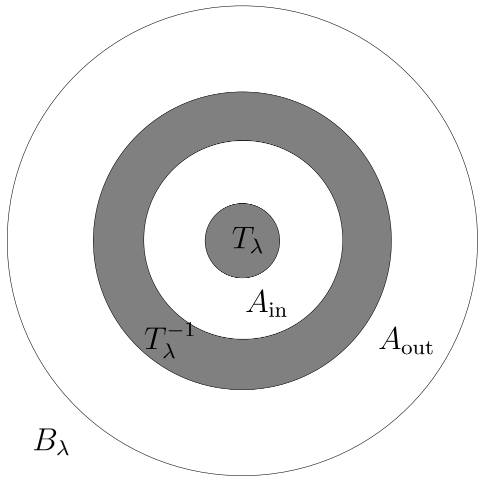


FIGURE 3. The sets A_{in}, A_{out}, T_λ, B_λ and T_λ^{-1}.

Since all of the critical points lie in T_λ^{-1}, the annuli A_{in} and A_{out} are mapped as coverings onto $\mathbb{C} - (\overline{B}_\lambda \cup \overline{T}_\lambda)$. Hence there exist preimages of T_λ^{-1} in each of these subannuli. Note that there will be two annular components of T_λ^{-2}, one in A_{in} and one in A_{out}. See Figure 4. Continuing in this fashion, we see that T_λ^{-n} consists of 2^{n-1} subannuli. In [8], quasiconformal surgery was used to show that the boundaries of B_λ, T_λ, and all of the preimages of T_λ are simple closed curves surrounding the origin. Hence the Julia set is given by a nested intersection of closed annuli and the result follows exactly as in the case described by McMullen in [14].

We remark that, by the covering properties of F_λ on A_{in} and A_{out}, we must have

$$\operatorname{mod} A > \operatorname{mod} A_{\text{in}} + \operatorname{mod} A_{\text{out}} = \left(\frac{1}{d} + \frac{1}{n}\right) \operatorname{mod} A$$

where $\operatorname{mod} A$ denotes the modulus of A. Hence, as in the McMullen result, we must have $1/d + 1/n < 1$ in order for v to lie in the trap door. Therefore, if $1/d + 1/n > 1$, then v cannot lie in the trap door, so part 2 of the Escape Trichotomy Theorem cannot occur if $d = n = 2$ or if either n or d is equal to 1. In Figure 1b we display a Julia set for which the critical values all lie in T_λ.

3.3. **Critical Values in a Preimage of T_λ.** We now describe the final case where the critical values have orbits that eventually escape through the trap door, but the critical values do not themselves lie in the trap door. In this case the Julia set is a Sierpinski curve. We first observe that the Julia set

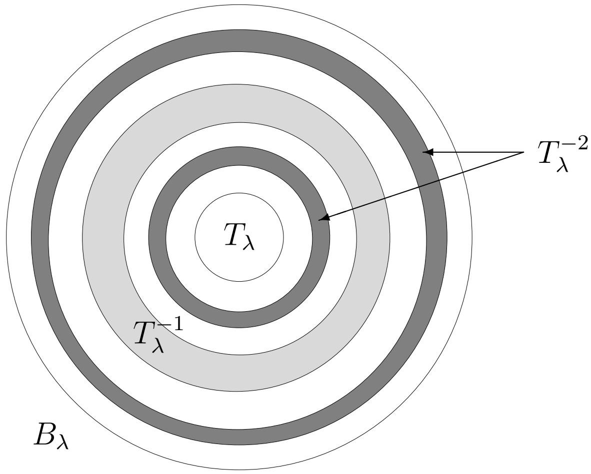


FIGURE 4. Inverse images of T_λ.

of F_λ is compact, connected, locally connected, and nowhere dense. Indeed, since we are assuming that the critical orbit eventually enters the basin of ∞, we have that the Julia set is given by $\mathbb{C} - \cup F_\lambda^{-j}(B_\lambda)$. That is, $J(F_\lambda)$ is $\mathbb{C}$ with countably many disjoint, simply connected, open sets removed. Hence $J(F_\lambda)$ is compact and connected. Since $J(F_\lambda) \neq \mathbb{C}$, $J(F_\lambda)$ cannot contain any open sets, so $J(F_\lambda)$ is also nowhere dense. Finally, since the critical orbits all tend to ∞ and hence do not lie in or accumulate on $J(F_\lambda)$, it follows that F_λ is hyperbolic on $J(F_\lambda)$ and standard arguments show that $J(F_\lambda)$ is locally connected (see [16]). In particular, since B_λ is a simply connected component of the Fatou set, it follows that the boundary of B_λ is locally connected. Hence $J(F_\lambda)$ fulfills the first four of the conditions to be a Sierpinski curve.

To finish showing that $J(F_\lambda)$ is a Sierpinski curve we need to show that the boundaries of B_λ as well as all of the preimages of B_λ are simple closed curves and that these boundary curves are pairwise disjoint. To see this, we first claim that $\mathbb{C} - \overline{B}_\lambda$ is a connected open set. This should be contrasted with the situation for quadratic polynomial Julia sets where $\mathbb{C} - \overline{B}_\lambda$ often consists of infinitely many disjoint open sets (consider the Julia sets known as the basilica or Douady's rabbit, for example). Assume that $\mathbb{C} - \overline{B}_\lambda$ has more than one component. Let W_0 be the component of $\mathbb{C} - \overline{B}_\lambda$ that contains the origin. Note that $T_\lambda \subset W_0$. Since $F_\lambda(\partial T_\lambda) = \partial B_\lambda \supset \partial W_0$, it follows that there are points in W_0 whose images also lie in W_0 and consequently $F_\lambda(W_0) \supset W_0$. Now if one of the prepoles lies in a component of $\mathbb{C} - \overline{B}_\lambda$ that is disjoint from W_0, then by symmetry all of the prepoles have this property. But this then gives us too many preimages of points in W_0, and so all of the prepoles must in fact lie in W_0. It then follows that all of the preimages of any point in W_0 lie in W_0.

If there were another component of $\mathbb{C} - \overline{B}_\lambda$, then the boundary of this set must eventually be mapped over the boundary of W_0 since $\partial W_0 \subset J(F_\lambda)$, and so there must be additional preimages of points in W_0. But again, this is impossible. Therefore W_0 is the only component of $\mathbb{C} - \overline{B}_\lambda$. Standard arguments using external rays then show that the boundary of W_0 must in fact be a simple closed curve. So too are the boundaries of all of the preimages of B_λ. One then checks that all of these curves are disjoint, for a point that lies in the intersection of one of these curves must either be a critical point or one of its preimages, but we know that all critical points have orbits that tend to ∞. This completes the proof that the Julia set is a Sierpinski curve.

In Figure 5 we show B_λ, T_λ and the first two preimages of T_λ in the special case where $n = d = 2$ and under the assumption that there are no critical points in T_λ^{-1} or T_λ^{-2}. An actual Julia set for which the critical points lie in T_λ^{-2} is depicted in Figure 1c.

