CHAPTER 8

Answers and Hints

These pages contain answers to every exercise appearing in the book. Before getting overly excited, though, keep in mind that a thorough understanding of an exercise encompasses much more than just the correct answer. It will include a mastery of notation and vocabulary, a working knowledge of the material on which the exercise is based, discussions with classmates that reveal alternate methods of solution, and so on. That being said, having an answer key at one's disposal does have its advantages. At the very least an answer serves as a blaze on a trail; it provides an indication that one is (or isn't, as the case may be) headed in the right direction.

Hints to all of the writing problems are also included below. They should be used sparingly, of course, and certainly never consulted before a sincere effort has been made to first solve a problem independently. Each hint will hopefully guide the solver in a fruitful direction, perhaps even past an obstacle or two. But good math problems will typically yield to several different approaches, only one of which is indicated by a hint. So don't become worried if you have found a solution that seems to strike out in a different direction than the one suggested by the hint. Just be sure that your solution addresses any issues that may be raised by the hint. Good luck with the problems, and enjoy the math!

Chapter 1: Logical Foundations

1.1 Statements and Open Sentences

Exercises

1. a) False b) True c) True d) True e) False
2. The open sentence is true for $n = 2$ and false for $n = 11$
3. The set of even numbers
4. $x = 40$ is the only solution
5. n is an integer, k is a positive integer, and x is a real number
6. The equation is satisfied if $x + 3 = 0$ or $x^2 - 9 = 0$
7. a) True b) False c) True d) False e) False f) True g) True
8. $n = 9$ and $n = 15$ also serve as counterexamples

WRITING
9. Consider the latest arrival time and the earliest departure time. Why must the latter occur later than the former?
10. Group the twenty students into ten pairs of adjacent students. (But don't argue along the lines that if the first student can recite π then the second one can't, so the third one must know π, etc. This approach doesn't consider all possible arrangements.)
11. If all the students received the same grade on the first quiz then we are done. Otherwise, two students got different grades on the first quiz; call them A and B. Hence we know that A and B got the same grade on the second quiz. Now show that any other student C must have gotten the same grade as A and B on the second quiz.

1.2 Logical Equivalence

EXERCISES
13. a) $\neg P \wedge \neg Q \wedge \neg R$ b) $P \wedge \neg R$ c) $(P \vee Q) \wedge R$ d) $\neg(Q \wedge R)$ e) $(P \wedge \neg Q) \vee (\neg P \wedge Q)$
14. The truth tables will not match; for instance, when P and Q are both false then $\neg(P \wedge Q)$ is true while $\neg P \wedge Q$ is false.
15. Both statements indicate that neither action will be taken. Letting P be "I will run for president," and Q be "I will stage a military coup," the two statements may be written as $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$.
16. The truth tables match; they will contain three F's and a T in that order.
17. a) Triangle ABC doesn't have a perimeter of 12 or doesn't have an area of 6.
b) Let k be an integer such that k is odd and $k > 10$.
c) I am younger than Al or older than Betty.
d) It is the case that $2x < y$ and $2y < x$.
e) Jack will not answer this question nor the next one.
f) There is not a new car behind any of the doors.
18. b) is correct; the truth tables for $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$ are identical.
19. The statement P EOR Q is true when P is true and Q is false, and also when P is false and Q is true. The statement is false in the other two cases.
20. One possibility is $(P \wedge \neg Q) \vee (\neg P \wedge Q)$.
21. The truth tables for $(P \wedge Q) \vee R$ and $P \wedge (Q \vee R)$ do not match. For instance, when P is false and R is true (and Q is either true or false) the former statement is true while the latter statement is false.
22. The statement $P \vee Q \vee R$ is true whenever at least one of P, Q or R is true. But the statement $P \wedge Q \wedge R$ is true only if all three of P, Q and R are true.
23. The truth table should contain all T's.
24. The truth table should contain all F's.
25. Create a truth table with two rows, where P is always false but Q can be either true or false. The outcome should be false in both cases.

WRITING
26. The oldest son only needs to buy a vote from two of his younger brothers (by giving them a better deal than dad would) in order to ensure that his plan is voted in.
27. This situation is considerably more intricate. The oldest son's best strategy is to allocate the dollar bills in a manner that will cause the second brother to increase the oldest son's share in order to buy his vote, say by proposing a 6–0–7–7–0 distribution.
28. There are relatively few products that Mr. Strump could have had in mind. For example, if he told Abby that the product was 14, Abby would know that his digits were 2 and 7, since this is the only pair of digits which multiply to give 14. Begin by finding all five products that can be obtained in more than one way.

