Maxwell Speed Distribution Classwork

Distribution Functions
For \( f(\text{something}) \) that describes the distribution of a particle quantity over something (like the number of particles distributed over energy or location over a span of space), then

\[
f(\text{something}) \, ds\text{something} = \text{Probability of finding a particle between } \text{something and something + dsomething}
\]

\[
f(x) \, dx = \text{Probability of finding a particle with a location between } x \text{ and } x + dx
\]

\[
f(E) \, dE = \text{Probability of finding a particle with energy between } E \text{ and } E + dE
\]

The expectation value of something described by this distribution is

\[
\langle \text{something} \rangle = \int_{-\infty}^{\infty} \text{something} \, f(\text{something}) \, ds\text{omething}
\]

Energy Distributions of Particles
To describe how the energy is distributed among particles in large collections (gases, liquids and solids), physicists developed different energy distributions based on the types of particles

Maxwell-Boltzmann Statistics: Classical Particles
Classical particles are distinguishable, only interact with each other through elastic collisions and are at a low enough density that the wave functions don't overlap.

The Maxwell-Boltzmann factor is

\[
F_{\text{MB}} = Ae^{-\beta E}
\]

Maxwell-Boltzmann Speed Distribution
The speed distribution of classical particles in a gas is

\[
F(v) \, dv = 4\pi \left( \frac{\beta m}{2\pi} \right)^{3/2} v^2 e^{-\frac{\beta m v^2}{2}} \, dv
\]

shown plotted to the right. This can be used to determine the most probable \((v^*)\), mean \((\bar{v})\), and root-mean-square \((v_{\text{rms}})\) speeds. The positions of these speeds are marked on the curve. You are to identify which is which and label them.
**v***: Most probable speed

This is the speed at the peak of the curve, the maximum of the function found by setting the derivative with respect to \( v \) equal to zero. Perform the operation to find \( v^* \).

\[
\frac{dF(v)}{dv} = 4\pi \left( \frac{\beta m}{2\pi} \right)^{3/2} \frac{d}{dv} \left( v^2 e^{-\frac{v^2}{2\beta m}} \right) \bigg|_{v=v^*} = 0
\]

\[
\left( 2v - \frac{v^2}{2} \beta m (2v) \right) e^{-\frac{v^2}{2\beta m}} \bigg|_{v=v^*} = \left( 2 - (v^*)^2 \right) \beta m = 0
\]

\[
v^* = \sqrt{2} \frac{2}{\beta m} = \sqrt{\frac{2kT}{m}}
\]

**\( \overline{v} \)**: Mean speed

This is the expectation value of the velocity (2), that can be solved using TZDII p. 683 (I3)

\[
\overline{v} = \int_0^\infty (v) F(v) dv
\]

\[
\overline{v} = 4\pi \left( \frac{\beta m}{2\pi} \right)^{3/2} \int_0^\infty (v)v^2 e^{-\frac{v^2}{2\beta m}} dv = 4\pi \left( \frac{\beta m}{2\pi} \right)^{3/2} \int_0^\infty v^3 e^{-\frac{v^2}{2\beta m}} dv
\]

\[
\overline{v} = 4\pi \left( \frac{\beta m}{2\pi} \right)^{3/2} \left( \frac{2}{\beta^2 m^2} \right) = \frac{4}{\sqrt{2\pi}\beta m} = \frac{4}{\sqrt{2\pi}} \sqrt{\frac{kT}{m}}
\]

**\( v_{rms} \)**: Root-mean-square speed: The Physicist’s Average!

\[
v_{rms}^2 = \langle v^2 \rangle = 4\pi \left( \frac{\beta m}{2\pi} \right)^{3/2} \int_0^\infty v^2 e^{-\frac{v^2}{2\beta m}} dv
\]

\[
v_{rms}^2 = 4\pi \left( \frac{\beta m}{2\pi} \right)^{3/2} \left[ \frac{32\pi}{8\beta^5 m^5} \right] = \frac{3}{2} \sqrt{\frac{\beta^3 m^3}{8\pi^3 \beta^5 m^5}} = \frac{3}{2} \beta m
\]

\[
v_{rms} = \sqrt{\frac{3}{\beta m}} = \sqrt{\frac{3kT}{m}}
\]

This is the average preferred by physicists because it gives the familiar mean kinetic energy

\[
\overline{K} = \langle \frac{1}{2}mv^2 \rangle = \frac{3}{2} kT
\]
When \( |z| \ll 1 \), one obtains a good approximation using just the first one or two terms of these five series. In particular, with \( |z| \ll 1 \), the binomial series reduces to the binomial approximation:

\[
(1 + z)^n \approx 1 + nz
\] 

**(binomial approximation)**

### SOME INTEGRALS

Integrals of the form

\[
I_n = \int_0^\infty x^n e^{-\lambda x^2} \, dx
\]

where \( \lambda \) is a positive number, occur frequently in several branches of physics. When \( n \) is a positive integer, their value can be found from the following:

\[
I_0 = \frac{\pi}{4\sqrt{\lambda}}, \quad I_1 = \frac{1}{2\sqrt{\lambda}}, \quad I_2 = \frac{\pi}{16\lambda^{3/2}}, \quad I_3 = \frac{1}{2\lambda^2}, \quad I_4 = \frac{3}{8\sqrt{\lambda^5}}
\]

and

\[
I_n = -\frac{dI_{n-2}}{d\lambda}
\]

Notice that the integral \( \int_0^\infty x^n e^{-\lambda x^2} \, dx \) equals \( 2I_n \) when \( n \) is even, but is zero if \( n \) is odd.

Another common integral is the indefinite integral

\[
J_n = \int x^n e^{-\frac{x}{b}} \, dx
\]

When \( n \) is a small integer, this is easily evaluated by parts, for example,

\[
J_0 = -be^{-\frac{x}{b}}, \quad J_1 = -(b^2 + bx)e^{-\frac{x}{b}}, \quad J_2 = -(2b^3 + 2b^2x + bx^2)e^{-\frac{x}{b}}
\]

In general,

\[
J_{n+1} = b^2 \frac{\partial J_n}{\partial b}
\]

Note, in particular, that

\[
\int_0^\infty e^{-\frac{x}{b}} \, dx = b, \quad \int_0^\infty xe^{-\frac{x}{b}} \, dx = b^2, \quad \int_0^\infty x^2 e^{-\frac{x}{b}} \, dx = 2b^3
\]

and also that

\[
\int_0^\infty \sqrt{x} e^{-\frac{x}{b}} \, dx = \frac{\sqrt{\pi}b^2}{2}
\]

### Plus the generalized form of the penultimate row of integrals

\[
\int_0^\infty x^n e^{-\frac{x}{b}} \, dx = n!b^{(n-1)}
\]