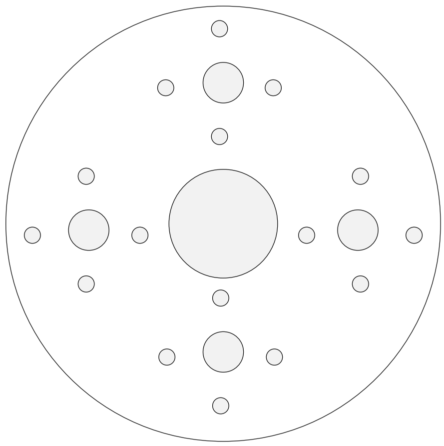


FIGURE 5. B_λ, T_λ, T_λ^{-1} and T_λ^{-2}.

In Figure 6, we show the λ plane in the case $n = d = 4$. The outside grey region in this image consists of λ-values for which $J(F_\lambda)$ is a Cantor set. The central grey region is the *McMullen domain* in which $J(F_\lambda)$ is a Cantor set of simple closed curves. The region between these two sets is called the connectedness locus as the Julia sets are always connected when λ lies in this region. The other grey regions in this figure correspond to *Sierpinski holes* in which the corresponding Julia sets are Sierpinski curves.

4. The Case $n = d = 2$

As mentioned earlier, the cases where $n = d = 2$ or $n > 1, d = 1$ are significantly different from the other cases where $n \geq 2$ because there is no McMullen domain in parameter space. In these cases, we instead have infinitely many open sets of parameters in any neighborhood $\lambda = 0$ in parameter space in which the critical orbits eventually enter B_λ and hence the Julia set

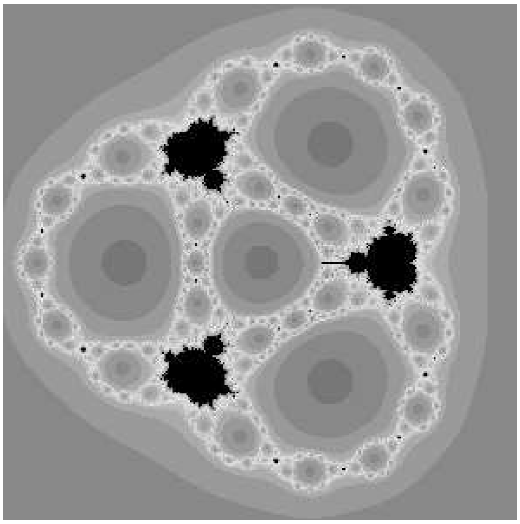


FIGURE 6. The parameter plane when $n = d = 4$.

is a Sierpinski curve. In each of these open sets the number of iterations that it takes for the critical orbit to enter B_λ is different, and so two maps drawn from different open sets are dynamically distinct in the sense that these maps are not topologically conjugate.

We sketch the proof of this when $n = d = 2$. We show that there are infinitely many open intervals in $\mathbb{R}^-$ in any neighborhood of the origin in parameter space in which the critical orbit eventually escapes. Similar results hold when $n > 1$, $d = 1$, though the real axis need not be the home of these open sets.

When $n = d = 2$, the four critical points and four prepoles of F_λ all lie on the circle of radius $|\lambda|^{1/4}$ centered at the origin. We call this circle the *critical circle*. The case $n = d = 2$ is especially simple since the second image of the critical points is given by

$$F_\lambda^2(c_\lambda) = 4\lambda + \frac{1}{4}$$

and so $\lambda \mapsto F_\lambda^2(c_\lambda)$ is an analytic function of λ that is a homeomorphism. If $-1/16 < \lambda < 0$, then one checks easily that the critical circle is mapped strictly inside itself. Therefore, as in the previous section, $J(F_\lambda)$ is a connected set and B_λ and T_λ are disjoint. In particular, the second image of the critical point lands on the real axis and lies in the complement of B_λ in $\mathbb{R}$.

Proposition. *There is an increasing sequence $\lambda_2, \lambda_3, \ldots$ in $\mathbb{R}^-$ with $\lambda_j \to 0$ and $F_{\lambda_j}^j(c_{\lambda_j}) = 0$.*

Proof: Since $F_\lambda^2(c_\lambda) = 4\lambda + 1/4$, this quantity increases monotonically toward $1/4$ as $\lambda \to 0$. Now the orbit of $1/4$ remains in $\mathbb{R}^+$ for all iterations of F_0 and

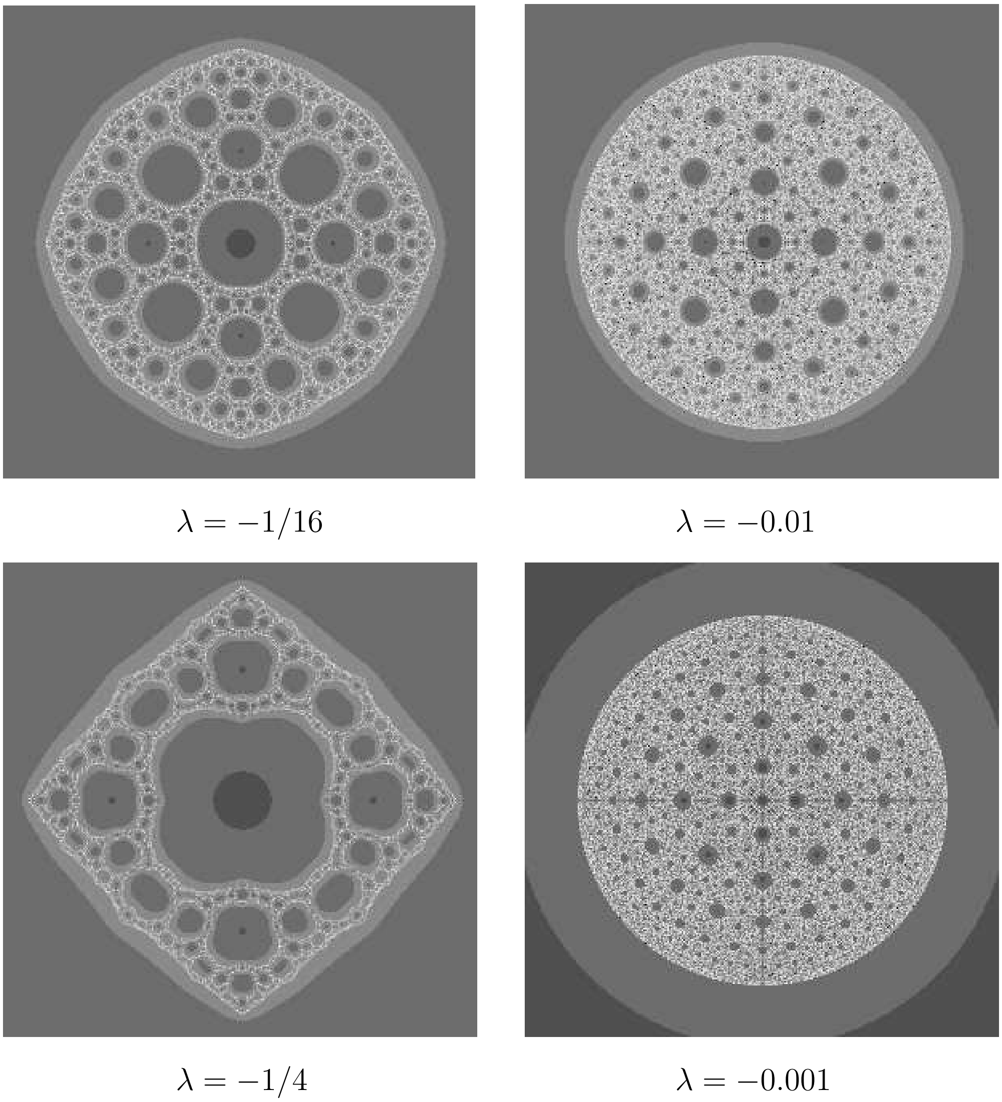


FIGURE 7. Sierpinski curve Julia sets for various negative values of λ when $n = d = 2$. All of these sets are homeomorphic, but the dynamics on each is different.

decreases monotonically to 0. Hence, given N, for λ sufficiently small, $F_\lambda^j(c_\lambda)$ also lies in $\mathbb{R}^+$ for $2 \le j \le N$ and moreover this finite sequence is decreasing.

