# CHAPTER 1

# Logical Foundations

## 1.1 Statements and Open Sentences

Certain words and phrases are ubiquitous in mathematical discourse because they convey the logical framework of the ideas being presented. For this reason they arise naturally in everyday language as well, although they may be used at times with a slightly different meaning in a conversational setting. Therefore our first task will be to introduce the standard terminology of logic and precisely define what these terms mean within the context of mathematics.

As a starting point, consider the following assertion:

*"For all positive integers $n$, it is the case that $n^2+2$ is a multiple of 3 or that $3n+7$ is a perfect square."* $(*)$

Before going on, decide for yourself whether or not this assertion is true. How confident are you of your answer? Would you bet \$1 that you are correct? Would you bet \$100? Throughout this text we will discuss strategies for demonstrating the validity or falsehood of various statements such as the one made above. We will begin by analyzing their structure.

> A **statement** is a mathematical assertion that can be assigned a **truth value**, either true or false.

Examples of statements include "The sum $3^3 + 4^3 + 5^3$ is equal to $6^3$," or "A triangle with sides of length 3, 4 and 5 has an area of 6." Clearly questions and commands should not qualify as statements. At the risk of omitting interesting propositions like "The Patriots are the best football team of the decade," or "It's freezing outside," we will restrict ourselves primarily to mathematical statements. In this way we avoid issues such as personal opinion or imprecisely defined terms which make ambiguous the truth value of a statement.

Note that it is not necessary to be able to establish the veracity of a given mathematical sentence in order for it to qualify as a statement. For this reason "There is a prime number between any two consecutive perfect squares," is a perfectly valid statement, even though nobody is completely sure whether it is true or false. (Recall that a prime number is a positive integer that has exactly two positive divisors: itself and 1. The first six prime numbers are 2, 3, 5, 7, 11 and 13.) In case you were wondering, the aforementioned statement is almost certainly true, although a proof has yet to be found.

a) Which of the following are mathematical statements?
*i.* Seven trillion is the largest number.
*ii.* Please compute $38^2$ in your head.
*iii.* Are all squares also rectangles?
*iv.* The integer $2^{1000}$ has 694 digits.
*v.* Both $n$ and $2n+1$ are primes.

One of the features of statement $(*)$ that makes it more complicated is the appearance of the variable $n$. Since the validity of a phrase such as "$3n+7$ is a perfect square" depends on what value is assigned to the variable, we cannot immediately ascertain its truth value. Therefore it would be inappropriate to dub it a statement.

> An assertion involving one or more variables is called an **open sentence**. Choosing a value for each variable reduces the open sentence to a statement that is either true or false, depending upon the values selected. The set of values that a variable may assume is known as the **domain** of the variable and is usually indicated at the start of the open sentence.

Examining statement $(*)$ we discover that it contains two shorter open sentences; namely "$n^2+2$ is a multiple of 3," and "$3n+7$ is a perfect square." We are also told that the domain of the variable $n$ is the set of positive integers. For a given value of $n$ in the domain each of these open sentences has a truth value. For example, when $n = 10$ we find that the first open sentence is true while the second is false. These two open sentences are joined into a single longer open sentence via the word OR. Common usage suggests that this compound sentence should be true when $n = 10$, since the first half is true.

> The **disjunction** of two statements or open sentences is obtained by joining them with the logical connective OR. A disjunction is true as long as at least one (and possibly both) of its components is true.

b) Is the disjunction "$n^2+2$ is a multiple of 3 or $3n+7$ is a perfect square" true or false when $n = 6$? How about when $n = 8$, $n = 12$ and $n = 14$?

In light of our definition, the statement "100 is a perfect square or 101 is a prime" is true. This convention might differ from the common notion of OR for some people, who hold that OR means that exactly one (but not both) of the individual statements is true. There is a mathematical term having this meaning, called 'exclusive or' (abbreviated to EOR), but it arises relatively infrequently. Just remember that whenever the word OR appears in a mathematical setting, it always means that at least one of the component statements is true.

The logical companion to the word OR is the word AND.

> The **conjunction** of two statements or open sentences is obtained by joining them with the logical connective AND. A conjunction is true when both of its components are true.

In this case there is no disagreement between the common and mathematical usage of the term AND. Thus the statement "100 is a perfect square and 101 is a prime" is true, while "$5 < 6$ and $-5 < -6$" is false.

**Concept Check** c) Create a conjunction of two open sentences involving a positive integer $n$ that is false when $n = 1$ and $n = 2$ but true when $n = 3$.

In our quest to dissect statement $(*)$, we finally come to the first two words of the sentence. The phrase 'for all' is known as a **quantifier**, in the sense that it prescribes the quantity of values of the variable for which the ensuing open sentence should be true; in this case, all of them. The other commonly employed quantifier is 'there exists,' which specifies that the open sentence should be true for at least one value of the variable. It is not hard to imagine other quantifiers, most of which are self-explanatory. For instance we have 'there does not exist' and 'there exists a unique'. The latter means that the open sentence should be true for exactly one value of the variable.

**Concept Check** d) Which of the following statements are true?

*i.* For all positive integers $n$ the number $n^2 + 2n + 2$ is a multiple of 5.
*ii.* There exists a unique circle passing through two given points in the plane.

We can now say with certainty that statement $(*)$ is false, because the value $n = 12$ provides a counterexample. As we have already seen, when $n = 12$ both of the open sentences "$n^2 + 2$ is a multiple of 3" and "$3n + 7$ is a perfect square" are false; hence so is their disjunction. In other words, the open sentence "$n^2 + 2$ is a multiple of 3 or $3n + 7$ is a perfect square" is not true for all values of $n$, as asserted. Note that we need only find a single counterexample in order to conclude that the entire statement is false.†

**All the Answers** a) The first and fourth sentences are mathematical statements. (Both happen to be false, though.) The second is a command, the third is a question, and the fifth is an open sentence.

b) The disjunction is true for $n = 6$, 8 and 14. However, it is false when $n = 12$ since $12^2 + 2 = 146$ is not a multiple of 3 nor is $3(12) + 7 = 43$ a perfect square.

c) A correct, but uninteresting conjunction is "$n > 2$ and $n < 10$." A more imaginative response might be "$n$ is odd and $n^2 + 1$ is divisible by 5."
d) When $n = 3$ we find that $3^2 + 2(3) + 2 = 17$ is not a multiple of 5, thus the first statement is false. The second statement is also false, because there is always more than one circle passing through two given points.

## Exercises

1. Decide whether the following statements are true or false.
a) The number 1776 is a perfect square.
b) It is the case that $3^3 + 4^3 + 5^3$ is equal to $6^3$.
c) A triangle with sides of length 3, 4, and 5 has an area of 6.
d) There are at least two prime numbers between $11^2$ and $12^2$.
e) A cube has six faces, eight vertices, and ten edges.

2. Find an integer value of $n$ for which the open sentence "$n^2 - n + 11$ is a prime number" is true. Find another value for $n$ that makes it false.

3. Find all positive integers $k$ for which the open sentence "$2^k + 1$ is a multiple of 3" is false. (Just check $k = 1, 2, \ldots, 8$ and describe the pattern.)

