Making a Mathematical Model for a Square Pendulum
Fall 2023

Introduction

In today's experiment, you have a collection of square pendulums of different sizes that can be suspended from one corner. The pendulums oscillate each with a certain period (the time for one complete back-and-forth oscillation). Your goal is to discover a mathematical relationship that connects the period, T, to the length, L, of the side of the square.

Experimental Details

The experiment itself is straightforward. The period is measured with a photogate set to "PEND" (pendulum) mode, memory "*on*" (Record these settings in your journal). Position the photogate so that the point of the pendulum interrupts the beam when at rest. Let the pendulum swing, and repeatedly hit the "Reset" button to collect your data (*don't stop and restart the pendulum!*) Be careful, however, to *use only small amplitude oscillations* – the pendulum point should swing just enough to break the photogate beam. Raise or lower the photogate on the timer **or** the pendulum support bar to accommodate different lengths. Use a meter stick with caliper jaws to measure the length of the side. *Be careful not to bend the small support wire on the corners; when you place the pendulum on the bench, do it so that the support wire hangs over the end of the table.*

Now you will use the laboratory and journal writing skills you have acquired, including the following:

- Drawing and labeling a Lᴚ picture of the apparatus.
- Setting up a data table. You'll find it easier in this experiment to keep your measurements in centimeters. Every pendulum is labeled, so keep track of the ones you use.
- Knowing the correct procedure for collecting measurements with the implement provided (*e.g.*, both meter stick caliper measurements for each pendulum must be recorded in your data table).
- The rules of data collection and graph creation to plot T vs. L. You don't know what the graph will look like before you start the experiment, but you should expect to find a smooth function.
- Using KaleidaGraph to try out various types of fits and make a final graph.
- Re-checking data points that deviate significantly from your best-fit function.
- Explaining the purpose of your measurements and calculations.
- The proper presentation of your results. Be sure to use T and L (not x and y) when presenting your model equation.
- Writing a ***thoughtful*** discussion.

A Hint on Finding the Best Fit

After trying a best-fit function, look at the value of the period when the pendulum length is zero. What do you expect the period be at this length? If something weird (un-physical) is predicted by the best fit as the length approaches zero, you can bet you have chosen the wrong function to fit your data.

KaleidaGraph Notes

1. Enter your data and create a scatter plot of the average period as a function of the pendulum side length $(\langle T \rangle \textit{ vs. } L)$; <u>don't add {0, 0} as a data point this week</u>. Change the axis scale so that the origin is shown (Select the graph, then choose **Plot → Axis Options**; set the *Minimum* value to 0 for the x- and y-axes).

2. Select one of the built-in functions from the Curve Fit menu (*Linear*, *Polynomial* (2nd order), *Exponential*, *Logarithmic* or *Power*), and determine which function best fits your data. This week you will be able to determine the best fit by visual inspection. Don't forget to deselect a fit before choosing another.

3. With the model of the best-fitting function chosen, you can then create your own "user-defined" fit, so that the uncertainties in the parameters will be calculated.

    a. Choose **Curve Fit → General → New Fit**. Note that this will keep the original fit on your graph, which will make it easier to ensure that you have created the user-defined fit correctly.

    b. Click the **Define…** button and replace the contents with your equation. For example, a linear fit would be entered as follows:

    ```
    a*x + b; a = 1; b = 1
    ```

    The statement "`a = 1; b = 1`" initializes 'a' and 'b' at unity, giving KaleidaGraph a place to begin its calculations. *Use 1 for your custom fit; don't use the values of 'a' and 'b' that were calculated using the built-in function from step 2!*

    c. Once you have the function defined, *deselect the fit from step 2* above so that only the user-defined fit remains on your graph.

4. Your best fit should extend across the width of the graph. Print your graph, <u>but don't close it yet</u>; you'll be making another plot in a few minutes.

5. Record your model equation in your journal; include the values of the parameters and their uncertainties.

Testing Your Model

Any law of nature is useful only if it can be used to predict the behavior of other experiments. If you have found a good model for the square pendulum, then you should be able to predict the period of *any* square pendulum, provided you know the side *L*.

You will select one pendulum with a different length than your original sample and then you will predict its period. Your prediction should be of the form $T = \underline{\hspace{1cm}} \pm \underline{\hspace{1cm}} sec$. Otherwise, the chances are nearly zero that you will get the exact value.

How do you estimate the uncertainty in the predicted *T*? You could look at how repeatable any given period measurement is. However, this ignores uncertainties in the length and slight variations in pendulum construction. The answer is hidden in our complete data, which lie above and below the best-fit line by various amounts (the *residuals*). Therefore, a good estimate is to find the "typical" residual, which will consider all of the experimental uncertainties. The best estimate of the "typical" residual is to calculate the *Root Mean Square Residual*:

$$\text{RMS Residual} = \sqrt{\frac{\sum_{i=1}^{N} R_i^2}{N}} \qquad\qquad (\text{Eqn.1})$$

where *N* is the number of data points, and *R* is the residual for each point (<u>*Note: this is different than the value of R calculated by KaleidaGraph. Ignore KaleidaGraph's value of R!*</u>) You will find the square root of the average of the squares of the residuals as follows:

The quantity in the numerator of Eqn. 1 is the sum of the squares of the residuals, SSR (also known as *Chi Square*, χ^2):

$$SSR = \sum_{i=1}^{N} R_i^2 \qquad\qquad (\text{Eqn.2})$$

Therefore, we can rewrite Eqn. 1 as:

$$\text{RMS Residual} = \sqrt{\frac{SSR}{N}} \qquad \text{(Eqn.3)}$$

(Recall that the SSR will appear on your graph (the *Chisq* value) when you create a "user-defined" fit.) The RMS Residual value indicates the typical uncertainty in a *single* measurement.

1. Use Eqn. 3 to calculate the RMS residual from your KaleidaGraph results (note that "1e–5" is computer-speak for "1×10^{-5}"). Your estimate of the uncertainty in *T* for your prediction will be *twice* the RMS Residual value you calculate.

2. Once you have your prediction, *explicitly write out the range of possible periods for the unknown pendulum*. Perform measurements (which should be recorded in a **new** data table – don't add the measurements to the previous table) of the unknown pendulum's period and calculate the average period.

Is There a Systematic Error in Your Data?

- You learned how to extract the residuals from a KaleidaGraph plot during the "*Fitting Data to a Mathematical Function*" experiment. **Create a new graph** of your residuals vs. *L* for the seven original pendula and *think* about the meaning of this graph. You can use the residuals to look for a *systematic* error that might occur during the experiment. A systematic error can affect the results in a way that is not random error. For example, an instrument may be calibrated incorrectly, or something about the apparatus used always causes values to be measured too high or too low. A pattern or trend in the residuals plot might indicate such an error.

- Set the vertical scale to range between ±0.01 *units* (you should have already figured out the proper *units*) to "zoom out" from the graph a little bit. Check that you haven't excluded any residuals that are outliers when changing the vertical scale.

- Print a copy of this graph for each lab partner's journal.

Discussion

- Summarize your results by presenting the following:

    o Your model equation. You should also discuss the reasons you rejected your next best choice for the model equation.

    o The parameters, their uncertainties and the SSR value (with appropriate units!)

    o The predicted period and its uncertainty.

    o The success of your model.

- Discuss the possible sources of error.

- Include a comment on the appearance of your residuals plot. Did you notice any pattern or trend in the residuals?