Now suppose $\beta < \alpha < 0$. We have $F_\beta(x) < F_\alpha(x)$ for all $x \in \mathbb{R}^+$. Also, $F_\beta^2(c_\beta) < F_\alpha^2(c_\alpha) < 1/4$. Hence $F_\beta^j(c_\beta) < F_\alpha^j(c_\alpha)$ for all j for which $F_\beta^j(c_\beta) \in \mathbb{R}^+$. The result then follows by continuity of F_λ with respect to λ.

$\square$

Note that $\lambda_2 = -1/16$. Using the previous Proposition, we may find open intervals I_j about λ_j for $j = 2, 3, \ldots$ having the property that, if $\lambda \in I_j$, then $F_\lambda^j(c_\lambda) \in T_\lambda$, and so $F_\lambda^{j+1}(c_\lambda) \in B_\lambda$. Therefore, $F_\lambda^n(c_\lambda) \to \infty$ as $n \to \infty$, and the Escape Trichotomy then shows that $J(F_\lambda)$ is a Sierpinski curve.

Now let $C(c_\lambda)$ denote the component of the Fatou set of F_λ containing c_λ. The map F_λ is two-to-one on each of the four sets $C(c_\lambda)$ containing these critical points, and we have $F_\lambda^j(C(c_\lambda)) = T_\lambda$ for some j. Now suppose that $F_\lambda|J(F_\lambda)$ is conjugate to $F_\alpha|J(F_\alpha)$ for some $\alpha \in \cup I_k$ for some $k > 1$. This conjugacy must take the boundaries of B_λ and T_λ to the corresponding boundaries of B_α and T_α. Similarly the boundaries of the four regions $C(c_\lambda)$ must be mapped to one of the corresponding regions by the conjugacy, since these are the only complementary domains (besides B_λ and T_λ) on which F_λ is two-to-one. If, however, $\lambda \in I_j$ and $\alpha \in I_k$ with $j \neq k$, then these maps cannot be conjugate, since a conjugacy maps each of the j^{th} preimages of T_λ to one of the j^{th} preimages of T_α. Such a conjugacy would also have to map boundaries of domains on which F_λ and F_α were two-to-one to each other. Since $j \neq k$, this is impossible. We therefore have:

Theorem. *Let $\lambda \in I_j$ and $\alpha \in I_k$ with $j \neq k$. Then F_λ is not conjugate to F_α on their corresponding Julia sets.*

In Figure 7 we display several dynamically distinct Sierpinski curve Julia sets for λ close to 0.

In Figure 8 we display the parameter plane for the case $n = d = 2$ as well as a magnification around $\lambda = 0$. In contrast to the image in Figure 6, all of the internal grey regions in this image are Sierpinski holes. There is no McMullen domain when $n = d = 2$.

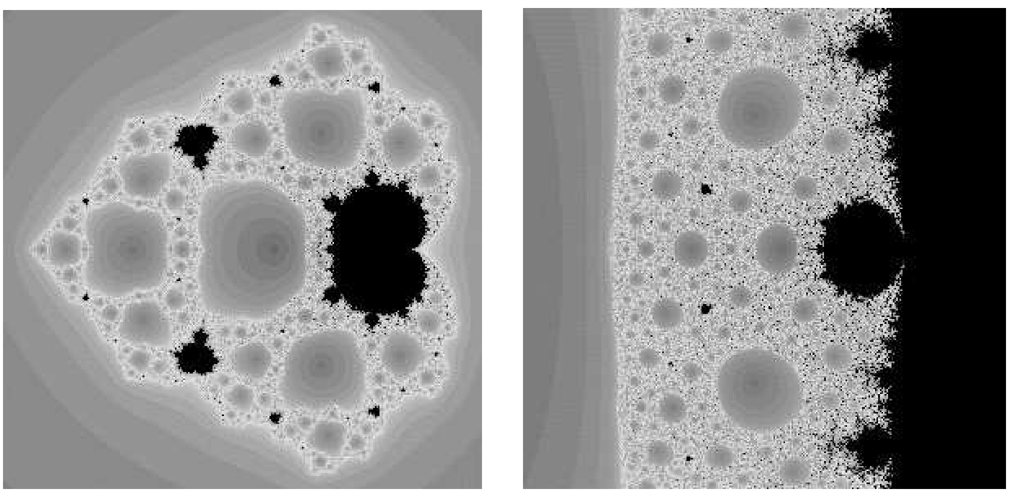


FIGURE 8. The parameter plane and a magnification when $n = d = 2$.

5. The case $n = 1$

In this section we restrict attention to the family of functions

$$F_\lambda(z) = z + \frac{\lambda}{z}$$

so that $n = 1$. The dynamics of these maps are quite different from those for which $n > 1$. First, one checks easily that, for each λ, the map F_λ is conjugate to the function

$$F_1(z) = z + \frac{1}{z}.$$

Hence this family does not really depend on a parameter. Therefore we change the family slightly so that we consider instead

$$F_\lambda(z) = \lambda\left(z + \frac{1}{z}\right).$$

This family is conjugate to the family

$$G_\lambda(z) = \lambda z + \frac{1}{z},$$

and so can be regarded as a linear perturbation of the involution $z \mapsto 1/z$.

The main difference between this family and our original family is that these functions have a repelling fixed point at infinity whenever $|\lambda| < 1$. Consequently, 0 lies in the Julia set and thus there is no trap door as in the case where $n > 1$.

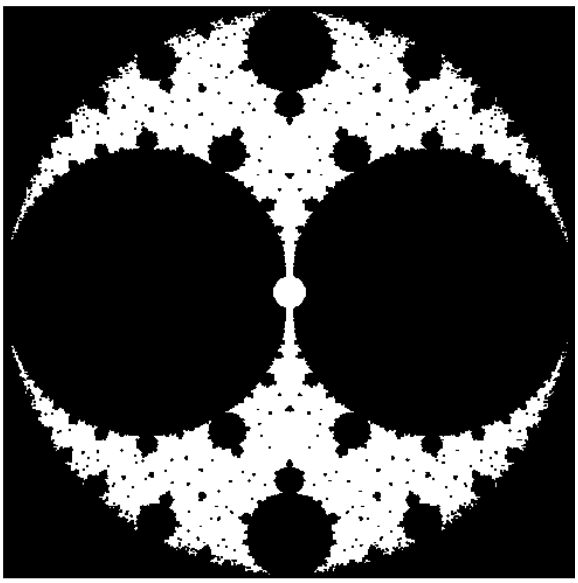


Figure 9. The λ plane for the function $F_\lambda(z) = \lambda(z + \frac{1}{z})$.