1.3 The Implication

Exercises

29. a) If it rains, then it pours. b) If I try escargot then Al must eat some first. c) If $7a$ is even then a is even. d) If a fire was started then a match was lit. e) If $\angle A \cong \angle B$ then $\triangle ABC$ is isosceles. f) If the discriminant is positive then the quadratic has two distinct solutions.
30. We would need to find a prime p for which $2^p - 1$ is not prime.
31. The implication $P \Rightarrow Q$ is logically equivalent to $\neg P \vee Q$.
32. Both truth tables should read F, T, F, F in that order.
33. a) Cinderella married the prince but I didn't eat my hat. b) The number a^2 is divisible by 12 but a is odd and a is not a multiple of 3. c) Quadrilateral $ABCD$ has four congruent sides but is not a square. d) One can make bread and omit at least one of flour, water or yeast.
34. a) $n = 9$ b) $n = 3$ c) $n = 73$ d) $n = 4$ (Other answers are possible.)
35. The only such value is $x = 2$.
36. The implication is true exactly when $y > -2$. For all $y \le -2$ the premise $y < 2$ is true but the conclusion $y^2 < 4$ is false.
37. The statement may be interpreted to say that if both P and $P \Rightarrow Q$ are true, then it follows that Q is true. The truth table reveals that the statement is a tautology, which validates it as a rule of inference.
38. The statement can be translated to mean that if P implies Q and Q implies R, then it follows that P implies R. Now finish as before.

Writing

39. Pick any student, whom we shall call A. Now explain why among the remaining students A either knows at least three of them or is unacquainted with at least three of them. Suppose A knows B, C and D. What happens if any of these three know each other? What happens if none of them are acquainted?
40. Notice that n handshakes involve $2n$ hands, which is an even number.
41. How many handshakes could a particular guest make? What is forced to occur if all ten guests shake hands a different number of times, and why is this a problem?

1.4 The Biconditional

Exercises

42. a) True b) False c) False d) False e) True
43. It would be a value of x for which one of P or Q is true while the other is false.
44. The truth tables should agree.
45. They are not logically equivalent. For example, when P is true but Q and R are false we find that $(P \vee Q) \Longleftrightarrow R$ is false while $(P \Longleftrightarrow R) \vee (Q \Longleftrightarrow R)$ is true.
46. a) For integers n, if $n^2 + 5$ is a multiple of 3, then n is not a multiple of 3.
b) If two rectangles have the same area then they are congruent.
c) For positive real numbers, if $1/x \le 1/y$ then $x \ge y$.
d) If R is in quadrant I and L is in quadrant II, then line RL has positive y-intercept.
e) For positive integers a and b, if the number $a + b$ involves the digit 0, then both a and b have the digit 0.
47. a) True, true, equivalent b) True, false, not equivalent c) True, true, equivalent d) False, true, not equivalent e) False, false, not equivalent
48. The only way for $P \Rightarrow Q$ to be false is for P to be true but for Q to be false. However, in this case $Q \Rightarrow P$ would be true.

49. The truth tables for $Q \Rightarrow P$ and $\neg P \Rightarrow \neg Q$ are identical.
50. a) Let k be a positive integer. If k is odd then $3^k + 1$ is a multiple of 4.
b) If $ABCD$ does not have four congruent sides then it is not a square.
c) For a real number a, if $2^x = a$ has a solution then a is positive.

WRITING
51. Clearly we have a winning position when $n \leq 4$, since the player about to move can win by taking all the pennies. However, $n = 5$ represents a losing position. (Why?) Use this fact to explain how to guarantee a win when $n = 6$, 7, 8 or 9.
52. Carefully examine each value of n, beginning with $n = 1$ and working your way upwards. You should find that of the first seven values, only $n = 2$ and $n = 7$ are losing positions. For instance, $n = 6$ is a winning position, since it is possible to take four pennies, thereby handing a losing position to the other player.

1.5 Quantifiers

EXERCISES
54. a) There exists an integer n such that n contains every odd digit.
b) There does not exist a real number x for which $x^2 + 4x + 5 = 0$.
c) There exists a unique point C on the lines $y = x$ and $y = 3x - 5$.
d) There exists a unique real number t such that $|t - 4| \leq 3$ and $|t + 5| \leq 6$.
e) For all integers k the number $6k + 5$ is odd.
f) There exists a point U such that the distance from U to the origin is positive.
55. For all real numbers x we have $\cos x \neq 3x$.
56. There exists a positive integer n such that n is prime but $2^n - 1$ is not prime. This is the case for $n = 11$, for instance, since $2^{11} - 1 = 2047 = 23 \cdot 89$.
57. There is a linear function $f(x)$ for which $f(1) + f(2) \neq f(3)$. When $f(x) = 2x + 5$ we have $f(1) = 7$, $f(2) = 9$ and $f(3) = 11$, and sure enough $7 + 9 \neq 11$.
58. The square is not unique—there are three such squares. Can you find them all?
59. Quadratic equations usually have two solutions or no solutions. But when $a = -16$ the equation has exactly one solution. Try applying the quadratic formula to see why.
60. There exists a rectangle in the plane such that no circle is tangent to all four sides. (Literally, “such that every circle is not tangent to all four sides.”)
61. The claim is true, as it asserts that given any real number, there is a larger one.
62. This claim is false, as there is no real number larger than every other real number.
63. For every positive integer N there is some integer $n > N$ with $\cos n \geq 0.99$.
64. The assertion is false; we could choose $x = 7$ and $y = 7$, for instance.
65. One can take $m = 3$ and $n = 5$, for example. Other answers are possible.
66. The statement does not necessarily follow. It might be the case that sometimes P is true and Q is false, while at other times P is false and Q is true. This occurs for the statements “m is odd” and “m is even,” for example.