4. Find all real numbers $x$ for which the open sentence $30(x - 20) = 20(x - 10)$ is true. By the way, this is commonly known as "solving the equation."

5. State the domain of the variables $n$, $k$ and $x$ in the previous three exercises.

6. In what way does the logical connective OR come into play when solving the equation $(x + 3)(x^2 - 9) = 0$?

7. Decide whether each statement is true or false.
a) There exists a real number $x$ such that $e^x = 2$.
b) There exist positive integers $p$ and $q$ such that $p/q = \pi$.
c) There does not exist an integer $k$ such that $k^2 + 2$ is divisible by 5.
d) For all real numbers $m$, the line having slope $m$ and $y$-intercept 1 intersects the circle of radius 1 centered at the origin in exactly two points.
e) For every positive integer $n$, there exists a smaller positive integer.
f) There exists a unique two-digit number that is twice the product of its digits.
g) There exists a unique line passing through any two distinct points.

8. Find two positive integer values of $n$ other than $n = 12$ which demonstrate that statement $(*)$ is false.

## Writing

9. Twenty students attend a math class one morning. Each student arrives at a certain time, stays for some portion of the class, then departs without returning. Suppose that given any pair of students, they are both present in the classroom together for some part of the lecture. Prove that at some point in time all twenty students are simultaneously present in the classroom.

10. Twenty students are lined up in a row for a math bee. Given any two adjacent students in the line, one or the other (or both) of them can recite the first fifty digits of $\pi$. Illustrate an arrangement in which exactly twelve students know how to recite $\pi$. Then prove that for any such arrangement at least ten students know how to recite the first fifty digits of $\pi$.

11. Twenty math students are comparing grades on their first two quizzes of the year. The class discovers that whenever any pair of students consult with one another, these two students received the same grade on their first quiz or they received the same grade on their second quiz (or both). Prove that the entire class received the same grade on at least one of the two quizzes.

### Further Exploration

12. It is standard in some programming languages for the number zero to represent one of the truth values (either true or false) and for positive numbers to represent the other truth value. If assigned correctly, the operations of addition and multiplication will then correspond to conjunction and disjunction, in some order. Figure out how to make this all work out neatly.

## 1.2 Logical Equivalence

The ultimatum "I will not both cook dinner and wash the dishes," is clearly equivalent to declaring that "I will not cook dinner or I will not wash the dishes." Notice that the first statement involves a conjunction while the second employs a disjunction. This observation is mildly troubling—apparently it is possible to say the same thing in two different ways! It is natural to wonder whether there is a systematic method for determining when two statements have the same logical meaning, especially since it is possible to construct far more complicated examples than the one given here. Happily, the answer is yes.

To describe this method efficiently we must first introduce some notation. We will commonly use the letters $P$, $Q$ and $R$ to represent statements. Thus a statement $P$ has a truth value, either true ($T$) or false ($F$). Another statement $Q$ also has two truth values, so there are four possible ways to assign truth values to both statements, listed in the **truth table** on the left below.

a) How many rows will a truth table for three statements have?

Next, we abbreviate "$P$ OR $Q$" as $P \vee Q$. Recall that a disjunction is true unless both component statements are false. This definition is summarized by the truth table in the middle.

| $P$ | $Q$ |
|---|---|
| $T$ | $T$ |
| $T$ | $F$ |
| $F$ | $T$ |
| $F$ | $F$ |

| $P$ | $Q$ | $P \vee Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ |

| $P$ | $Q$ | $P \wedge Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ |

On the other hand, the conjunction "$P$ AND $Q$," abbreviated as $P \wedge Q$, is only true when both $P$ and $Q$ are true. This fact is reflected by the truth table on the right. The **negation** "NOT $P$," written compactly as $\neg P$, has a particularly simple truth table. Thus if $P$ is true then $\neg P$ is false, while if $P$ is false then $\neg P$ is true. (Negation becomes more interesting when combined with other logical operations.) We introduce this notation because it is helpful to be familiar with these standard symbols. However, we will use symbolic notation outside of this chapter only rarely, such as when validating proof techniques.

b) The disjunction $P \vee \neg Q$ will be true except in one case. What truth values for $P$ and $Q$ make $P \vee \neg Q$ false?

c) Let $P$ and $Q$ be the statements "We won our first game," and "We won our second game," respectively. Translate the following statements into logical notation, using the symbols $P$, $Q$, $\wedge$, $\vee$ and $\neg$.
*i.* We won both of our first two games.
*ii.* We lost both of our first two games.
*iii.* We won at least one of our first two games.
*iv.* We lost at least one of our first two games.
*v.* We didn't win both of our first two games.

We are now in a position to show conclusively that the two statements made earlier mean the same thing, i.e. are logically equivalent.

> Suppose that two statements are constructed from the same set of component statements. The two statements are said to be **logically equivalent** if they have the same resulting truth value regardless of the manner in which truth values are assigned to the component statements.

To see how this plays out in practice, let $P$ be the statement "I will cook dinner" and let $Q$ be the statement "I will wash the dishes." Then "I will not both cook dinner and wash the dishes," can be translated as $\neg(P \wedge Q)$ while "I will not cook dinner or I will not wash the dishes," becomes $\neg P \vee \neg Q$. We next create a single truth table to compare the truth values of these two statements for all possible truth values for $P$ and $Q$. Sure enough, in every case the outcomes are identical. We indicate their equivalence by writing $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$.†

| $P$ | $Q$ | $P \wedge Q$ | $\neg(P \wedge Q)$ | $\neg P$ | $\neg Q$ | $\neg P \vee \neg Q$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |

It should not come as a surprise to discover that the statements $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ are also logically equivalent, as you will confirm in the exercises.

These two rules for negating a conjunction or disjunction frequently come in handy, so we highlight them below.

> **DeMorgan's Laws** indicate how negation distributes over conjunction and disjunction. They assert that
>
> $$\neg(P \wedge Q) \equiv \neg P \vee \neg Q \qquad \text{and} \qquad \neg(P \vee Q) \equiv \neg P \wedge \neg Q.$$

Among other things, DeMorgan's Laws indicate how we should negate certain statements. Thus the opposite of "Let $x$ be a real number such that $x \geq 0$ and $x^2 = 9$," would be "Let $x$ be a real number such that $x < 0$ or $x^2 \neq 9$." (Note that the domain of the variable does not change.)

d) Determine the negative of the statement "Sink or swim."

The use of parentheses in the above examples was crucial to clarifying the scope of the NOT symbol. As you will discover in the exercises, $\neg(P \wedge Q)$ and $\neg P \wedge Q$ are not logically equivalent. Parentheses are also important when it comes to specifying order of operation, just as with algebraic expressions. For example, the statement $P \vee Q \wedge R$ is ambiguous—does this mean $(P \vee Q) \wedge R$ or $P \vee (Q \wedge R)$? The distinction is necessary, because these statements have different logical meanings. We may demonstrate this fact via a truth table.

| $P$ | $Q$ | $R$ | $(P \vee Q) \wedge R$ | $P \vee (Q \wedge R)$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $F$ |

Since the right-hand columns are not identical we conclude that the two statements are not logically equivalent; that is, $(P \vee Q) \wedge R \not\equiv P \vee (Q \wedge R)$.