As in the previous cases, we are mainly concerned with the case of λ small, so that we are perturbing away from the identically zero function. It has been shown by Yongcheng [25] that for $0 < |\lambda| \leq 1$ the Julia set is connected while if $|\lambda| > 1$, it is a Cantor set. The parameter space is plotted in Figure 9. Similar figures have been produced by Hawkins [12] and Milnor [17]. Note that most of the interesting behavior seems to occur as we approach the parameter 0 along the imaginary axis. In fact, it is easy to check that, for $0 < \lambda \leq 1$, $J(F_\lambda)$ is the imaginary axis, and all other points have orbits that are attracted to one of two attracting fixed points. For $-1 \leq \lambda < 0$, $J(F_\lambda)$ is the real axis, and this set separates the basins of an attracting two-cycle. In both of the large black circular regions in parameter space flanking the origin, the Julia sets are similar curves passing through the origin and ∞. In contrast, the dynamical behavior along the imaginary axis is much more complicated.

Given nonzero λ, the function F_λ is a degree-two rational map with two critical points at ± 1. The orbits of these critical points behave symmetrically under F_λ. For purely imaginary parameter values, this function has the desirable property that, in the dynamical plane, the real axis is mapped to the imaginary axis and vice versa. Therefore, for such parameter values we will consider the second iterate map restricted to the real axis, that is, we restrict attention to the behavior of $F_{i\lambda}^2$ on $\mathbb{R}$, where λ is now a real parameter.

We compute

$$F_{i\lambda}^2(x) = -\lambda^2 \left(x + \frac{1}{x} \right) + \frac{1}{x + \frac{1}{x}}.$$

Note that, for small λ, this second iterate map can be viewed as a perturbation of the λ-independent function

$$x \mapsto \frac{1}{x + \frac{1}{x}}.$$

When one and hence both of the critical points land on the repelling fixed point at ∞, the Julia set is known to be the entire Riemann sphere [16]. We will refer to such parameter values as *blowup* points, with the convention that a blowup point of order n is one such that $F_{i\lambda}^{2n}(1) = 0$. Parameter values for which this occurs are also known as m-ergodic rational maps (although m-ergodicity describes a larger set of maps than just those for which a critical point lands on a repelling cycle). Rees [21] has proved that m-ergodic maps comprise a set of positive Lebesgue measure in the parameter space of most rational maps. Hawkins [12] developed a computer algorithm for finding and plotting these parameter values. In that paper, it was shown numerically that the m-ergodic maps accumulate on the origin along the imaginary axis in parameter space. We formalize this observation via the following theorem.

Theorem. *For the family of functions $F_{i\lambda}(z) = i\lambda(z + 1/z)$, in any neighborhood of $\lambda = 0$, there exists:*

(1) *A countably infinite set of λ-values lying in $(-1, 1)$ for which the Julia set is the entire Riemann sphere;*

(2) *A countably infinite set of λ-values lying in $(-1,1)$ for which the critical point is part of a superattracting cycle.*

Proof: To prove the first assertion, we will define a function $G_n : \mathbb{R} \to \mathbb{R}$ via $G_n(\lambda) = F_{i\lambda}^{2n}(1)$ where $\lambda \in \mathbb{R}$. For $\lambda_1 = .5$, $G_1(\lambda_1) = 0$. Also, $G_1(0) = 1/2$. Further, note that $G_m(\lambda)$ is continuous except at blowup points of order less than m. We now see that G_2 maps $(0, \lambda_1)$ to $(-\infty, 1/2)$. Thus, by continuity of G_2 in this interval, there exists a $\lambda_2 \in (0, \lambda_1)$ such that $G_2(\lambda_2) = 0$. If more than one λ value exists, we will chose the smallest to be λ_2. This ensures that G_3 will be continuous on $(0, \lambda_2)$. Iterating this process we obtain the desired sequence.

Now suppose that this sequence does not accumulate on the origin. In other words, there exists some interval $(0, \hat{\lambda})$ such that $G_n(\lambda) > 0$ for all n and $\lambda \in (0, \hat{\lambda})$. Since the graph of $F_{i\lambda}^2$ lies strictly below the diagonal on $(0,1)$ and $F_{i\lambda}^2$ is monotonically increasing there, the interval $(0,1)$ is mapped inside itself. Thus, by the contraction mapping principle there exists a fixed point in $(0,1)$, which is a contradiction.

To prove the second part of the assertion, let λ_n and λ_m be blowup points of order n and m. Assume $n < m$. For fixed m there are a finite number of discontinuities of G_m in the interval (λ_m, λ_n). Furthermore, these discontinuities represent blowup points of order less than m. Therefore, we will restrict ourselves to a subinterval on which G_m is continuous and note that the result holding here is sufficient to establish the result in the general setting. Thus, without loss of generality, assume that G_m is continuous on (λ_m, λ_n). Therefore, $G_m(\lambda_n) = \infty$ and $G_m(\lambda_m) = 0$. By continuity of G_m there exists $\lambda_p \in (\lambda_m, \lambda_n)$ such that $G_m(\lambda_p) = 1$.

□

We will now briefly turn our attention to the case where $d > 1$. In this case the critical points are $c = d^{\frac{1}{d+1}}$. As in the $d = 1$ case the critical points do not depend on the parameter value. Also there exist lines, analogous to the imaginary axis for the case $d = 1$ and passing through the origin in parameter space for which F_λ^{d+1} is invariant over $\mathbb{R}$. The parameter planes for several of these functions are plotted in Figure 10. For this class of rational functions, the results of Rees [21] guarantee a set of positive Lebesgue measure in parameter space for which the Julia set is the whole Riemann sphere. However, it is unknown whether this behavior accumulates on the origin and hence whether a corollary to the Theorem is true for $d > 1$.

6. Buried Sierpinski Curves

In this section, we discuss an infinite collection of dynamically distinct Sierpinski curve Julia sets for the family F_λ where the Fatou components are quite different than those described in previous sections. Instead of being preimages of a single superattracting basin at ∞, we give examples where the

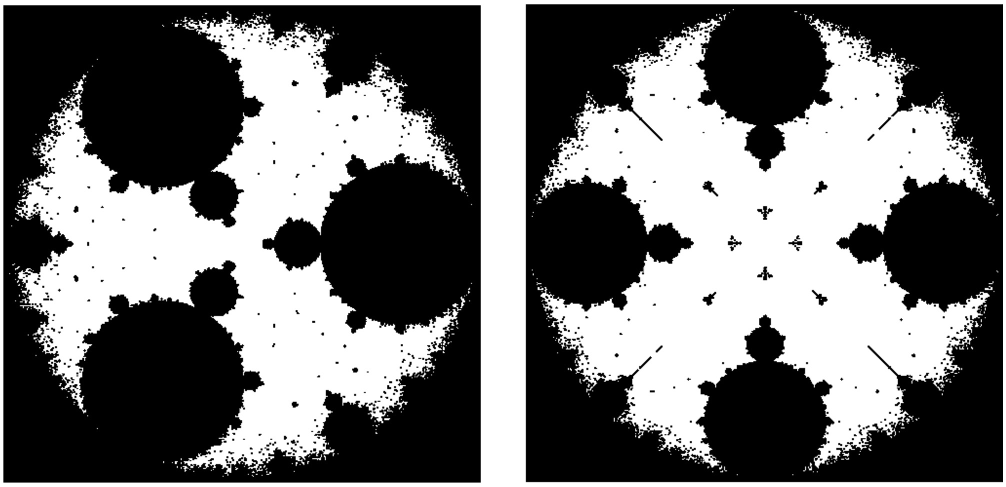


FIGURE 10. The λ plane for the functions $F_\lambda(z) = \lambda(z+1/z^2)$ and $F_\lambda(z) = \lambda(z + 1/z^3)$

complementary domains consist of a collection of different attracting basins together with the basin at ∞ and all of the preimages of these basins. As before, we sketch a proof that the dynamics on these Julia sets are all distinct from one another as well as from those mentioned above, but again, all of these Julia sets are homeomorphic.

