8.47) a) Show that $\Theta_{1,\pm1} = \sin\theta$ is a solution for the 2p states ($\ell = 1$, $m = \pm 1$)

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right)\Theta \qquad (8.65)$$

b) Show that the sum of the wave functions

$$\psi_{2,1,\pm1}(r,\theta,\phi) = R_{2p}(r)\sin\theta\, e^{\pm i\phi}$$

is the $2p_x$ state, whereas the difference is $2ip_y$.

a) For $\ell = 1$, $m = \pm 1$, the DE becomes $[m^2 = (\pm1)^2 = 1]$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(2 - \frac{1}{\sin^2\theta}\right)\Theta = 0$$

For $\Theta = \sin\theta$, $\Theta' = \cos\theta$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\cos\theta\right) + \left(2\sin\theta - \frac{1}{\sin\theta}\right) = 0$$

$$\frac{1}{\sin\theta}\left(\cos^2\theta - \sin^2\theta\right) + \left(2\sin\theta - \frac{1}{\sin\theta}\right) = 0$$

Multiplying through by $\sin\theta$

$$\cos^2\theta - \sin^2\theta + 2\sin^2\theta - 1 = 0$$

$$\cos^2\theta + \sin^2\theta - 1 = 0$$

$$1 - 1 = 0$$ QED! $\sin\theta$ is a solution!

b) Take the sum of $\psi_{2,1,\pm1} = R_{2p}(r)\sin\theta\, e^{\pm i\phi}$

Table 8.2 has $R_{2p}(r) = \frac{1}{\sqrt{24a_B^3}}\rho e^{-\rho/2} = \frac{r}{\sqrt{24a_B^5}} e^{-r/2a_B}$

Thus

$$\psi_{2,1,\pm1} = \frac{1}{\sqrt{24a_B^5}} r\sin\theta\, e^{-r/2a_B} e^{\pm i\phi}$$

Taking the sum

$$\psi_{2,1,1} + \psi_{2,1,-1} = \frac{1}{\sqrt{24a_B^5}} e^{-r/2a_B} r\sin\theta\left(e^{i\phi} + e^{-i\phi}\right)$$

$\longrightarrow$

8.47) CONTINUED

NOTE THAT $\cos\phi = \frac{1}{2}\left(e^{i\phi} + e^{-i\phi}\right)$

THUS

$$\psi_{2,1,1} + \psi_{2,1,-1} = \frac{1}{\sqrt{24 a_B^5}}\, e^{-r/2a_B}\, \underbrace{2r\sin\theta\cos\phi}_{x = r\sin\theta\cos\phi\,!}$$

$$\boxed{\psi_{2,1,1} + \psi_{2,1,-1} = \frac{1}{\sqrt{24 a_B^5}}\, e^{-r/2a_B}\,(2x) = 2\psi_{2x}}$$ THE SUM GIVES $2\psi_x$

TAKING THE DIFFERENCE

$$\psi_{2,1,1} - \psi_{2,1,-1} = \frac{1}{\sqrt{24 a_B^5}}\, e^{-r/2a_B}\, r\sin\theta\left(e^{i\phi} - e^{-i\phi}\right)$$

NOTING THAT $\sin\phi = \frac{1}{2i}\left(e^{i\phi} - e^{-i\phi}\right)$

$$\psi_{2,1,1} - \psi_{2,1,-1} = \frac{1}{\sqrt{24 a_B^5}}\, e^{-r/2a_B}\, 2i\,\underbrace{r\sin\theta\sin\phi}_{y = r\sin\theta\sin\phi\,!}$$

$$\boxed{\psi_{2,1,1} - \psi_{2,1,-1} = \frac{1}{\sqrt{24 a_B^5}}\, e^{-r/2a_B}\,(2iy) = 2\psi_{2y}}$$ THE DIFFERENCE GIVES $2\psi_y$

THUS, FOLLOWING TZD II'S DISCUSSION ON PAGES 276 & 277, THE 2p WAVE FUNCTIONS ARE

$$\left.\begin{aligned} \psi_{2p_z} &= A z\, e^{-r/2a_B} \\ \psi_{2p_y} &= A(2iy)\, e^{-r/2a_B} \\ \psi_{2p_x} &= A(2x)\, e^{-r/2a_B} \end{aligned}\right\}$$

THOUGH $|\psi|^2$ IN EACH CARTESIAN COORDINATE DIRECTION HAS TWO SYMMETRIC MAXIMA ON EACH AXIS (e.g. FIG 8.20), THE OVERALL PROBABILITY DISTRIBUTION IN ALL COORDINATE DIRECTIONS IS SPHERICALLY SYMMETRIC. THIS IS EXPECTED FOR A PURELY RADIAL POTENTIAL.