Certain statements, like "We'll get there when we get there," and "Either I'll pay you the money back, or I won't," are amusing because they manage to be undeniably true without really saying anything new.

> A statement that is always true is called a **tautology**, while a statement that is always false is known as a **contradiction**.

One can confirm that a given logical statement is a tautology by constructing its truth table and checking that every possible outcome is true; similarly, every entry in the truth table for a contradiction will be $F$. One example of a tautology is $P \vee \neg P$; for if $P$ is true then so is $P \vee \neg P$, but if $P$ is false then $\neg P$ is true, hence $P \vee \neg P$ is again true.

**Concept Check** e) Why is $P \wedge \neg P$ is a contradiction?

**Concept Check** f) What is the negation of a contradiction?

Tautologies and contradictions usually involve more than one statement, such as the tautology $(P \wedge \neg Q) \vee (\neg P \vee Q)$ appearing in the exercises. Bear in mind that these examples are relatively rare; most logical statements are sometimes true and sometimes false.

**All the Answers**
a) A truth table for three statements has eight rows.
b) If $P$ is false while $Q$ is true then $P \vee \neg Q$ will be false.
c) The translations, in order, are $P \wedge Q$, $\neg P \wedge \neg Q$, $P \vee Q$, $\neg P \vee \neg Q$, and $\neg(P \wedge Q)$. Note that the last two statements actually mean the same thing.
d) "Don't sink and don't swim," or perhaps "Float."
e) It is impossible for $P$ and $\neg P$ to both be true, hence $P \wedge \neg P$ will always be false.
f) The negation of a contradiction is a tautology.

## Exercises

13. Let $P$ be the statement "The cat is outside," let $Q$ be the statement "The dog is outside," and let $R$ be the statement "It is bright and sunny today." Translate the following statements into logical syntax.
a) The cat and dog are both inside, as it is raining today.
b) The cat is outside even though it is raining today.
c) At least one pet is outside on this sunny day.
d) It is not the case that the dog is outside in the sunshine.
e) The cat and dog are in different locations.

14. Establish that $\neg(P \wedge Q) \not\equiv \neg P \wedge Q$.

15. Consider the statements "It is not the case that I will run for president or stage a military coup," and "I will not run for president and I will not stage a military coup." Explain why these statements mean the same thing. Then translate both statements into compact logical notation.

16. Demonstrate that $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$.

17. Negate each of the following statements.
a) Triangle $ABC$ has a perimeter of 12 and an area of 6.
b) Let $k$ be an integer such that $k$ is even or $k \leq 10$.
c) I am older than Al but younger than Betty.
d) It is not the case that $2x < y$ and $2y < x$.
e) Jack will answer this question or the next one.
f) There is a new car behind at least one of the doors.

18. Which of the following gives a valid logical equivalent to $P \wedge (Q \vee R)$? Create a truth table to confirm that your choice is right. (Don't write out truth tables for the other two options, though.)
a) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee R$
b) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
c) $P \wedge (Q \vee R) \equiv (P \vee Q) \wedge (P \vee R)$

19. Create a truth table for the exclusive or operation "$P$ EOR $Q$." Recall that this statement is true whenever exactly one of $P$ and $Q$ is true.

20. Write EOR in terms of OR, AND, NOT. In other words, create a statement involving only $P$, $Q$, $\vee$, $\wedge$, $\neg$, (, and ) that is logically equivalent to "$P$ EOR $Q$."

21. Show that the statement $P \wedge Q \vee R$ is ambiguous by demonstrating that the statements $(P \wedge Q) \vee R$ and $P \wedge (Q \vee R)$ are not logically equivalent.

22. Describe in words how to predict when the disjunction $P \vee Q \vee R$ is true. Then do the same for the conjunction $P \wedge Q \wedge R$.

23. Demonstrate that $(P \wedge \neg Q) \vee (\neg P \vee Q)$ is a tautology.

24. Verify that $((P \vee \neg Q) \wedge \neg R) \wedge (R \vee (Q \wedge \neg P))$ is a contradiction.

25. Suppose that statement $P$ is a contradiction. Show that $(P \vee Q) \wedge (P \vee \neg Q)$ is also a contradiction.

## Writing

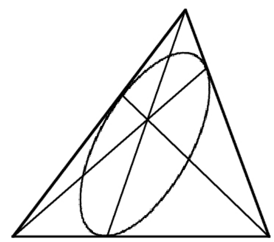
26. A father has 20 one dollar bills to distribute among his five sons. He declares that the oldest son will propose a scheme for dividing up the money and all five sons will vote on the plan. If a majority agree to the plan, then it will be implemented, otherwise dad will simply split the money evenly among his sons. Assume that all the sons act in a manner to maximize their monetary gain but will opt for evenly splitting the money, all else being equal. What proposal will the oldest son put forth, and why?

27. Imagine that in the scenario of the previous problem the father decides that after the oldest son's plan is unveiled, the second son will have the opportunity to propose a different division of funds. The sons will then vote on which plan they prefer. Assume that the sons still act to maximize their monetary gain, but will vote for the older son's plan if they stand to receive the same amount of money either way. What will transpire in this case, and why?

28. As part of an arithmetic exercise, Mr. Strump chooses two different digits from 1 to 9, tells Abby their product, then challenges Abby to figure out which two digits he has chosen. After a moment, Abby complains that there could be more than one answer. Realizing that she is correct, Mr. Strump helpfully mentions that the sum of the digits is not equal to 10. Abby is then able to correctly deduce the two digits. Explain how it is possible to precisely determine Mr. Strump's two digits based on this story.

## 1.3 The Implication

One of the aspects of mathematics that makes it such an exciting subject is the manner in which a given set of assumptions can lead to surprising or unexpected conclusions. For example, suppose that an ellipse is inscribed within a triangle, meaning that the ellipse is tangent to all three sides of the triangle. If we draw a segment joining each vertex of the triangle to the point of tangency on the opposite side as shown at right, then remarkably these three segments all intersect at a single point. This result is relatively simple to describe, but not nearly so easy to prove. The clearest explanation involves an operation on the diagram known as an affine transformation.



Or consider this delightful fact from number theory. Let $n \geq 2$ be a positive integer. Now multiply together all the numbers from $n-1$ down to 1 and then add 1 to the product. This quantity is written as $(n-1)!+1$ in mathematical notation. If $n$ is prime, the result will always be an exact multiple of $n$. For example, taking $n=7$ we find that $6!+1=(6)(5)(4)(3)(2)(1)+1=721$, which is a multiple of 7 as predicted. This result is known as Wilson's Theorem.

a) Confirm that Wilson's Theorem holds for $n=3$ and $n=5$.

Most mathematical results follow the format just described: if certain statements or conditions hold, then a result follows.

> An **implication** has the form "If $P$ then $Q$," where $P$ and $Q$ are statements or open sentences. We write $P \Rightarrow Q$ for short, and refer to $P$ as the **hypothesis** or **premise**, whereas $Q$ is known as the **conclusion**.

In practice, $P$ encapsulates the facts we are given, while $Q$ represents the result to be proved.