WRITING
67. Find values of b that work for $a = 1$, 2, 3 and 4. Do you notice a pattern?
68. Choose the length of the rectangle to guarantee that the perimeter is at least $4r$.
69. Such a path will need to meander into every part of the plane and always pass relatively close to itself. What sort of curve will accomplish this?
70. Choose a polynomial that factors to guarantee that $f(n)$ also factors.
71. The desired triangle will need to be small. What quantity can we use to measure the size of a triangle? Now examine the triangle for which this quantity is the smallest.

Chapter 2: Set Theory

2.1 Presenting Sets

Exercises

1. The empty set, since the Jamaican flag is yellow, green and black.
2. The numbers appearing in this set are powers of positive integers; i.e. the perfect squares, cubes, fourth powers, and so on.
3. This set consists of those months having 31 days.
4. One answer could be $\{-2x+6, 4x^2-12x-100, 6x^3+8x^2+10\}$.
5. We could take $C = \{\{a,b,c,d\}, \{a,c,e,g\}\}$, for instance.
6. We have $D = \{2, 6, 12, 20, 30, 42, \ldots\}$ or $D = \{n(n+1) \mid n \in \mathbb{N}\}$. The second option is probably better as it clarifies the method by which the elements of D are generated.
7. a) False b) True c) False d) False e) Depends f) True
8. There are six such sets, for instance {NH, ME, MA, RI, CT}.
9. a) $A = \{7k \mid k \in \mathbb{Z}\}$ b) $B = \{2^n+1 \mid n \in \mathbb{N}\}$ c) $C = \{x \mid x \in \mathbb{R},\ \sqrt{2} < x < \pi\}$ d) $D = \{1/(m+1) \mid m \in \mathbb{N}\}$
10. The first set is $\{m/n \mid m, n \in \mathbb{N} \text{ and } m < n\}$. The second set may be described as $\{m/2^k \mid m, k \in \mathbb{Z} \text{ and } k \geq 0\}$.
11. These objects are equations of non-vertical lines passing through the point $(1, 0)$. (Note that the objects are *equations*, not actually lines.)
12. We have $10 \notin B$ but $13 \in B$.

Writing

13. Given any three integers, why must two of them have the same parity? (I.e. be both even or both odd.)
14. What would go wrong if every letter of the alphabet were either in all three sets or else in none of them?
15. Suppose that 1 and 2 are in the same set, say $1, 2 \in A$. Then we must have $3 \in B$. Now explain why there must be some other number $n \geq 6$ also in A, then argue that $n-2$, $n-1$, $n+1$, and $n+2$ must all be in B. Why is this a problem? Finally, handle the case where 1 and 2 are in different sets.
16. First subdivide the points in C into four smaller 3×3 squares, then explain why it is not possible for all five chosen points to occupy different squares.

2.2 Combining Sets

Exercises

17. The set is $A \cup \overline{B}$.
18. One possibility is $A = \{a,b,c,d,e\}$ and $B = \{d,e,f,g\}$. Regardless of your choice, you should find that $|\overline{A} \cap \overline{B}| = 1$.
19. The first statement requires that $u \in A$ and $u \in B$. So must have $u \in A \cup B$.
20. a) True b) True c) False d) True e) False f) False g) False
21. $A \cup C = \{x \mid 1 < x < 6\}$, $(A \cup B) \cap C = \{x \mid 2 < x < 3 \text{ or } 5 \leq x < 6\}$ and finally $B \cap \overline{C} = \{x \mid 6 \leq x \leq 7\}$
22. We have $x \in A$ and $x \in B$ (region II), $x \in A$ and $x \notin B$ (region I), $x \notin A$ and $x \in B$ (region III), $x \notin A$ and $x \notin B$ (region IV)
23. The Venn diagram for either set consists solely of the region outside of all three circles. (Diagram omitted.)
24. The Venn diagram for either set consists of that portion of circle A which overlaps with the other two circles. (Diagram omitted.)

Writing

25. The only way two sets can fail to be equal is if some object belongs to one of the sets but not the other. Explain why this cannot occur.

