FOR THE GENERAL 3-D SCHRÖDINGER EQUATION

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Psi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}=\left[\frac{1}{a_B^2n^2}-\frac{2}{a_Br}\right]\Psi$$

SHOW $\Psi_{2,1,-1}(r,\theta,\phi)$ IS A SOLUTION FOR $n=2,\ l=1,\ m=-1$.

a) FROM TZDII's TABLES

$$\Phi_{-1}(\phi)=e^{-i\phi} \qquad \text{(p. 265)}$$

$$\Theta_{1,-1}(\theta)=+\sqrt{\frac{3}{8\pi}}\sin\theta \qquad \text{(p. 269)}$$

$$R_{2,1}(r)=\frac{1}{\sqrt{24a_B^3}}\frac{r}{a_B}e^{-\frac{r}{2a_B}} \qquad \text{(p. 279)}$$

THUS

$$\Psi_{2,1,-1}(r,\theta,\phi)=\sqrt{\left(\frac{1}{\underset{8}{24}a_B^3}\right)\left(\frac{3}{8\pi}\right)}\frac{r}{a_B}\sin\theta\, e^{-\frac{r}{2a_B}}e^{-i\phi}$$

$$\Psi_{2,1,-1}(r,\theta,\phi)=\sqrt{\frac{1}{64a_B^5\pi}}\,r\sin\theta\, e^{-\left(\frac{r}{2a_B}+i\phi\right)}$$

$$\boxed{\Psi_{2,1,-1}(r,\theta,\phi)=\frac{1}{8}\sqrt{\frac{1}{a_B^5\pi}}\,r\sin\theta\, e^{-\left(\frac{r}{2a_B}+i\phi\right)}} \qquad (1)$$

b) SHOW THIS SOLVES THE SCHRODINGER EQUATION FOR 2, 1, -1

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Psi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}=\left[\frac{1}{4a_B^2}-\frac{2}{a_Br}\right]\Psi$$

TREAT THIS TERM BY TERM. $A=\frac{1}{8}\sqrt{\frac{1}{a_B^5\pi}}$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Psi)=\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(Ar^2\sin\theta\, e^{-\left(\frac{r}{2a_B}+i\phi\right)}\right)$$

$$=\frac{A}{r}\sin\theta\, e^{-i\phi}\frac{\partial^2}{\partial r^2}\left(r^2e^{-\frac{r}{2a_B}}\right)$$

$$=\frac{A}{r}\sin\theta\, e^{-i\phi}\frac{\partial}{\partial r}\left(2r-\frac{r^2}{2a_B}\right)e^{-\frac{r}{2a_B}}$$

$$=\frac{A}{r}\sin\theta\, e^{-i\phi}\left[\left(2-\frac{r}{a_B}\right)-\left(2r-\frac{r^2}{2a_B}\right)\frac{1}{2a_B}\right]e^{-\frac{r}{2a_B}}$$

$$=\frac{A}{r}\sin\theta\, e^{-i\phi}\left[2-\frac{r}{a_B}-\frac{r}{a_B}+\frac{r^2}{4a_B^2}\right]e^{-\frac{r}{2a_B}}$$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Psi)=A\sin\theta\, e^{-i\phi}\left(\frac{2}{r}-\frac{2}{a_B}+\frac{r}{4a_B^2}\right)e^{-\frac{r}{2a_B}} \qquad (2)$$

THE SECOND TERM:

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}Ar\sin\theta\, e^{-\left(\frac{r}{2a_B}+i\phi\right)}\right]$$

$$= \frac{A}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\sin\theta\right]e^{-\left(\frac{r}{2a_B}+i\phi\right)}$$

$$= \frac{A}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\cos\theta\right]e^{-\left(\frac{r}{2a_B}+i\phi\right)}$$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) = \frac{A}{r\sin\theta}\left(\cos^2\theta-\sin^2\theta\right)e^{-\left(\frac{r}{2a_B}+i\phi\right)} \qquad (3)$$

THE THIRD TERM:

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\Psi = \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\left[Ar\sin\theta\, e^{-\frac{r}{2a_B}}e^{-i\phi}\right]$$

$$= \frac{A}{r\sin\theta}e^{-\frac{r}{2a_B}}\frac{\partial^2}{\partial\phi^2}\left(e^{-i\phi}\right)$$

$$= \frac{A}{r\sin\theta}e^{-\frac{r}{2a_B}}\frac{\partial}{\partial\theta}\left(-ie^{-i\phi}\right)$$

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\Psi = \frac{-A}{r\sin\theta}e^{-\frac{r}{2a_B}}e^{-i\phi} \qquad (4)$$

PUTTING THESE INTO (1) GIVES

$$\cancel{A}\,\cancel{e^{-\frac{r}{2a_B}}}\,\cancel{e^{-i\phi}}\left\{\sin\theta\left(\frac{2}{r}-\frac{2}{a_B}+\frac{r}{4a_B^2}\right)+\frac{\cos^2\theta-\sin^2\theta}{r\sin\theta}-\frac{1}{r\sin\theta}\right\} =$$

$$= \left[\frac{1}{4a_B^2}-\frac{2}{a_B r}\right]\cancel{A}r\sin\theta\,\cancel{e^{-\frac{r}{2a_B}}}\,\cancel{e^{-i\phi}}$$

MULTIPLY THROUGH BY $\frac{r}{\sin\theta}$

$$2-\cancel{\frac{2r}{a_B}}+\cancel{\frac{r^2}{4a_B^2}}+\frac{(1-\sin^2\theta)-\sin^2\theta}{\sin^2\theta}-\frac{1}{\sin^2\theta} = \cancel{\frac{r^2}{4a_B^2}}-\cancel{\frac{2r}{a_B}}$$

$$2-\frac{(1-\sin^2\theta)-\sin^2\theta}{\sin^2\theta}-\frac{1}{\sin^2\theta} = 2-\cancel{\frac{1}{\sin^2\theta}}-\frac{2\sin^2\theta}{\sin^2\theta}-\cancel{\frac{1}{\sin^2\theta}}$$

$$\boxed{2-2=0} \quad \text{QED!}$$

So $\Psi_{2,1,-1}(r,\theta,\phi) = \frac{1}{8}\sqrt{\frac{1}{a_B^5\pi}}\,r\sin\theta\, e^{-\frac{r}{2a_B}}e^{-i\phi}$ IS A SOLUTION!