It is important to have a sound understanding of the logical meaning of implication, since it will inform the techniques we develop for proving mathematical statements. However, the implication has the potential to be confusing at first, for several reasons. To begin, there are many ways of expressing the implication in our language. Thus $P \Rightarrow Q$ can be written as "If $P$ then $Q$" or "$P$ implies $Q$" or "$Q$ whenever $P$" or "$Q$ follows from $P$," among many other possible ways to phrase this fundamental idea.

b) State the following implications in if-then form.
*i.* In order for photosynthesis to take place it is necessary to have light.
*ii.* We always have a great time when Laszlo comes over.
*iii.* For $n^3$ to be even it is sufficient for $n$ to be a multiple of 6.

Another cause for the uncertainty that may accompany the implication is the fact that it is not immediately obvious how its truth table should be defined.

Of course, it makes sense that if $P$ is true and the implication $P \Rightarrow Q$ is to hold, then $Q$ must also be true. But other rows of the truth table would seem to be more debatable, at least on the surface. The following illustration will help to clarify the issue.

Suppose that Kate and Nate are playing a game of checkers. Nate promises that "If you beat me at checkers, then I will give you a chocolate bar." Let's consider each of four possible scenarios in turn and decide whether or not Nate has broken his promise.

- *Kate wins and Nate gives her a chocolate bar.*
  Clearly Nate has kept his promise, so his statement was true.

- *Kate wins but Nate does not give her a chocolate bar.*
  Just as clearly, Nate has broken his promise, thus his statement was false.

- *Kate loses but Nate gives her a chocolate bar anyway.*
  In this case Nate is being generous. (Perhaps he likes Kate.) At any rate, one can hardly claim that he has broken his promise.

- *Kate loses and Nate does not give her a chocolate bar.*
  Nate is not so generous here, but there are no grounds on which to accuse him of breaking his promise. Once again, his statement was valid.

Because of examples like the one above (and for various other good reasons), we define the truth table for $P \Rightarrow Q$ as shown at right. The final two rows are a bit counter-intuitive, where we declare that when $P$ is false, the implication $P \Rightarrow Q$ is nonetheless true. These correspond to the final two scenarios above, in which we decided that Nate's statement was true, since he did not break his promise.

| $P$ | $Q$ | $P \Rightarrow Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

CONCEPT CHECK c) Consider the claim that "If quadrilateral $ABCD$ has right angles at vertices $A$ and $B$, then it is a rectangle." Draw a quadrilateral which shows that this implication can be false. Which row of the truth table came into play?

When mathematical results are stated as implications they typically involve open sentences (i.e. one or more variables). This was the case for Wilson's Theorem mentioned earlier, which is restated below in several forms.

a. If $n$ is prime, then $(n-1)! + 1$ is a multiple of $n$.

b. For all positive integers $n$, if $n$ is prime then $(n-1)! + 1$ is a multiple of $n$.

c. For all primes $p$, the number $(p-1)! + 1$ is a multiple of $p$.

The first version would seem to be adequate. Technically, though, this version is an open sentence, not a statement. Of course, it is understood that we are asserting that the implication is true for every positive integer $n$. Therefore a more precise wording is given by statement b. It is possible to streamline the wording without losing any of the meaning, as illustrated by the final version.

## Mathematical Outing ⋆ ⋆ ⋆

Each of the four cards below has a digit printed on one side and a letter printed on the other side. Imagine that a classmate makes the assertion that "If there is a vowel on one side of a card then there is an odd number on the other side." Which cards *must* be turned over to check whether or not this is a true statement?

$D$ $\quad$ $E$ $\quad$ 2 $\quad$ 3

Now consider the following situation, concocted by psychologist Leda Cosmides and described by Malcolm Gladwell in his book *The Tipping Point.*

> Suppose four people are drinking in a bar. One is drinking Coke. One is sixteen. One is drinking beer and one is twenty-five. Given the rule that no one under twenty-one is allowed to drink beer, which of those people's IDs do we have to check to make sure the law is being observed?

If you felt that the first question was considerably harder than the second, you are not alone. But in fact they are equivalent puzzles. The point made by Gladwell is that as human beings most of us are hardwired to draw logical conclusions in relational as opposed to abstract contexts.

To convincingly argue that an implication such as the one above is true, we would in theory need to compute $(p-1)!+1$ for each prime $p$ and check that it is a multiple of $p$ in every case. This is hardly feasible, since there are infinitely many primes. To circumvent this difficulty, mathematicians have developed general methods of argument that apply equally well to any prime. In this way a single proof can handle all the cases simultaneously. We shall develop some of these techniques in Chapter 4.

On the other hand, to demonstrate that such an implication is false, we need only find a single counterexample. According to our truth table, $P \Rightarrow Q$ fails to be true when $P$ is true but $Q$ is false. Therefore we should seek a value of the variable for which $P$ holds but $Q$ does not.

The **negation of the implication** "$P$ implies $Q$" is given by "$P$ and not $Q$." In symbolic notation we have $\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$. Hence to find a counterexample to the statement "$P$ implies $Q$," it suffices to find an instance for which $P$ is true but $Q$ is false.

**Concept Check** d) State the negation of each implication.
*i.* If I eat another bite then I'll burst.
*ii.* If $(x-1)(x-3)=3$ then $x-1=3$ or $x-3=3$.

Algebra instructors are accustomed to seeing claims such as "If $x^2 = 25$ then $x = 5$." This implication looks good, but is actually not always valid. To discover a counterexample, we must find a value of $x$ for which the premise $x^2 = 25$ is true while the conclusion $x = 5$ is false. Put another way, we must find a value of $x$ which satisfies the negation, which states that "$x^2 = 25$ and $x \neq 5$." The table below tests several values of $x$.

| $x$-value | $x^2 = 25$ | $x = 5$ | $x^2 = 25 \Rightarrow x = 5$ |
|---|---|---|---|
| $x = 5$ | $T$ | $T$ | $T$ |
| $x = 1$ | $F$ | $F$ | $T$ |
| $x = -5$ | $T$ | $F$ | $F$ |

Therefore $x = -5$ provides a counterexample to the claim. This is the logical analysis behind the mistake known as overlooking a solution.†

**Concept Check** e) For which value of $x$ is the implication "If $(x-1)(x-3)=3$ then $x-1=3$ or $x-3=3$" false?

**All the Answers** a) For $n = 3$ we have $(2)(1)+1 = 3$, which is divisible by 3. When $n = 5$ we have $(4)(3)(2)(1)+1 = 25$, which is divisible by 5.
b) *i.* If photosynthesis takes place then light is present. *ii.* If Laszlo comes over then we have a great time. *iii.* If $n$ is a multiple of 6 then $n^3$ is even.
c) One example would be a trapezoid having angles of 90°, 90°, 135°, and 45°, in that order. The second row of the table applies.
d) *i.* I ate another bite and I didn't burst. *ii.* We have $(x-1)(x-3)=3$ and $x-1 \neq 3$ and $x-3 \neq 3$. (In other words, $(x-1)(x-3)=3$ but $x \neq 4$ and $x \neq 6$.)
e) When $x = 0$ we have $(x-1)(x-3)=3$, but neither $x-1=3$ nor $x-3=3$ holds.
Two cards must be flipped over to verify the assertion: the E and the 2. (The latter because if there were a vowel on the reverse side then the assertion would be false.) Similarly, we must check IDs for the sixteen-year-old and the beer drinker.