26. According to our strategy, we should first show that if $x \in \overline{A \cup B}$ then $x \in \overline{A} \cap \overline{B}$. Begin by noting that $x \in \overline{A \cup B}$ means that $x \notin A \cup B$, so x can be in neither A nor B, since it is not in their union.

27. Suppose that $x \in A \cup (B \cap C)$. This means that either $x \in A$ or $x \in B \cap C$. Now argue that in either case x is an element of both $A \cup B$ and $A \cup C$.

28. Adapt the proof given in the text so that it applies to four sets instead of just two.

29. We need to count how many letters are in neither of the sets A nor B. Why can't we just subtract off the number of elements of A and B from the total number of letters in the alphabet to get the correct answer?

30. Let x_1 through x_8 represent the number of elements within each of the eight regions in a Venn diagram for three sets. Now rewrite each side of the inequality in terms of x_1 through x_8.

2.3 Subsets and Power Sets

Exercises

32. We should write $A - \{l\}$.

33. We would have $\overline{B} \subseteq \overline{A}$.

34. The set $B - A$ would be nonempty.

35. Both are valid. Since the elements of $\mathcal{P}(A)$ are subsets of A, in $\emptyset \subseteq \mathcal{P}(A)$ the empty set means "the set containing no subsets of A." On the other hand, in $\emptyset \in \mathcal{P}(A)$ the empty set here plays the role of a subset of A, and hence an element in $\mathcal{P}(A)$.

36. a) False b) True c) True d) False e) False

37. We might have $B = \{d, e, l, i, g, h, t\}$ and $A = \{v, a, l, i, d\}$, for instance.

38. We must have $|A - B| = 44$.

39. The empty set.

40. There are 15 nonempty subsets of B that are disjoint from the set $\{s, t, r, e, a, m\}$.

41. There are 10 subsets of $\{s, a, t, i, n\}$ having exactly two elements.

42. Exactly 64 subsets of A contain the element m.

43. There are 48 subsets in $\mathcal{P}(B) - \mathcal{P}(A)$.

Writing

44. Suppose that $x \in A \cap B$, so that $x \in A$ and $x \in B$. Figure out why $x \notin B - A$.

45. If we have $x \in \overline{B \cup C}$ then $x \notin B \cup C$, which only occurs when $x \notin B$ and $x \notin C$. Now use these facts to argue that $x \notin (A \cap B) \cup C$.

46. Begin your proof "We must show that if $x \in (A \cup B) \cap C$ then $x \in A \cup (B \cap C)$." But if $x \in (A \cup B) \cap C$ we know that $x \in A \cup B$ and $x \in C$. Now consider two cases; either $x \in A$ and $x \in C$ or $x \in B$ and $x \in C$. Show that $x \in A \cup (B \cap C)$ regardless.

47. Begin your proof by writing "We must show that if $x \in (A - B) - C$ then $x \in A - (B - C)$." Figure out why $x \in (A - B) - C$ leads to $x \in A$, $x \notin B$ and $x \notin C$, then employ these facts to show that $x \in A - (B - C)$.

48. We need to prove that $A \cup \overline{C} \subseteq A \cup \overline{B}$, i.e. we need to show that if $x \in A \cup \overline{C}$ then it follows that $x \in A \cup \overline{B}$. So suppose that $x \in A \cup \overline{C}$. This means that $x \in A$ or $x \in \overline{C}$. Now consider each case separately and show that $x \in A \cup \overline{B}$ regardless.

49. We need to prove that $\overline{C} \subseteq \overline{A} \cup B$, so begin by supposing that $x \in \overline{C}$. This means that $x \notin C$, so deduce that $x \notin A - B$ either. The trickiest step in this argument is to figure out how to deal with $x \notin A - B$. This could occur in two ways—either $x \notin A$ at all, or else $x \in A$ but $x \in B$ also.

50. We need to show that if X is an element of $\mathcal{P}(A)$ (which means that $X \subseteq A$) or if X is an element of $\mathcal{P}(B)$, then X is in $\mathcal{P}(A \cup B)$. Consider each case separately.
51. Note that we have a set equality to prove here, which involves two separate arguments. Begin by explaining why it is the case that if $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$ (i.e. that $X \subseteq A$ and $X \subseteq B$) then it follows that $X \in \mathcal{P}(A \cap B)$.
52. One could prove that each set is a subset of the other, but there is an alternate approach available. Show that the only subsets removed from $\mathcal{P}(B)$ when subtracting subsets in $\mathcal{P}(A)$ are those subsets all of whose elements are contained in $A \cap B$.
53. Consider the fate of each piece of candy separately, just as we did in the process of building a subset earlier in the section.

