8.22) SUBSTITUTE $\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$ INTO (8.49)

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = \frac{2M}{\hbar^2}\left[U(r) - E\right]\psi$$

a) DERIVE $\Phi''(\phi) = -m^2\Phi$ (8.51)

b) DERIVE

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = (\text{FUNCTION OF } r)$$

AND EXPLAIN WHY EACH SIDE MUST EQUAL $-k$, A CONSTANT.

c) DERIVE

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(k - \frac{m^2}{\sin^2\theta}\right)\Theta = 0 \qquad (8.53)$$

AND

$$\frac{d^2}{dr^2}(rR) = \frac{2M}{\hbar^2}\left[U(r) + \frac{k\hbar^2}{2mr^2} - E\right](rR) \qquad (8.54)$$

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a) TREAT EACH TERM SEPARATELY

$$\frac{1}{r}\frac{\partial}{\partial r^2}(rR\Theta\Phi) = \Theta\Phi\frac{1}{r}\frac{\partial}{\partial r^2}(rR)$$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}R\Theta\Phi\right) = \frac{R\Phi}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)$$

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}R\Theta\Phi = \frac{R\Theta}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\Phi$$

MULTIPLY EACH BY $\frac{r^2\sin^2\theta}{R\Theta\Phi}$

$$\frac{r^{\cancel{2}}\sin^2\theta}{R\cancel{\Theta}\cancel{\Phi}}\left[\frac{\cancel{\Theta}\cancel{\Phi}}{\cancel{r}}\frac{\partial}{\partial r^2}(rR)\right] = \frac{r\sin^2\theta}{R}\frac{\partial^2}{\partial r^2}(rR)$$

$$\frac{\cancel{r^2}\sin^{\cancel{2}}\theta}{\cancel{R}\Theta\cancel{\Phi}}\left[\frac{\cancel{R}\cancel{\Phi}}{\cancel{r^2}\cancel{\sin\theta}}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)\right] = \frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)$$

$$\frac{\cancel{r^2\sin^2\theta}}{\cancel{R}\cancel{\Theta}\Phi}\left[\frac{\cancel{R}\cancel{\Theta}}{\cancel{r^2\sin^2\theta}}\frac{\partial^2}{\partial\phi^2}\Phi\right] = \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}$$

THE RIGHT HAND SIDE BECOMES

$$\frac{r^2\sin^2\theta}{R\Theta\Phi}\left\{\frac{2M}{\hbar^2}\left[U(r) - E\right]R\Theta\Phi\right\} = \frac{2M}{\hbar^2}\left[U(r) - E\right]r^2\sin^2\theta$$

$\longrightarrow$

PUTTING THESE TOGETHER,

$$\frac{r\sin^2\theta}{R}\frac{\partial^2}{\partial r^2}(rR)+\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\Theta}{\partial\theta}\right]+\frac{1}{\Phi}\frac{\partial\Phi}{\partial\phi}=\frac{2M}{\hbar^2}\left[U(r)-E\right]r^2\sin^2\theta$$

ISOLATING THE Φ TERM ON THE LEFT GIVES

$$\underbrace{\frac{1}{\Phi}\frac{\partial\Phi}{\partial\phi}}_{\text{FUNCTION OF } \phi}=\underbrace{\frac{2M}{\hbar^2}\left[U(r)-E\right]r^2\sin^2\theta-\frac{r\sin^2\theta}{R}\frac{\partial^2}{\partial r^2}(rR)-\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\Theta}{\partial\theta}\right]}_{\text{FUNCTIONS OF } r \,\&\, \theta}$$

SINCE THE SIDES ARE FUNCTIONS OF DIFFERENT VARIABLES, EACH MUST EQUAL A CONSTANT, $-m^2$

$$\Rightarrow \frac{1}{\Phi}\frac{\partial\Phi}{\partial\phi}=-m^2 \Rightarrow \boxed{\Phi''=-m^2\Phi} \text{ THE } \Phi \text{ EQUATION}$$

b) SETTING THE RIGHT HAND SIDE EQUAL TO $-m^2$, DIVIDING THROUGH BY $\sin^2\theta$ GIVES

$$\underbrace{r^2\frac{2M}{\hbar^2}\left[U(r)-E\right]-\frac{r}{R}\frac{\partial^2}{\partial r^2}(rR)}_{\text{FUNCTION OF } r}=\underbrace{-\frac{m^2}{\sin^2\theta}+\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\Theta}{\partial\theta}\right]}_{\text{FUNCTION OF } \theta}$$

AGAIN, EACH SIDE IS A FUNCTION OF A DIFFERENT VARIABLE, SO BOTH MUST EQUAL A CONSTANT, $-k$

c) REARRANGING THE Θ EQUATION GIVES

$$\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\Theta}{\partial\theta}\right]-\frac{m^2}{\sin^2\theta}=-k$$

AND

$$\boxed{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\Theta}{\partial\theta}\right]+\left(k-\frac{m^2}{\sin^2\theta}\right)\Theta=0} \qquad (8.53)$$

THE r SIDE BECOMES

$$r^2\frac{2M}{\hbar^2}\left[U(r)-E\right]-\frac{r}{R}\frac{\partial^2}{\partial r^2}(rR)=-k$$

OR

$$\frac{\partial^2}{\partial r^2}(rR)=\frac{2M}{\hbar^2}\left[U(r)-E\right](rR)+\frac{kR}{r}$$

OR

$$\boxed{\frac{\partial^2}{\partial r^2}(rR)=\frac{2M}{\hbar^2}\left[U(r)+\frac{k\hbar^2}{2Mr^2}-E\right](rR)} \qquad (8.54)$$

QED!