# Exercises

29. Write the following implications in "If $P$ then $Q$" form.
a) When it rains, it pours.
b) I'll try escargot only if Al eats some first.
c) That $a$ is even follows from the fact that $7a$ is even.
d) In order to start a fire it is necessary to light a match.
e) For triangle $ABC$ to be isosceles it is sufficient to have $\angle A \cong \angle B$.
f) A positive discriminant implies that a quadratic has two distinct solutions.

30. Experimentation suggests that if $p$ is a prime then $2^p - 1$ is also a prime. What would be required to show that this implication can be false?

31. Create a statement that is logically equivalent to $P \Rightarrow Q$ using only the symbols $P$, $Q$, $\wedge$, $\vee$, $\neg$. (Not necessarily all of them.)

32. Validate the definition of the negation of an implication by verifying that $\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$ is a logical equivalence.

33. Write out the negation of each of the following implications.
a) If Cinderella marries the prince then I'll eat my hat.
b) If $a^2$ is divisible by 12 then $a$ is even or $a$ is a multiple of 3.
c) Quadrilateral $ABCD$ is a square whenever it has four congruent sides.
d) To bake bread it is necessary to use flour, water and yeast.

34. Let $P$ and $Q$ be the open sentences "$7n + 1$ is a perfect cube" and "$n$ is a perfect square," respectively. Find positive integer values for $n$ for which
a) $P$ and $Q$ are both true, b) $P$ and $Q$ are both false,
c) $P$ is true while $Q$ is false, d) $P$ is false while $Q$ is true.
In which cases is the implication $P \Rightarrow Q$ true?

35. For what value of $x$ is the implication "If $|x - 3| = 1$ then $|x - 2| = 2$" false?

36. Find all values of $y$ for which the implication "If $y < 2$ then $y^2 < 4$" is true. (In particular, note that it is *not* true for all values of $y$; in other words, squaring an inequality is not a valid algebraic step.)

37. In plain English, what does $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ say? This rule of inference in propositional logic is known as *modus ponens*. Construct a truth table for this statement. How does the truth table demonstrate that *modus ponens* is a valid rule of inference?

38. Repeat the previous exercise for $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$. This is another staple rule of inference in propositional logic, known as syllogism.

## Writing

*While the following problems are not directly related to the material presented in this section, they represent a selection of classic problems with which every student should be familiar.*

39. Six students get together to study for a math exam. Each pair of students are either acquainted with one another or else are unacquainted. Prove that it is possible to find three of the students all of whom are acquainted with one another, or else all of whom are unacquainted.

40. Seven mathematicians get together for a dinner party. Prove that it is not possible for each mathematician to shake hands with exactly three others.

41. Suppose that eight people attend a math mixer. Prove that if each person shakes hands with some (possible none) of the other guests, and no pair of individuals shakes hands more than once, then there must exist two guests who shook the same total number of hands.

## 1.4 The Biconditional

Let us return to the case of the student faced with the equation $x^2 = 25$. Imagine that instead of just writing $x = 5$, this student gave their answer as $x = 5, -5$ or 0; in other words, included an extraneous solution. We justifiably feel that this response should not receive full credit because $x = 0$ does not solve the equation. More precisely, a number should appear in the list of solutions *if* it satisfies $x^2 = 25$, which requires that we include both $x = 5$ and $x = -5$, but also *only if* it satisfies the equation, which rules out all other numbers.

> A statement of the form "$P$ if and only if $Q$," is a **biconditional**. For convenience, the phrase "if and only if" is often shortened to just **iff**. A biconditional essentially declares that two statements are **equivalent**, meaning that one statement is true exactly when the other is.

Incidentally, the popular abbreviation "iff" was invented by Paul Halmos, a beloved writer who is best known among budding mathematicians for his book *I Want to Be a Mathematician.*

CONCEPT CHECK a) The following statements illustrate some of the different ways that the biconditional can be expressed. Determine a suitable way to complete each sentence to create valid statements.
*i.* A triangle has three congruent angles exactly when it has...
*ii.* For an integer $n$, a necessary and sufficient condition for $n$ to be both odd and 1 greater than a multiple of 3 is that...
*iii.* That a real number $x$ satisfies $x^2 < 4$ is equivalent to...
*iv.* Pigs can fly if and only if...

Since a biconditional asserts that one statement is true exactly when the other is true, the truth table for a biconditional has the entries shown at left.

| $P$ | $Q$ | $P \Longleftrightarrow Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ |

Although not immediately obvious, one way to express the notion that two statements $P$ and $Q$ are equivalent is to require that each statement imply the other. This explains the genesis of the term 'biconditional,' since the validity of each statement is conditional upon the validity of the other. It also explains the notation $P \Longleftrightarrow Q$ used for the biconditional—it's a combination of $P \Rightarrow Q$ and $P \Leftarrow Q$. This fundamental connection between implication and equivalence can be established by means of a truth table, as will be done in the exercises.

Whenever we discover that one statement implies another, it is natural to wonder whether the implication is true in the other direction.

> The implication $Q \Rightarrow P$ is known as the **converse** of $P \Rightarrow Q$.

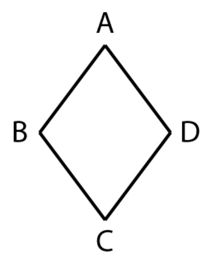
## Mathematical Outing ⋆ ⋆ ⋆

We discovered earlier that when $n$ is a prime the quantity $(n-1)!+1$ is divisible by $n$. Let's see what happens when $n$ is not a prime. Compute $(n-1)!+1$ for $n = 6$, 8 and 9. What do you notice? Make a conjecture based on your findings. Does it appear that the given expression is divisible by $n$ if and only if $n$ is prime?

The next few questions will help to explain the pattern you just noticed. What would you predict will be the remainder when $47!+1$ is divided by 48? Explain why your prediction should be true. Then repeat this exercise for the case in which $48!+1$ is divided by 49.

b) Using if-then form, state the converse of each implication.
*i.* My flight is delayed whenever I reach the airport early.
*ii.* The angle measure $m\angle ACB = 90°$ implies that $(AC)^2 + (BC)^2 = (AB)^2$.

It is a common mistake to assume that just because an implication is true, then its converse will also be true. In reality, the converse is sometimes true and sometimes false; it depends on the particular implication under consideration. For example, we all know that a square has four sides of equal length. This can be phrased as an implication by saying "If quadrilateral $ABCD$ is a square, then it has four congruent sides." As mathematicians, we should immediately ask ourselves whether it is also true that "If $ABCD$ has four congruent sides, then it is a square." A moment's thought reveals that the converse if false, since a rhombus (diamond) has four sides of equal length but is not a square. We conclude that "$ABCD$ is a square" and "$ABCD$ has four congruent sides" are *not* equivalent statements.†




We are also familiar with the fact that if a positive integer $a$ is even, then $a^2$ will also be even. The converse of this implication would read, "For any positive integer $a$, if $a^2$ is even then $a$ is also even." In this case both the original statement and the converse happen to be true. According to our earlier discussion, we may now conclude that $a$ is even if and only if $a^2$ is even. In general, whenever a statement and its converse are both true we have a pair of equivalent statements.

a) *i.* …when it has three congruent sides. *ii.* …is that $n$ is 1 greater than a multiple of 6. *iii.* …is equivalent to $-2 < x < 2$. *iv.* …elephants can jump (or any other situation that never occurs).
b) *i.* If my flight is delayed, then I reached the airport early. *ii.* If we have the equality $(AC)^2 + (BC)^2 = (AB)^2$ then $m\angle ACB = 90°$.
The value of $(n-1)!+1$ is equal to 121, 5041 and 40321 when $n = 6$, 8 and 9. In each case the number is 1 more than a multiple of $n$, rather than equal to a multiple of $n$. Hence it appears that $(n-1)!+1$ is divisible by $n$ iff $n$ is prime.