2.4 Cartesian Products

Exercises
54. It must be the case that $A = \emptyset$ or $B = \emptyset$.
55. We define $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.
56. The set of cards is the Cartesian product of the suits (♠, ♡, ♢, ♣) with the possible card values (A, 2, 3, . . . , Q, K).
57. a) 81 b) 15 c) 24 d) 66 e) 12 f) 9
58. The two sets cannot be equal, since they have a different number of elements.
59. An ellipse, or more precisely, the graph of an equation of an ellipse.
60. a) The region is a rectangle in the first quadrant having width 2 and height 3. b) This region is also a rectangle, but with width 3 and height 2. c) The intersection will be the square with vertices $(2, 2)$, $(2, 3)$, $(3, 3)$ and $(3, 2)$.
61. The intersection will be the Cartesian product $\{c, a, p, e, r\} \times \{2, 3, 5\}$, a set consisting of 15 ordered pairs.
62. The set $A \times B \times C$ is all ordered triples (a, b, c) with $a \in A$, $b \in B$ and $c \in C$.
63. We have $|A \times B \times C| = |A| \cdot |B| \cdot |C| = 5 \cdot 6 \cdot 3 = 90$.

Writing
64. Suppose that $(x, y) \in \overline{A} \times \overline{B}$. This means that $x \in \overline{A}$ and $y \in \overline{B}$. In particular, we may conclude that $x \notin A$. Consequently, is $(x, y) \in A \times B$?
65. We must show that each set is a subset of the other. To begin, suppose that $(x, y) \in A \times (B \cup C)$. This means that $x \in A$ and $y \in B \cup C$; that is, $y \in B$ or $y \in C$. Now consider the case $y \in B$ and show that $(x, y) \in (A \times B) \cup (A \times C)$, etc.
66. One conjectures that the two sets are equal. To prove this, we show that each set is a subset of the other. For instance, if $(x, y) \in (A \times B) - (A \times C)$, then $(x, y) \in A \times B$ but $(x, y) \notin A \times C$. The former statement means that $x \in A$ and $y \in B$, while the latter means that we have $x \notin A$ or $y \notin C$. (Why?) And so on.
67. As usual, we show that each set is a subset of the other. For the first half of the proof, argue that if $(x, y) \in (A \times C) \cap (B \times D)$ then $(x, y) \in (A \times C)$ and $(x, y) \in (B \times D)$, which means that $x \in A$, $x \in B$, $y \in C$, and $y \in D$. We may now deduce that (x, y) is an element of the right-hand set.
68. Consider what it would mean for (x, y) to be an ordered pair in $(A \times B) \cap (B \times A)$. For starters, we must have $(x, y) \in A \times B$, meaning that $x \in A$ and $y \in B$. What else must be true about (x, y)?
69. Suppose we group all the ordered pairs in $A \times A$ according to the sum of their coordinates. How many different groups will result? How does this help?
70. The first move introduces two characters into the list; one letter and one number. How many new characters are included in the list of plays on each subsequent move? How does this limit the length of the game?

2.5 Index Sets

EXERCISES

71. The sets are circles, so we could call such a set C_r. We are told that $r \geq 2$, so the index set is $J = \{r \in \mathbb{R} \mid r \geq 2\}$.
72. We could write $\bigcup_{r \in J'} C_r$, where $J' = [3, 5]$. (Other answers are possible.) This set of points looks like a solid ring, with inner radius 3 and outer radius 5.
73. $\bigcup_{k \in I} A_k = \{x \mid x \in A_k \text{ for some } k \in I\}$
74. a) sixtytwo b) vivacious c) mississippi d) boldface
75. All words that contain at least one of the letters a, b, c, d, e, f.
76. a) $\bigcup_{n=8}^{12} A_n$ b) $\bigcap_{n=1}^{\infty} B_{3n}$ c) $\bigcup_{n=2}^{\infty} C_n$ d) $\bigcap_{n=5}^{8} D_n$
77. a) $\{1, 2, 3, 4, 5, 7, 8, 14, 15, 16\}$ b) $\{4, 20, 100\}$ c) $\{1, 2, 3, \ldots, 99, 100\}$ d) $\{1\}$
78. We omit the sketch of the intervals. One finds that $\bigcap_{k=4}^{6} C_k = \{x \in \mathbb{R} \mid 6 \leq x \leq 8\}$ and similarly that $\bigcup_{k=4}^{6} C_k = \{x \in \mathbb{R} \mid 4 \leq x \leq 12\}$.
79. We have $\bigcap_{r \in J} C_r = \emptyset$ and $\bigcup_{r \in J} C_r = \{x \mid x \geq 1\}$.
80. In this case $\bigcap_{r \in J'} C_r = [4.5, 6]$ and $\bigcup_{r \in J} C_r = [3, 9]$.
81. We find that $M_3 \cap M_5 = M_{15}$, $M_4 \cap M_6 = M_{12}$, and $M_{10} \cap M_{15} \cap M_{20} = M_{60}$.
82. We have $M_a \cap M_b = M_c$, where c is the least common multiple of a and b. We also find that $\bigcap_{n \in \mathbb{N}} M_n = \{0\}$.
83. The intersection is the set $\{x \mid 4 \leq x < 10\}$.
84. One possible method is to write $\bigcap_{r \in J} B_r$, where $J = [3, 6]$. (The 3 could be replaced by any other number less than 6.)