For simplicity, we restrict attention in this section to the special family $F_\lambda(z) = z^2 + \lambda/z$ with $\lambda \in \mathbb{R}^-$. In Figure 11, we display the Julia set of F_λ when $\lambda = -0.327$. For this map, there are attracting basins of period 3 and period 6 together with the basin at ∞. We also display the case where $\lambda = -0.5066$ for which there are three different attracting basins of period 4 together with the basin at ∞. The basins of the finite cycles are displayed in black.

There is a positive real fixed point for F_λ which we denote by $p(\lambda)$. Also, $c(\lambda) = (\lambda/2)^{1/3}$ is a critical point and

$$v(\lambda) = \frac{3}{2^{2/3}}\lambda^{2/3}$$

is a critical value. Note that, for $\lambda \in \mathbb{R}^-$, both $c(\lambda)$ and $v(\lambda)$ are real.

Let $\lambda^* = -16/27$. Straightforward calculations show that $p(\lambda^*) = 4/3$ and $p(\lambda^*)$ is repelling. Further, the real critical point $c(\lambda^*) = -2/3$ is pre-fixed, i.e., $F_{\lambda^*}(c(\lambda^*)) = 4/3 = p(\lambda^*)$. For λ-values slightly larger than λ^*, the real critical value lies to the left of $p(\lambda)$ and hence subsequent points on the orbit of the critical value begin to decrease. Graphical iteration shows that there is a sequence of λ-values tending to λ^* for which the critical orbit decreases along the positive axis and then, at the next iteration, lands back at $c(\lambda)$. See Figure 12. Thus, for these λ-values, we have a superattracting cycle. More precisely, we have:

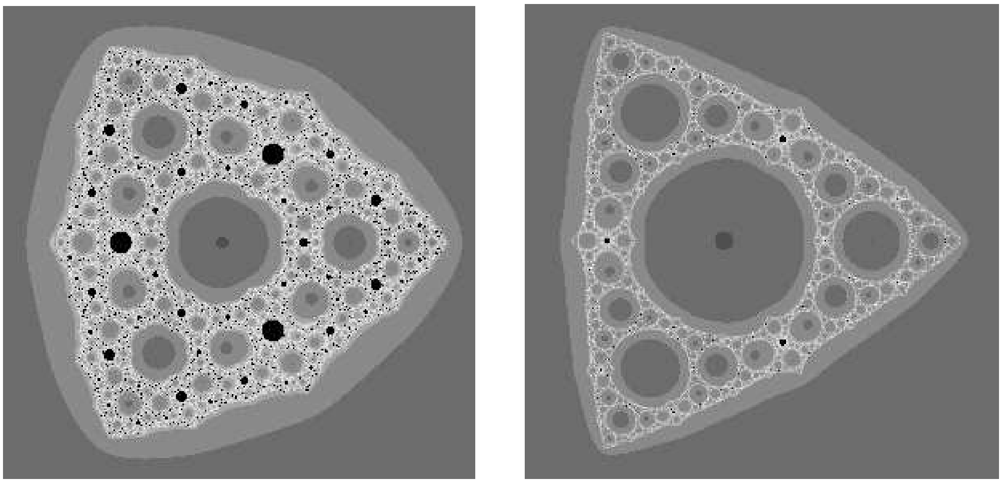


FIGURE 11. The Julia sets for $F_\lambda(z) = z^2 + \lambda/z$ where $\lambda = -0.327$ and $\lambda = -0.5066$.

Theorem. *There is a decreasing sequence $\lambda_n \in \mathbb{R}^-$ for $n \geq 3$, such that $\lambda_n \to \lambda^* = -16/27$, and having the property that F_{λ_n} has a superattracting cycle of period n given by $x_j(\lambda_n) = F_{\lambda_n}(x_{j-1}(\lambda_n))$, where*

(1) *$x_0(\lambda_n) = x_n(\lambda_n) = c(\lambda_n)$, and*
(2) *$x_0 < 0 < x_{n-1} < x_{n-2} < \cdots < x_1 = v(\lambda_n) < p(\lambda_n)$.*

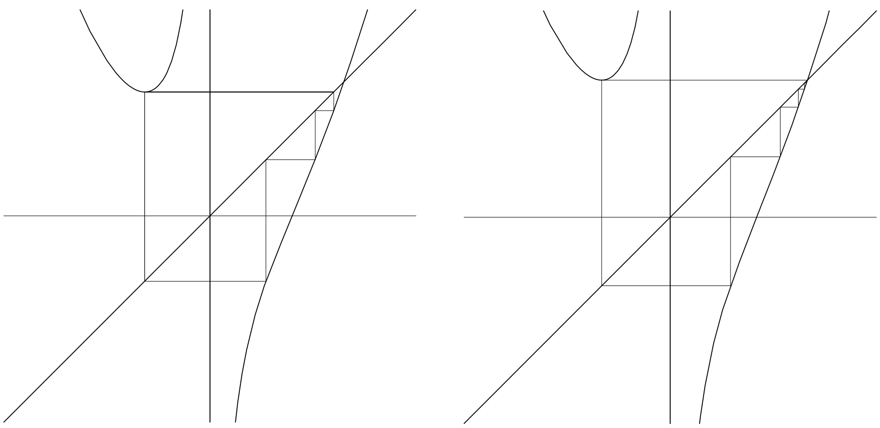


FIGURE 12. The graphs of $F_\lambda(x) = x^2 + \lambda/x$ where $\lambda = \lambda_4$ and $\lambda = \lambda_7$.

For a proof see [6]. Now fix a particular parameter value $\lambda = \lambda_n$ for which F_λ has a superattracting periodic point x_0 lying in $\mathbb{R}^-$ as described in the previous Theorem. We say that a basin of attraction of F_λ is *buried* if the boundary of this basin is disjoint from the boundaries of all other basins of attraction (including B_λ). Note that, if the basin of one point on an attracting cycle is buried, then so too are all forward and backward images of this basin, so the entire basin of the cycle is buried. In [6] the following was shown:

Theorem. *All of the basins of F_λ are buried and $J(F_\lambda)$ is a Sierpinski curve.*

As discussed earlier, any two Sierpinski curves are homeomorphic. Hence $J(F_{\lambda_n})$ is topologically equivalent to $J(F_{\lambda_m})$ for any n and m. However, each of these Julia sets is dynamically distinct from the others since the periods of the superattracting cycles are different.

In Figure 13 we display the parameter plane for the degree three family

$$F_\lambda(z) = z^2 + \frac{\lambda}{z}$$

together with a magnification of a certain region along the negative real axis.