It makes sense that 47! would be a multiple of 48, since the product includes both 6 and 8. So adding 1 yields a number that is not a multiple of 48. Similarly, 48! is a multiple of 49 since this product includes 7 and 14.

## Exercises

42. Briefly explain why these biconditional statements are true or false.
a) A necessary and sufficient condition for a triangle to have two congruent sides is for it to have two congruent angles.
b) For $x$ a real number, $x^2 + x - 2 = 0$ if and only if $x = 1$ or $x = 2$.
c) Let $a$ and $b$ be positive integers. Then $ab$ is a multiple of 10 exactly when $a$ is a multiple of 10 or $b$ is a multiple of 10.
d) We have that $x \neq y$ is equivalent to $x^2 \neq y^2$ for real numbers $x$ and $y$.
e) A positive integer is divisible by 3 iff its reverse is. (The 'reverse' is obtained by writing the digits in the opposite order.)

43. Let $P$ and $Q$ be open sentences involving the variable $x$. What would a counterexample to the claim "For all $x$ we have $P$ iff $Q$," look like?

44. Create a truth table for the statement $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ and confirm that it is identical to the one given above for $P \Longleftrightarrow Q$.

45. Are the statements $(P \vee Q) \Longleftrightarrow R$ and $(P \Longleftrightarrow R) \vee (Q \Longleftrightarrow R)$ logically equivalent? Why or why not?

46. Write the converse of the following implications.
a) Let $n$ be an integer. If $n$ is not a multiple of 3, then $n^2 + 5$ is a multiple of 3.
b) When two rectangles are congruent it follows that they have the same area.
c) For positive real numbers $x$ and $y$, if $x \geq y$ then $1/x \leq 1/y$.
d) Let $R$ and $L$ be points to the right and left of the $y$-axis, respectively. If line $RL$ has positive $y$-intercept, then $R$ is in quadrant $I$ and $L$ is in quadrant $II$.
e) For positive integers $a$ and $b$, the number $a + b$ involves the digit 0 whenever both $a$ and $b$ use the digit 0.

47. For each of the implications in the previous problem, determine whether the implication is true or false and then decide whether its converse is true or false. Consequently, in which instances do we have a pair of equivalent statements?

48. Let $P$ and $Q$ be statements. Show that either $P \Rightarrow Q$ or its converse is true.

49. Show that the statements $Q \Rightarrow P$ and $\neg P \Rightarrow \neg Q$ are logically equivalent. Hence an equivalent way to state the converse of "$P$ implies $Q$" is "not $P$ implies not $Q$." (The statement $\neg P \Rightarrow \neg Q$ is known as the **inverse** of $P \Rightarrow Q$.)

50. Write out the inverse of each implication below. Which statement do you find to be more clear, the inverse or the converse?
a) Let $k$ be a positive integer. If $3^k + 1$ is not a multiple of 4 then $k$ is even.
b) Quadrilateral $ABCD$ is a square whenever it has four congruent sides.
c) For a real number $a$, the equation $2^x = a$ has no solution if $a$ is not positive.

## Writing

51. The game of Snatch involves two players who take turns removing either 1, 2, 3 or 4 pennies from a pile of pennies. The winner is the player to take the last penny. Depending on the number $n$ of pennies in the pile, the player about to move can either be guaranteed of eventually taking the last penny (a winning position) or cannot prevent the other player from doing so (a losing position). Which values of $n$ constitute a losing position for the person about to play? Begin your answer with "A pile of $n$ pennies represents a losing position if and only if $n$ is. . . " Then explain why your answer is correct.

52. Suppose instead that in the game of Snatch players may only remove 1, 3 or 4 pennies on each turn. Now which values of $n$ constitute a losing position? Write your answer and explanation as before.

## Further Exploration

53. There are many possible ways to modify the game of Snatch. For instance, one might allow the players to remove 1, 2, 4, 8, 16, . . . pennies on each turn. Or one might have two separate piles, with the stipulation that from 1 to 4 pennies may be removed, but only from one of the piles. Invent your own variation of Snatch and analyze your game by carefully describing the winning and losing positions and justifying your description.

# 1.5 Quantifiers

We briefly discussed quantified statement in the first section. At that point we were interested in positive integer values of $n$ for which $n^2 + 2$ was a multiple of 3 or $3n + 7$ was a perfect square. Naturally we were curious as to how many values of $n$ met these conditions. All values of $n$? At least one value of $n$? Exactly one value of $n$? No values of $n$?

> A phrase that indicates the number of values of a variable satisfying an open sentence is known as a **quantifier**. The two most common such phrases are the **universal quantifier** "for all $x$" ($\forall x$) and the **existential quantifier** "there exists an $x$" ($\exists x$).

Quantifiers are an indispensable part of our mathematical vocabulary. For example, theorems often assert that some result holds for all values of a variable. This is understood to be the case even when the phrase 'for all' is omitted. Thus it is the case that if $2^n - 1$ is a prime, then $n$ itself must be a prime. A more precise rendering of this fact would state "For all positive integers $n$, if $2^n - 1$ is prime then $n$ itself is prime." (Primes of the form $2^n - 1$ are known as *Mersenne primes*. The largest known primes are of this form. As of this writing the current record-holder is $2^{43,112,609} - 1$, a number having over ten million digits.)

Although the concept of quantified statements is intuitively clear, it takes some careful thought to keep track of what happens when we negate a quantified statement or analyze statements containing two variables and two quantifiers.

a) Determine a statement which asserts the exact opposite of the quantified statement "For every positive integer $n$, $n^2 - n + 41$ is prime."

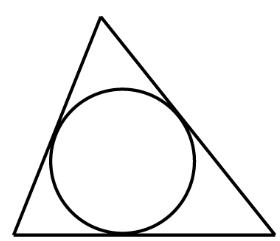
An appealing but incorrect way to phrase the negative would be "For every positive integer $n$, it is the case that $n^2 - n + 41$ is not prime." The reason that this option does not suffice is that it swings from one extreme to the other rather than encompassing all possibilities not covered by the original statement. The given statement claims that there are no exceptions to the rule that $n^2 - n + 41$ is prime. The opposite stance would be that there is an exception to this rule; in other words, there does exist a value of $n$ for which $n^2 - n + 41$ is not prime. (Observe that this is a considerably less stringent condition than requiring that $n^2 - n + 41$ is not prime for all $n$.)