WRITING

85. The same reasoning that was used to show that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ applies here. Thus suppose that x is an element of the left-hand side. Then x is not in the intersection of this family of sets, which can only happen if x fails to be in at least one of the sets, say $x \notin A_k$ for some $k \in I$. Continue in this manner.
86. Let $x > 8$ be a real number. Explain why x is not in the intersection by showing that we can always choose r small enough so that $x \notin (5 + r, 8 + r)$. Use the same argument for any $x < 6$ by choosing r close enough to 1. Finally, demonstrate that any x in the range $6 \leq x \leq 8$ is in all of the sets B_r.
87. You should be able to guess the answer by figuring out the ten smallest elements of $M_3 \cup M_5 \cup M_7 \cup M_9 \cup M_{11} \cup M_{13}$ and noticing a pattern among the integers that are missing from this set. To explain your answer, argue that if $n \in \mathbb{Z}$ has any odd divisor, then it is excluded when we subtract the set $\bigcup_{n=1}^{\infty} M_{2n+1}$ from the set of all integers. What sorts of numbers remain?

2.6 Reference

Show that $A - B = \emptyset$ means that $A \subseteq B$. (This is a whole separate little proof in and of itself.) How does this help to finish the main proof?

Chapter 3: Proof Techniques

3.2 Mathematical Writing

EXERCISES

1. a) The sum of two 3's and four 5's is 26.
b) Let x be a real number satisfying $x \geq 0$ or $x \leq -1$.
c) We use a truth table to show that $\neg(P \wedge Q \wedge R) \equiv \neg P \vee \neg Q \vee \neg R$.

d) There are three kinds of people: those who can count and those who can't.
e) The set relation $A \supseteq B$ can also be written as $B \subseteq A$.
f) To solve the equation $x^2 = 36$ one may take the square root of both sides.
g) If $f(x) = x^2$ and $g(x) = 3x + 4$, then $f(x) = g(x)$ when $x = 4$.
h) The set $A \cup B$ contains more elements than B as long as $B \not\subseteq A$.
i) It comes as a surprise to learn that

$$\sqrt{\frac{6}{1^2} + \frac{6}{2^2} + \frac{6}{3^2} + \frac{6}{4^2} + \cdots} = \pi.$$

3.3 Sudoku Interlude

Exercises

2.

1	4	2	3	5
4	3	5	2	1
5	2	1	4	3
2	5	3	1	4
3	1	4	5	2

3. There are a variety of possible answers. Check that every row, column and pentomino contains every digit from 1 to 5, and verify that squares `c`, `l`, `i`, `p`, `y` do not all contain the same digit.

Writing

4. The solution might continue in this manner. "The only squares in the second column that may contain a 2 are `b` and `l`. But we know $\texttt{b} \neq 2$, thus $\texttt{l} = 2$, and now `c` is the only location for the remaining 2." The last sentence can be as short as "Now by process of elimination we deduce that $\texttt{e} = 5$, $\texttt{j} = 1$, $\texttt{o} = 3$, and $\texttt{t} = 4$."
5. First convince yourself that swapping any two particular digits in the grid will produce another solution. (For instance, replacing all 4's by 5's and vice-versa.) What does this fact have to do with the question?

3.4 Indirect Proofs

Exercises

7. Each truth table should contain all T's except for when P is true and Q is false.
8. a) For positive integers m and n, if m is divisible by n then m^2 is divisible by n^2.
b) For a real number x, if $3^x \leq 3x$ then $x \geq 1$.
c) For finite sets A and B, if $|A| = 0$ then $A \subseteq B$.
d) If a triangle has two congruent angles, then it has two congruent sides.
A proof by contrapositive is preferable for parts a), c) and d).
9. a) Suppose to the contrary that $\sqrt{7}$ is rational. Then $\sqrt{7} = p/q$ for integers p and q.
b) Suppose to the contrary that there are only finitely many primes.
c) Suppose to the contrary that each sister has less than 400 grams of chocolate.
d) Suppose to the contrary that every angle formed by three of the points is acute.
10. a) Contradiction b) Contrapositive c) Direct proof d) Contrapositive e) Direct proof f) Contradiction