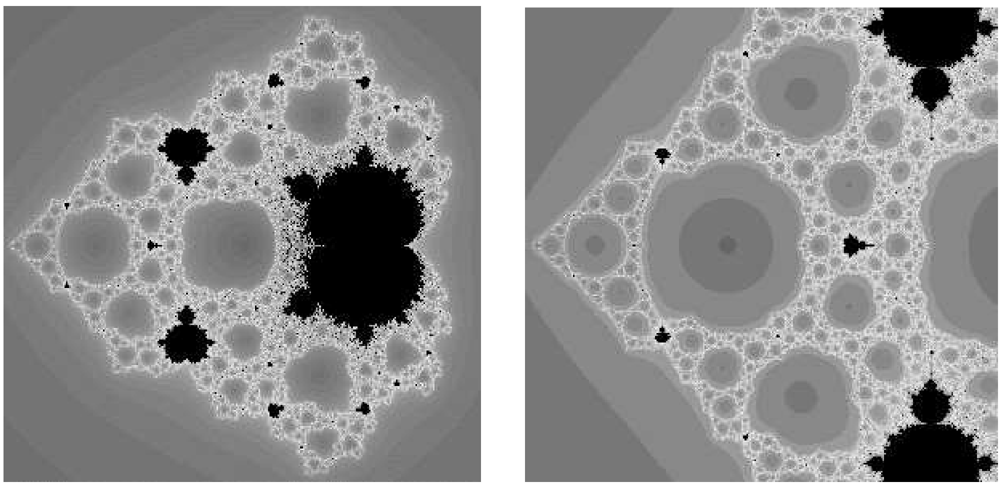


FIGURE 13. The parameter plane for the degree three family of rational maps and a magnification.

The grey holes in this parameter plane correspond to parameter values for which the critical orbit eventually escapes to ∞ through the trap door, so the Julia set is a Sierpinski curve as discussed in Section 3. These are the Sierpinski holes. Note the existence of a small copy of a Mandelbrot set along the negative real axis in this image. In fact, there are infinitely many such Mandelbrot sets converging to the left tip of the parameter space, which is the parameter λ^*. The parameters for which we have the superattracting cycles constructed above form the centers of the main cardioids of certain of these Mandelbrot sets.

We remark that there appear to be two very different types of baby Mandelbrot sets in this picture, some of which touch the outer boundary of the connectedness locus, and some that do not. It is known [3] that those Mandelbrot sets that touch the outer boundary actually touch infinitely many of the Sierpinski holes as well. We conjecture that the Mandelbrot sets corresponding to the λ_n in this section are also buried, this time in the sense that these sets do not touch any of the Sierpinski holes, nor the outer boundary.

7. Sierpinski Gasket-like Julia Sets

One of the outstanding theorems in the study of the families of polynomials $z \mapsto z^d + c$ is the Landing Theorem, due to A. Douady and J. H. Hubbard [11], which states that every external ray in the parameter plane whose external angle is rational lands at a unique point in the boundary of the connectedness locus. Recently, C. Petersen and G. Ryd [20] have shown that this result may be extended to many other one-parameter families of maps with a single free critical orbit, including the family F_λ when $n \geq 2$. In this section we will concentrate on λ-values that correspond to external rays whose external angles are of the special form p/n^j with $p, j \in \mathbb{Z}$. The Landing Theorem implies that such a λ-value is a parameter for which the critical orbits eventually land on a fixed point in the boundary of B_λ. We call the corresponding maps *Misiurewicz-Sierpinski maps*, or MS maps, for short.

In Figures 14 and 15 we display several examples of Julia sets corresponding to Misiurewicz parameters for $z \mapsto z^2 + \lambda/z$ and $z \mapsto z^2 + \lambda/z^2$ respectively. Clearly, these sets are no longer homeomorphic to the Sierpinski curve, as infinitely many complementary boundaries meet other complementary boundaries at one or more points. In particular, the Julia set in the left-hand side of Figure 14 is homeomorphic to the well-known *Sierpinski gasket* (or triangle). Although the second Julia set in Figure 14 looks similar to the Sierpinski gasket, these two Julia sets are not homeomorphic, as we explain below.

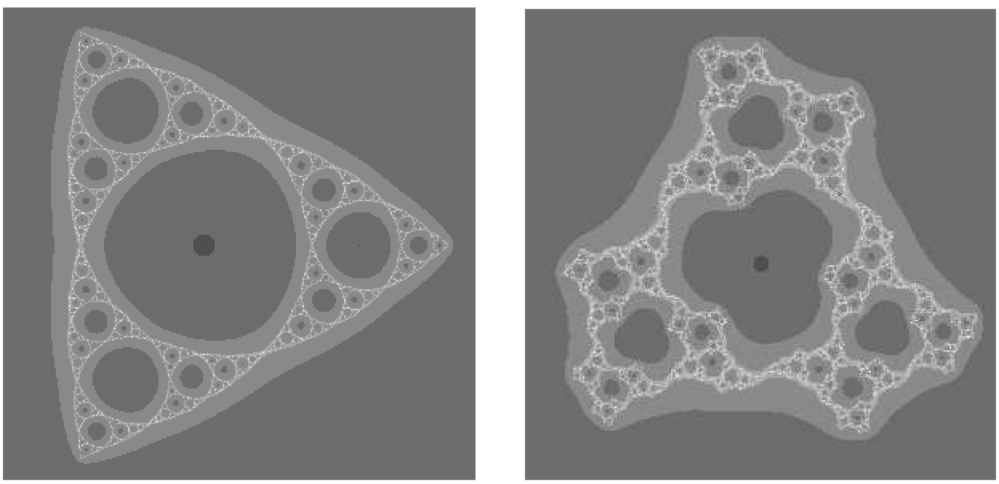


FIGURE 14. Julia sets from the family $z \mapsto z^2 + \lambda/z$ with $\lambda \approx -0.59257$ and $-0.03804 + i0.42622$.

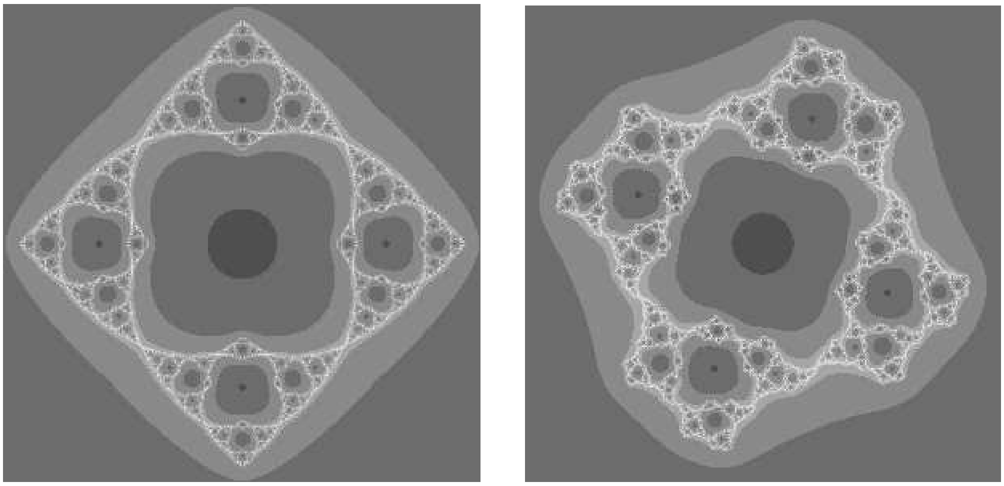


FIGURE 15. Julia sets from the family $z \mapsto z^2 + \lambda/z^2$ with $\lambda \approx -0.36428$ and $\lambda \approx -0.01965 + i0.2754$.