> The **negation of a quantified statement** of the form "For all $x$, we have $P$," can be written as "There exists an $x$ such that $\neg P$." Similarly, the negation of the statement "There exists an $x$ such that $P$," takes the form "For all $x$, we have $\neg P$."

The negation of an existential quantifier might also be written "There does not exist $x$ such that $P$." However, this is generally not as useful a way to convey the opposite meaning. Also, if $P$ happens to be a conjunction, disjunction or implication one must also take care to write $\neg P$ correctly.

b) Write the negation of the statement "There exists a real number $x$ such that $|x + 3| \leq 1$ and $|x - 4| \leq 2$."

A third quantifier occurs regularly in mathematical discussions, although perhaps not quite as frequently as the universal and existential quantifiers just discussed. It is the phrase 'there exists a unique,' also indicated by writing 'there is one and only one.' Uniqueness is one of the most elegant and appealing characteristics in mathematics. For instance, given a triangle, there exists a unique circle within the triangle that is tangent to all three sides, as illustrated at right. Or given any positive integer $n$, there is one and only one way to write $n$ as a product of primes.



c) Given any three points in the plane, is it true that there exists a unique circle passing through all three points?

As a brief illustration, we argue that there is a unique solution to the equation $(x+4)(x-2) = -9$. This is because rearranging and factoring the given equation leads to $(x + 1)^2 = 0$. Clearly $x = -1$ is the one and only solution.

## Mathematical Outing ⋆ ⋆ ⋆

Self-referential statements are a recipe for trouble when it comes to logical consistency. Consider

$P_1$) The other statement is true.
$P_2$) The other statement is false.

There is no consistent way to assign a truth value to each of these statements. For if $P_1$ is true then $P_2$ will be true, which means that $P_1$ is false, a contradiction. Likewise, if $P_1$ is false then $P_2$ will be also, implying that $P_1$ is true after all, so we again reach a contradiction.

Self-referential statements can form the basis for entertaining logic puzzles, though. See if you can deduce the unique way of assigning truth values to the following five statements in a logically consistent manner, and consequently determine whether or not I like spaghetti and meatballs.

$P_1$) I like spaghetti and meatballs.
$P_2$) All odd-numbered statements are false.
$P_3$) All even-numbered statements are true.
$P_4$) At least one of $P_2$ or $P_3$ is true.
$P_5$) If $P_1$ is false then $P_2$ is true.

The situation becomes even more exciting for open sentences involving two or more variables, since we can quantify each variable separately. For instance, Bertrand's Postulate states that for all positive integers $n \geq 2$, there exists a prime $p$ between $n$ and $2n$. It is important to understand that these quantifiers are *nested*, meaning that the second one ("there exists a prime $p$") falls within the scope of the first one ("For all positive integers $n \geq 2$"). We first select any permissible value of $n$, say $n = 13$. We are then guaranteed a prime between $n$ and $2n$; in this case, between 13 and 26. Sure enough, such a prime does exist. In fact, there are a total of three such primes, namely $p = 17$, 19, and 23.

**Concept Check** d) Confirm that Bertrand's Postulate holds for $n = 2, 3, \ldots, 10$.

The structure of Bertrand's Postulate is "For all $n \geq 2$, we have $P$," where $P$ itself is the quantified statement "There exists a prime $p$ such that $Q$," and $Q$ is the statement "$p$ is between $n$ and $2n$." Identifying this structure permits us to determine the negation of the result. We begin by writing "There exists an $n \geq 2$ such that $\neg P$." The negation of $P$ is "For all primes $p$ we have $\neg Q$." Finally, the negation of $Q$ is "$p$ is not between $n$ and $2n$." Putting this all together and writing the result smoothly, the negation states that "There exists an integer $n \geq 2$ such that there are no primes $p$ between $n$ and $2n$."

**Quick Query** e) To appreciate how crucial the order of quantifiers can be, consider the claim "There exists a prime $p$ such that for all integers $n \geq 2$ the prime $p$ lies between $n$ and $2n$." Is the new statement true? Why or why not?

The order in which quantifiers appear does not matter when they are of the same type. Thus we could claim that for all real numbers $x$, it is the case that for all real numbers $y$ the quantity $(x-y)^2$ is positive. The meaning would remain unchanged if we were to place the "for all $y$" quantifier ahead of the "for all $x$" quantifier. Because of this fact, the given statement is usually shortened to just "For all real numbers $x$ and $y$, we have $(x-y)^2 > 0$." In the same manner, rather than writing "There exists a positive integer $m$ for which there is a positive integer $n$ such that $m^2+mn+n^2$ is a perfect square," we typically combine the two existential quantifiers to obtain "There exist positive integers $m$ and $n$ such that $m^2+mn+n^2$ is a perfect square."

Finally, note that when the universal quantifier is applied to an implication, it makes sense to consider quantifying the converse of the implication. For instance, we claimed earlier that for all positive integers $n$, if $2^n-1$ is prime then $n$ is also prime. Quantifying the converse would read "For all $n$, if $n$ is prime then $2^n-1$ is also prime." If both of these statements were true, we could conclude that "For all $n$, $2^n-1$ is prime if and only if $n$ is prime." As one of the exercises will reveal, the converse does not hold for all values of $n$, so the statements "$2^n-1$ is prime" and "$n$ is prime" are not equivalent.

**ALL THE ANSWERS** a) There is a positive integer $n$ such that $n^2-n+41$ is not prime.
b) Every real number $x$ satisfies $|x+3|>1$ or $|x-4|>2$.
c) No, it is not always true. If the three points are situated along a line, then there is no circle that passes through all three of them.
d) In each case there is a prime between $n$ and $2n$.
e) The new statement is false; there is no such prime. Thus $p=23$ won't do because 23 does not lie between $n$ and $2n$ for all $n$—certainly not for $n=10$, for instance.
If $P_3$ is true then $P_2$ will be also, which leads to a contradiction. Hence $P_3$ is false. Now if $P_4$ is true then $P_2$ is also, contradicting the fact that $P_3$ is false. Thus $P_4$ is also false, which means that $P_2$ is false. Now suppose that $P_1$ is false. This would make $P_5$ false also, which contradicts $P_2$. So $P_1$ is true, hence $P_5$ is also. When the dust settles, it turns out that I do like spaghetti and meatballs.

## Exercises

54. Determine the most accurate way to quantify each open sentence. Choose from among for all, there exists, there exists a unique, or there does not exist. Rewrite each quantified statement so that it reads nicely.
a) ________ (integer $n$): $n$ contains every odd digit.
b) ________ (real number $x$): $x^2+4x+5=0$.
c) ________ (point $C$): $C$ lies on the lines $y=x$ and $y=3x-5$.
d) ________ (real number $t$): $|t-4|\le 3$ and $|t+5|\le 6$.
e) ________ (integer $k$): the number $6k+5$ is odd.
f) ________ (point $U$): the distance from $U$ to the origin is positive.

55. Find the negation of "There exists a real number $x$ such that $\cos x = 3x$." Employ the universal quantifier in your statement.

56. Determine the negation of “For all positive integers $n$, if $n$ is prime, then $2^n - 1$ is prime.” Then find a value for $n$ that satisfies your negation.