WRITING

11. Prove the contrapositive. You should be able to argue that there are elements $x \in A$ and $y \in B$. How does this help prove that $A \times B$ is nonempty?
12. First state the contrapositive of the given implication. To show that $A \subseteq B \cup C$ suppose that $x \in A$. Then handle the cases $x \in B$ and $x \notin B$ separately, and show that $x \in B \cup C$ regardless.
13. To prove that $\overline{B \cup C} \subseteq \overline{A}$ you need to show that if $x \in \overline{B \cup C}$ then $x \in \overline{A}$. Show the contrapositive of this implication to complete the proof.
14. We are given that $A - C \nsubseteq A - B$, which means there is some element x satisfying $x \in A - C$ but $x \notin A - B$. Now $x \in A - C$ means that $x \in A$ but $x \notin C$. And so on.
15. Suppose to the contrary that $x \geq 3$. What can we say about the value of x^3? What about the value of $5x$? Combine these results to reach a statement that contradicts the fact that $x^3 + 5x = 40$.
16. Prove the contrapositive instead. When you simplify the equality $\frac{x}{2x-1} = \frac{y}{2y-1}$ several terms should conveniently cancel.
17. Suppose to the contrary that $\frac{x}{x+2y} < \frac{1}{3}$ and $\frac{y}{y+2x} < \frac{1}{3}$. Simplifying both inequalities should lead to an impossible situation.
18. Suppose to the contrary that $\beta - 4$ is actually rational, so that we can write $\beta - 4 = m/n$. How does this lead to a contradiction with the fact that we are given that β is irrational?
19. The contrapositive states that if $\sqrt{\beta}$ is rational then β is also rational. Prove this statement instead.
20. Suppose to the contrary that m/n is the smallest positive rational number. Obtain a contradiction by demonstrating how to use m/n to create a smaller positive rational.
21. Begin by noting that since $A \in \mathcal{P}(A)$ we can deduce that $A \in \mathcal{P}(C)$, which means that $A \subseteq C$. Similarly we find that $B \subseteq C$. Now suppose that A and B are both proper subsets of C and reach a contradiction.

3.5 Biconditional, Vacuous and Trivial Proofs

EXERCISES

23. Because $Q \Rightarrow P$ and $\neg P \Rightarrow \neg Q$ are logically equivalent statements. (The latter is the contrapositive of the former.)
24. a) False b) True c) True d) False
25. The set of odd integers.
26. The set of rational numbers.
27. We must show that if $f(x)$ is a linear function, then $f(x-a) + f(x+a) = 2f(x)$ for all $a, x \in \mathbb{R}$. Conversely we must also show that if $f(x-a) + f(x+a) = 2f(x)$ for all $a, x \in \mathbb{R}$ then $f(x)$ is a linear function.
28. This set of points is characterized by the property that they are the same distance from the x-axis as the y-axis. (Other answers are possible.)
29. a) Trivially true b) Vacuously true c) Vacuously true d) Trivially true
e) Vacuously true f) Trivially true g) Trivially true h) Vacuously true

WRITING

30. We omit the outline here. The example given in the text gives a good indication of how to frame the outline.
31. We must prove that if $A \subseteq B$ then $A \cap B = A$ and also that if $A \cap B = A$ then $A \subseteq B$. To establish the latter implication, suppose that $x \in A$. We wish to show that $x \in B$ also. Now use the fact that $A = A \cap B$ to your advantage.

32. Let the circles have radii R_1 and R_2, respectively. We must show that if $\pi R_1^2 = \pi R_2^2$ then $2\pi R_1 = 2\pi R_2$ and conversely. To prove the stated implication, begin by dividing by π and taking square roots. Continue in this manner to reach the desired conclusion.
33. Transform each equation into the other. For instance, multiply both sides of $\frac{2}{x} + \frac{3}{y} = 1$ by xy, move all terms to one side, then add 6 to both sides.
34. For this argument it is best to adopt our alternate approach to biconditional proofs. In other words, show that if a is even then a^2 ends with 0, 4 or 6; while if a is not even (i.e. a is odd) then a^2 does not end with one of these digits.
35. We must prove that if $A \subseteq B$ then $A \cap \overline{B} = \emptyset$ and also that if $A \cap \overline{B} = \emptyset$ then $A \subseteq B$. In the latter case we must show that if $x \in A$ then $x \in B$. So suppose that $x \in A$. Since $A \cap \overline{B} = \emptyset$, it must be the case that $x \notin \overline{B}$, which is equivalent to $x \in B$.
36. We must prove that if $-1 \leq x \leq 1$ then x^2 is at least as close to 0 as x is, and conversely. It helps to rephrase the problem in terms of absolute values. So we must show that $|x| \leq 1$ if and only if $|x^2| \leq |x|$.
37. We provide a proof of the first part to illustrate how a proof may be phrased. We will show that the statement is trivially true. Since we may rewrite $n^2 + 2n + 1$ as $(n+1)^2$, it is clear that this expression is a perfect square for all values of n. Hence the conclusion is always true, so the claim is trivially true.