The Julia sets in Figure 15 can be thought of as generalizations of a Sierpinski gasket set with four distinguished vertices. We will see that these Julia sets are again not homeomorphic to each other. A *generalized Sierpinski gasket* set with four distinguished vertices is constructed as follows. Consider the closed unit disk in the plane from which we remove an open rectangular region whose vertices lie in the boundary of the disk. We assume that the removed rectangle is symmetric under rotation of the disk by angle $\pi/2$. We are left with four symmetric closed sets which we denote by I_0, I_1, I_2 and I_3. From each of the I_j we next remove an open "generalized" rectangle whose vertices lie on the boundary of I_j. We stipulate that exactly two of these vertices lie on the boundary of the previously removed rectangle and that the newly removed sets are all symmetrically arranged. This leaves sixteen sets whose only intersection points are vertices of the removed rectangles. We continue in this fashion by removing at each stage open generalized rectangles with exactly two vertices lying in the boundary of the previously removed rectangle. In the limit this produces a set which we call a generalized Sierpinski gasket or a *Sierpinski gasket-like* set.

For simplicity, in this section we consider only the special case where $n = d = 2$, although all of the results go over with minor modifications to the more general family of maps with $n \geq 2$, $d \geq 1$. See [9].

Theorem. *Let $F_\lambda(z) = z^2 + \lambda/z^2$ be an MS map. Then the Julia set $J(F_\lambda)$ is a generalized Sierpinski gasket-like set with four distinguished vertices. Moreover, if we assume that λ and μ are chosen so that F_λ and F_μ are MS maps from the same family, then their Julia sets are homeomorphic if and only if $\lambda = \overline{\mu}$.*

7.1. **Topology of Julia Sets.** Suppose F_λ is an MS map with $n = d = 2$. Since the post-critical orbit is finite, the map is sub-hyperbolic and thus the boundary of each Fatou component is locally connected [16]. Moreover, as shown in [8], there is only one component to the set $\mathbb{C} - \overline{B}_\lambda$ and the boundary of this set is a simple closed curve which is also the boundary of B_λ. Denote the boundary of B_λ by β_λ and the boundary of the trap door by τ_λ. Our assumption implies that the four finite and non-zero critical points $c_\lambda = \lambda^{1/4}$ lie in both β_λ and τ_λ. A straightforward argument given in [7] shows that if the set $\beta_\lambda \cap \tau_\lambda$ is non-empty, then the critical points are the only points in this intersection. We call these points the *corners* of the trap door. The four corners separate τ_λ into four *edges*.

Using the fact that F_λ is conjugate to $z \mapsto z^2$ in B_λ, and that this conjugacy extends to β_λ, there exist four disjoint smooth curves, γ_j for $j = 0, 1, 2, 3$, connecting each of the critical points c_j to ∞ in B_λ. The γ_j are the external rays landing at c_j. Let $H_\lambda(z) = \sqrt{\lambda}/z$. One checks easily that the two involutions H_λ interchange B_λ and T_λ and satisfy $F_\lambda((H_\lambda(z)) = F_\lambda(z)$. Let ν_j denote the image of γ_j under the involution H_λ that fixes c_j. Then the curve $\eta_j = \gamma_j \cup \nu_j$ connects 0 to ∞ and meets $J(F_\lambda)$ only at c_j. Moreover, the η_j are pairwise disjoint (except at 0 and ∞). Hence these four curves divide the Julia set into four symmetric pieces $I_0, \ldots, I_3$ where we assume that $c_j \in I_j$ but c_j does not lie in the other three regions. Let I_0 be the component that contains the repelling fixed point $p(\lambda)$ that lies in β_λ. Note that the I_j are neither open nor closed subsets of $J(F_\lambda)$.

Since there are no critical points in any of the preimages of the trap door, it follows that each of its preimages is mapped in one-to-one fashion onto the trap door by F_λ. Hence each component of $F_\lambda^{-k}(\tau_\lambda)$ also has four corners and edges, and each of these corners is mapped by F_λ^k onto a distinct critical point in τ_λ.

To see that $J(F_\lambda)$ is a Sierpinski gasket-like set, we require the following lemma.

Lemma. *Let τ_λ^k be the union of all of the components of $F_\lambda^{-k}(\tau_\lambda)$ and let A be a particular component in τ_λ^k with $k \geq 1$. Then exactly two of the corner points of A lie in a particular edge of a single component of τ_λ^{k-1}.*

Proof: The case $k = 1$ is seen as follows. We have that F_λ maps each I_j for $j = 0, \ldots, 3$ in one-to-one fashion onto all of $J(F_\lambda)$, with $F_\lambda(I_j \cap \beta_\lambda)$ mapped onto one of the two halves of β_λ lying between two critical values (which, by assumption, are not equal to any of the critical points). Hence $F_\lambda(I_j \cap \beta_\lambda)$ contains exactly two critical points. Similarly, $F_\lambda(I_j \cap \tau_\lambda)$ maps onto the other half of β_λ and so also meets two critical points. The preimages of these latter two critical points in τ_λ are precisely the corners of the component of τ_λ^1 that lies in I_j. Thus we see that each component in τ_λ^1 meets the boundary of one of the I_j's in two points lying in β_λ and two points lying in τ_λ. In particular, two of the corners lie in the edge of τ_λ that meets I_j.

Now consider a component in τ_λ^k with $k > 1$. F_λ^k maps each component in τ_λ^k onto τ_λ and therefore F_λ^{k-1} maps the components in τ_λ^k onto one of the four components of τ_λ^1. Since each of these four components meets a particular edge of τ_λ in exactly two corner points, it follows that each component of τ_λ^k meets an edge of one of the components of τ_λ^{k-1} in exactly two corner points as claimed.

□

We may now show the Julia set of an MS map is a gasket-like set as follows. Let $K_0 = \overline{\mathbb{C}} - B_\lambda$ and $K_1 = K_0 - T_\lambda$. Then K_1 consists of the union of the four sectors I_j which are mapped in a one-to-one fashion onto K_0. Define recursively the sets $K_{n+1} = K_n - F_\lambda^{-n}(T_\lambda)$. Each K_n is a nested collection of closed and connected subsets of the Riemann sphere with exactly 4^n generalized rectangles removed at each nth step. Moreover, the above lemma shows that for each n, the removed rectangles satisfy the two corner restriction given in the definition of gasket-like sets. Is not hard to see that $\cap_{n=0}^{\infty} K_n$ coincides with $J(F_\lambda)$ and hence is a Sierpinski gasket-like set.

7.2. **Homeomorphisms Between Julia Sets.** Before proceeding with the discussion of homeomorphisms between Julia sets of MS maps, we provide a topological characterization of the critical points and the corners of every τ_λ^k. Proofs of the following propositions may be found in [9].