57. Write down the negation of the assertion “If $f(x)$ is a linear function then $f(1) + f(2) = f(3)$.” Now use the linear function $f(x) = 2x + 5$ to show that the negation is true.

58. Decide whether or not it is true that given fixed points $A$ and $B$, there exists a unique square having these points as two of its vertices.

59. Find the value of $a$ for which the equation $(x - 3)(x + 5) = a$ has a unique solution; i.e. is satisfied by a unique real number $x$.

60. Determine the negation of the statement “For all rectangles in the plane there exists a circle inside the rectangle that is tangent to all four sides.”

61. Consider the claim “For all real numbers $x$ there exists a real number $y$ such that $y > x$.” Is this claim true or false? Explain.

62. Now consider the closely related claim “There exists a real number $y$ such that for all real numbers $x$ we have $y > x$.” Is this claim true or false? Explain.

63. Write the negation of “There exists a positive integer $N$ such that for all integers $n > N$ we have $\cos n < 0.99$.”

64. Consider the statement “For all real numbers $x$ and $y$, we have $(x-y)^2 > 0$.” Is this assertion true or false?

65. Show that there exist positive integers $m$ and $n$ such that $m^2 + mn + n^2$ is a perfect square.

66. Let $P$ and $Q$ be open sentences involving a variable $m$. Suppose that it is the case that “There exists an integer $m$ such that $P \Rightarrow Q$,” and it is also true that “There exists an integer $m$ such that $Q \Rightarrow P$.” Does it necessarily follow that the statement “There exists an integer $m$ such that $P \Longleftrightarrow Q$” is true?

## Writing

67. Show that for every positive integer $a$ there exists a positive integer $b$ such that $ab + 1$ is a perfect square.

68. Prove that for all positive real numbers $r$ there exists a rectangle whose area is equal to $r$ and whose perimeter is greater than $4r$.

69. Demonstrate that there exists an infinitely long path in the plane, starting at the origin, such that from any point $(x, y)$ in the plane one can reach the path by moving a total distance of less than one unit.

70. Prove that there exists a polynomial of the form $f(n) = n^2 + bn + c$, where $b$ and $c$ are positive integers, such that $f(n)$ is composite (i.e. not prime) for all positive integers $n$.

71. Explain why given any finite collection of points in the plane there exists a triangle having three of the points as its vertices which contains none of the other points in its interior.

## 1.6 Reference

The purpose of this section is to provide a condensed summary of the most important facts and techniques from this chapter, as a reference when studying or working on material from later chapters.

• *Vocabulary* statement, truth value, open sentence, domain, disjunction, conjunction, truth table, negation, logically equivalent, DeMorgan's Laws, tautology, contradiction, implication, hypothesis, premise, conclusion, biconditional, iff, equivalent statements, converse, inverse, universal/existential quantifier

• *Compound statements* A statement is a mathematical sentence that is either true or false, while an open sentence is an assertion involving one or more variables. Given statements or open sentences $P$, $Q$ we may form their conjunction $P \wedge Q$ ("$P$ and $Q$"), their disjunction $P \vee Q$ ("$P$ or $Q$"), the implication $P \Rightarrow Q$ ("$P$ implies $Q$"), its converse $Q \Rightarrow P$ ("$Q$ implies $P$"), its inverse $\neg P \Rightarrow \neg Q$, and the biconditional $P \Longleftrightarrow Q$ ("$P$ if and only if $Q$").

• *Truth tables* A truth table contains one row for each possible set of truth values of its components. For complicated statements, create several columns to determine the truth value of each part of the statement first. Two statements are logically equivalent if they have identical truth tables. A tautology is a statement that is true in every case while a contradiction is false in every case.

• *Implication* The statement "$P$ implies $Q$" may be expressed as $P$ is sufficient for $Q$, $Q$ whenever $P$, $Q$ follows from $P$, or when $P$ we have $Q$. This implication is true unless $P$ is true while $Q$ is false. The converse is written "$Q$ implies $P$." If an implication and its converse are both true then the component statements are equivalent, meaning that each is true or false exactly when the other is; in this case we say "$P$ if and only if $Q$."

• *Quantified statements* An open sentence contains variables. By inserting a quantifier such as 'For all' (universal quantifier $\forall$) or 'There exists' (existential quantifier $\exists$) or 'There exists a unique' ($\exists!$) or 'There does not exist' ($\nexists$) we obtain a statement. The statement "For all $x$ there exists a $y$ such that..." has a different meaning than "There exists a $y$ such that for all $x$..."

• *Negation* The table below indicates how to negate a variety of statements.

| Statement | Negation |
|---|---|
| $P$ and $Q$ | $\neg P$ or $\neg Q$ |
| $P$ or $Q$ | $\neg P$ and $\neg Q$ |
| if $P$ then $Q$ | $P$ and $\neg Q$ |
| for all $x$ we have $P$ | there exists $x$ such that $\neg P$ |
| there exists $x$ such that $P$ | for all $x$ we have $\neg P$ |

Furthermore, the negation of "For all $x$ there exists a $y$ such that $P$," is written "There exists an $x$ such that for all $y$ we have $\neg P$." Similarly, the negation of the statement "There exists an $x$ such that for all $y$ we have $P$," is written "For all $x$ there exists a $y$ such that $\neg P$.

## Sample Proofs

The following proofs provide concise explanations for results discussed within this chapter. They are meant to serve as an illustration for how proofs of similar statements could be phrased. The boldface numbers indicate the section containing each result; the location of that result within the section is marked by a dagger (†).

**1.1** Show that the following assertion is false: "For all positive integers $n$, it is the case that $n^2 + 2$ is a multiple of 3 or that $3n + 7$ is a perfect square."

*Proof* We exhibit a counterexample to show that the given assertion is false. Taking $n = 9$ we find that $n^2+2 = 83$, which is not a multiple of 3. Furthermore $3n+7 = 34$, which is not a perfect square. Therefore it is not the case that $n^2+2$ is a multiple of 3 or that $3n + 7$ is a perfect square for all positive integers $n$.

**1.2** Show that the statements $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ are logically equivalent.

*Proof* To show that these statements are logically equivalent we construct a truth table for each, shown below.

| $P$ | $Q$ | $P \wedge Q$ | $\neg(P \wedge Q)$ | $\neg P$ | $\neg Q$ | $\neg P \vee \neg Q$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |

Since the truth values for the two statements match in every case, they are logically equivalent.

**1.3** Explain why the implication "For all $x$, if $x^2 = 25$ then $x = 5$" is false.

*Proof* We will demonstrate a value of $x$ for which the hypothesis is true but the conclusion is false. Consider $x = -5$; in this case $x^2 = 25$ holds, but $x = 5$ does not. Therefore this implication is not valid for all real numbers $x$.

**1.4** Determine whether the claim "If $ABCD$ is a square then $ABCD$ has four congruent sides" is true, whether the converse is true, and whether we have a pair a equivalent statements.

*Proof* We know from elementary geometry that a square has four congruent sides, so the given implication is true. The converse asserts that "If $ABCD$ has four congruent sides then $ABCD$ is a square." This claim is false, since a rhombus (diamond) has four congruent sides but is not a square. Therefore the two statements are not equivalent, since they do not each imply the other.