3.6 Conjecture and Disproof

Exercises

38. Pairs satisfying $3m^2 = n^2 \pm 1$ are $(1, 2)$, $(4, 7)$ and $(15, 26)$.
39. Perhaps the most noticeable observations include the fact that the minus sign is always used, the sum $2m + n$ gives the next m value, and the sum $3m + 2n$ gives the next n value. (Several other conjectures are possible.) The next two pairs are $(56, 97)$ and $(209, 362)$.
40. One should find that in each case $x + y + z \geq 3\sqrt[3]{xyz}$. It is possible for the two values to be equal, but only when $x = y = z$.
41. We find that $11 \to 34 \to 17$, at which point we are done because we have already seen that the sequence beginning with 17 eventually reaches 1. We omit the verification of the remaining numbers.
42. Of all the positive integers from 1 to 20 the numbers 2, 6, 10, 14 and 18 cannot be written as a difference of two squares. This list consists of every fourth natural number beginning with 2. Equivalently (and more relevantly), these are the natural numbers with exactly one factor of 2.
43. One should find that $AG = 2(GL)$ and $BG = 2(GM)$ for any triangle ABC. Furthermore, segment $\overline{CN}$ passes through G.

Writing

44. Substitute the values $m = k^2 - 1$ for $k = 1, 2, 3, \ldots$ into $5m + 1$. You should find another perfect square fairly soon.
45. Try drawing a long skinny kite-shaped quadrilateral to find a counterexample.
46. First write down the negative of the given statement. Next take $x = 1$ and find a value of y other than 1 which satisfies the equation.
47. We are looking for a prime p and a perfect square n^2 such that $p = n^2 - 1$. Now use the fact that the right-hand side factors.
48. We know that three points equidistant from one another must form an equilateral triangle. Now consider where the fourth point might be located.

3.7 Existence

Exercises

50. The numbers $x = \frac{1}{2}(\pi + \sqrt{2})$ and $y = \frac{1}{2}(\pi - \sqrt{2})$ fill the bill.
51. If we take $x = 10$ and $y = 5$ then $x + 2y = 20 = \sqrt{8xy}$. (Other pairs can work.)
52. The smallest such trio of positive integers is 98, 99 and 100.
53. The only other solution is to use the numbers 4, 4 and -5.
54. As x increases from 0 to 1 the quantity $8^x + 9x$ increases from 1 to 17, so at some point it must equal 10. In fact, this occurs at $x = \frac{2}{3}$.
55. The line must pass through the center of the circle. Thus we should take the line passing through the center of both circles.
56. Take the line passing through the center of both rectangles. If these centers happen to coincide, then such a line will not be unique.

Writing

57. Consider the sums $10 + 0$, $11 + 1$, $12 + 2$ and so on.
58. What would go wrong if every child received less than eight pieces of candy?
59. Slightly modify the proof that there are 1000 consecutive composite numbers.
60. Construct the decimal expansion of r in a way that guarantees that every finite string of digits appear. For instance, if we start $r = 0.123456789\ldots$ then so far r contains every possible one-digit string.
61. Adapt the proof given in the text. At what point should we stop rotating the line to solve the problem?
62. Let 2^k be the largest power of 2 that is less than all the numbers on the list. (Why does such a power of 2 exist?) What can be said of the number s 2^{k+1} and 2^{k+2}?
63. First show that for almost any direction there is a line parallel to that direction which divides all the blue points in half. Now rotate this line 180°, all the while evenly splitting the blue points, and keep track of how the line divides the red points throughout the process.

Chapter 4: Number Theory

4.1 Divisibility

Exercises

1. a) True b) False c) True d) True e) False f) True g) True h) False i) True j) False k) True l) True
2. We have $d \mid 23$ for $d = -23, -1, 1$ and 23. Next, $d \mid 24$ iff $d = \pm1, \pm2, \pm3, \pm4, \pm6, \pm12$, or ±24. Finally, $d \mid (-25)$ for $d = \pm1, \pm5$, and ±25.
3. a) $n = (n)(1)$ b) $n^3 = (-n^2)(-n)$ c) $0 = (0)(n)$ d) $-3n = (-3)(n)$ e) either n or $n + 1$ will be even, hence $n(n + 1)$ is divisible by 2
4. a) 7 b) 10 c) 1 d) 25 e) 1
5. We find that $(1, n) = 1$, $(3n, 2n^2) = n$ if $3 \nmid n$ while $(3n, 2n^2) = 3n$ if $3 \mid n$, and $(n, 7n + 1) = 1$.
6. There are two such integers, 1 and -1.
7. For instance, take $a = 4$, $b = 5$ and $c = 6$.
8. One possibility is to use $a = 4$ and $b = 5$.
9. With \$22 and \$60 bills one can buy any item costing an even number of dollars. And with \$9 and \$14 bills one can spend any whole number of dollars. We have 11(\$22) − 4(\$60) = \$2 and 2(\$14) − 3(\$9) = \$1.