Proposition. (The Disconnection Property.) *The four corners of the trap door are the only set of four points in the Julia set whose removal disconnects $J(F_\lambda)$ into exactly four components. Any other set of four points removed from $J(F_\lambda)$ will yield at most three components.*

Clearly the corners of each component A in τ_λ^k inherit the disconnection property when restricted to the largest connected component of τ_λ^{k-1} that contains A. A homeomorphism between Julia sets of MS maps must then preserve this topological invariant as described in the following result.

Proposition. *Suppose F_λ and F_μ are MS maps. If there exists a homeomorphism $h : J(F_\lambda) \to J(F_\mu)$, then*

(1) *The map h takes the corners of $F_\lambda^{-k}(\tau_\lambda)$ to the corners of $F_\mu^{-k}(\tau_\mu)$ when $k \geq 0$.*
(2) *For $k \geq 1$, each component of $F_\lambda^{-k}(\tau_\lambda)$ is mapped to a unique component of $F_\mu^{-k}(\tau_\mu)$.*

Suppose λ and μ are given parameters that correspond to MS maps of the degree four family. Unless these parameters are complex conjugate, the main theorem in this section states their Julia sets are not homeomorphic. To prove this assertion, we have developed a recursive algorithm based solely on the configuration of the corners of a finite number of preimages of τ_λ along β_λ. The configuration is completely determined by the itinerary associated to the finite critical orbit. If the itineraries for λ and μ disagree at the $(n+1)^{\text{st}}$

entry, then the algorithm shows that the corner configurations of τ_λ^n and τ_μ^n differ along the respective boundaries of the basin at infinity. Hence there is no homeomorphism between these Julia sets. We illustrate this algorithm with the two examples given in Figure 15.

Using the partition given by the sectors I_j we define the itinerary of a point $z \in J(F_\lambda)$ as the infinite sequence $S(z) = (s_0 s_1 s_2 \dots) \in \{0, 1, 2, 3\}^N$ defined in the natural way by its orbit in the regions I_j. Hence the itinerary of the accessible fixed point p_λ is $\overline{0} = (000\dots)$, while the itinerary of $-p_\lambda$ is $2\overline{0} = (2000\dots)$, and so forth.

By assumption the itinerary of any critical point of a MS map ends with an infinite string of 0's. Due to the four-fold symmetry and the existence of a unique free critical orbit, we will only concentrate on the itinerary of the critical point c_λ that lies in the first quadrant.

The two examples displayed in Figure 15, in which $\lambda \approx -0.36428$ and $\mu \approx -0.01965{+}0.2754\,i$, correspond to the landing points of external rays with arguments 1/2 and 1/4 respectively. The extension of the Landing Theorem for the rational families implies that the external rays of the same argument must land in the dynamical plane at the second iterate of the critical point. Thus, the itinerary of $F_\lambda^2(c_\lambda)$ is $(2\overline{0})$ and the itinerary of $F_\mu^2(c_\mu)$ is $(12\overline{0})$. It follows that the itinerary of c_λ is $(112\overline{0})$ while the itinerary of c_μ is $(1112\overline{0})$.

Since these itineraries differ at the third entry, we only need to look at the configuration of the corners of the second preimage of the trap door.

We start with the case $\lambda \approx -0.36428$. The ray 1/8 lands at the critical point $c_1 = c_\lambda$. By symmetry, the ray 7/8 lands at c_0. Thus, the preimages of c_1 and c_0 in I_0 are landing points of the rays 1/16 and 15/16 respectively. Note that these points are two corners of the component of τ_λ^1 that lies in I_0. The remaining two corners of this component lie in the arc of τ_λ contained in I_0 and are mapped onto the critical points c_2 and c_3.

By four-fold symmetry, we can compute the external rays landing on the corners of each component of τ_λ^1 in each remaining sector I_j by adding a proper multiple of $\pi/2$. In particular, two corners of the component of τ_λ^1 in I_1 correspond to landing points of the rational rays 5/16 and 3/16.

Now we compute the configurations of the components of τ_λ^2. For our purposes it suffices to find the configuration of the corners of $B \subset \tau_\lambda^2$ lying along the arc $\gamma \subset \beta_\lambda$ bounded by the rays 1/16 and 1/8. Under F_λ, γ is mapped onto an arc bounded by rays 1/8 and 1/4. Since the ray 3/16 lands at a corner of the component of τ_λ^1 in I_1, this implies that the ray 3/32 lands at a corner of the component B in τ_λ^2 along γ. A similar analysis can be done to compute the locations of the remaining three corners of B. See Figure 16.

For the case $\mu \approx -0.01965 + i0.2754$, let $c_1 = c_\mu$ be the critical point lying in the first quadrant which is the landing point of the ray 1/16. By symmetry, the ray 13/16 lands at c_0. Hence the first preimages of c_1 and c_0 in I_0 are landing points of the rays 1/32 and 29/32 respectively. We may compute the external rays of the remaining corners in τ_μ^1 by addition of a multiple of $\pi/2$ as before. In particular the external rays landing at corner points of τ_μ^1 in I_1 are 9/32 and 5/32.

Let γ denote the arc of β_μ bounded by the landing points of the rays 1/16 and 1/32. Then γ is mapped onto the arc bounded by the rays 1/8 and 1/16. In this case, the image of γ fails to contain a corner point of τ_μ^1 in I_1 as $1/16 < 5/32$. This implies that there is no corners of the component B in τ_μ^2 along γ. See Figure 16.

The previous proposition implies that a homeomorphism between $J(F_\lambda)$ and $J(F_\mu)$ must preserve the configurations shown in Figure 16, which is impossible. Therefore these Julia sets cannot be homeomorphic.

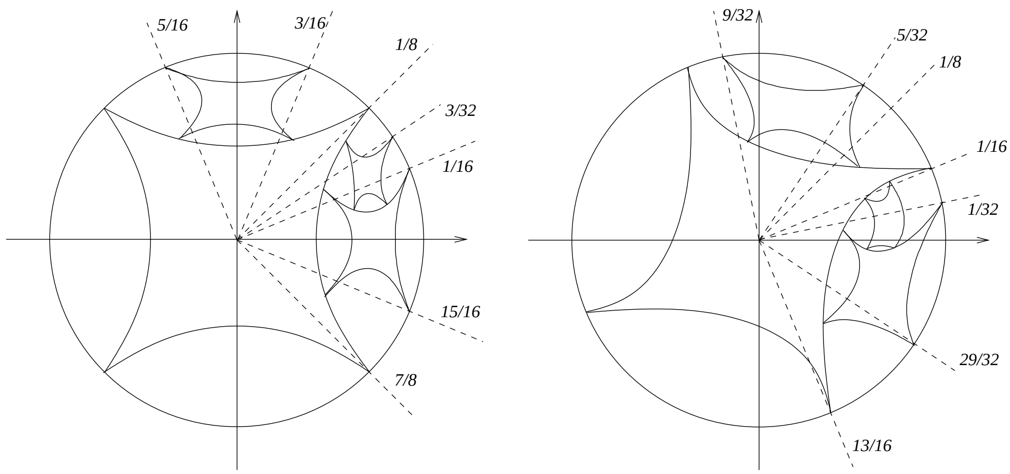


FIGURE 16. A schematic representation of the Julia set $J(F_\lambda)$ and $J(F_\mu)$, respectively, up to second preimage of the trap door. For clarity, only the relevant rational rays and certain preimages of the trap door in sectors I_0 and I_1 have been displayed.

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