

FOR THE GENERAL 3-D SCHRÖDINGER EQUATION

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} (r\psi) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial r} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2} = \left[\frac{1}{a_B^2 n^2} - \frac{Z}{a_B r} \right] \psi$$

SHOW $\Psi_{2,1,-1}(r,\theta,\phi)$ IS A SOLUTION FOR $n=2, \ell=1, m=-1$.

a) FROM T&D II'S TABLES

$$\Phi_{-1}(\phi) = e^{-i\phi} \quad (\text{p. 265})$$

$$\Theta_{1,-1}(\theta) = +\sqrt{\frac{3}{8\pi}} \sin\theta \quad (\text{p. 269})$$

$$R_{2,1}(r) = \frac{1}{\sqrt{24} a_B^3} \frac{r}{a_B} e^{-\frac{r}{2a_B}} \quad (\text{p. 279})$$

THUS

$$\Psi_{2,1,-1}(r,\theta,\phi) = \left[\left(\frac{1}{24 a_B^3} \right) \left(\frac{3}{8\pi} \right) \right]^{1/2} \frac{r}{a_B} \sin\theta e^{-\frac{r}{2a_B}} e^{-i\phi}$$

$$\Psi_{2,1,-1}(r,\theta,\phi) = \sqrt{\frac{1}{64 a_B^5 \pi}} r \sin\theta e^{-\left(\frac{r}{2a_B} + i\phi\right)}$$

$$\boxed{\Psi_{2,1,-1}(r,\theta,\phi) = \frac{1}{8} \sqrt{\frac{1}{a_B^5 \pi}} r \sin\theta e^{-\left(\frac{r}{2a_B} + i\phi\right)}} \quad (1)$$

b) SHOW THIS SOLVES THE SCHRÖDINGER EQUATION FOR $2, 1, -1$

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} (r\psi) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial r} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2} = \left[\frac{1}{4a_B^2} - \frac{Z}{a_B r} \right] \psi$$

TREAT THIS TERM BY TERM: $A = \frac{1}{8} \sqrt{\frac{1}{a_B^5 \pi}}$

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} (r\psi) = \frac{1}{r} \frac{\partial^2}{\partial r^2} \left(A r^2 \sin\theta e^{-\left(\frac{r}{2a_B} + i\phi\right)} \right)$$

$$= \frac{A}{r} \sin\theta e^{-i\phi} \frac{\partial^2}{\partial r^2} \left(r^2 e^{-\frac{r}{2a_B}} \right)$$

$$= \frac{A}{r} \sin\theta e^{-i\phi} \frac{\partial}{\partial r} \left(2r - \frac{r^2}{2a_B} \right) e^{-\frac{r}{2a_B}}$$

$$= \frac{A}{r} \sin\theta e^{-i\phi} \left[\left(2 - \frac{r}{a_B} \right) - \left(2r - \frac{r^2}{2a_B} \right) \frac{1}{2a_B} \right] e^{-\frac{r}{2a_B}}$$

$$= \frac{A}{r} \sin\theta e^{-i\phi} \left[2 - \frac{r}{a_B} - \frac{r}{a_B} + \frac{r^2}{4a_B^2} \right] e^{-\frac{r}{2a_B}}$$

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} (r\psi) = A \sin\theta e^{-i\phi} \left(\frac{2}{r} - \frac{2}{a_B} + \frac{r}{4a_B^2} \right) e^{-\frac{r}{2a_B}} \quad (2)$$

THE SECOND TERM:

$$\begin{aligned} \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Psi}{\partial \theta} \right) &= \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \frac{\partial}{\partial \theta} A r \sin \theta e^{-\left(\frac{r}{2a_B} + i\phi\right)} \right] \\ &= \frac{A}{r \sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \frac{\partial}{\partial \theta} \sin \theta \right] e^{-\left(\frac{r}{2a_B} + i\phi\right)} \\ &= \frac{A}{r \sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \cos \theta \right] e^{-\left(\frac{r}{2a_B} + i\phi\right)} \\ \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Psi}{\partial \theta} \right) &= \frac{A}{r \sin \theta} (\cos^2 \theta - \sin^2 \theta) e^{-\left(\frac{r}{2a_B} + i\phi\right)} \quad (3) \end{aligned}$$

THE THIRD TERM:

$$\begin{aligned} \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Psi}{\partial \phi^2} &= \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \left[A r \sin \theta e^{-\frac{r}{2a_B}} e^{-i\phi} \right] \\ &= \frac{A}{r \sin \theta} e^{-\frac{r}{2a_B}} \frac{\partial^2}{\partial \phi^2} (e^{-i\phi}) \\ &= \frac{A}{r \sin \theta} e^{-\frac{r}{2a_B}} \frac{\partial}{\partial \theta} (-ie^{-i\phi}) \\ \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Psi}{\partial \phi^2} &= \frac{-A}{r \sin \theta} e^{-\frac{r}{2a_B}} e^{-i\phi} \quad (4) \end{aligned}$$

PUTTING THESE INTO (1) GIVES

$$\begin{aligned} A e^{-\frac{r}{2a_B}} e^{-i\phi} \left\{ \sin \theta \left(\frac{2}{r} - \frac{2}{a_B} + \frac{1}{4a_B^2} \right) + \frac{\cos^2 \theta - \sin^2 \theta}{r \sin \theta} - \frac{1}{r \sin \theta} \right\} &= \\ &= \left[\frac{1}{4a_B^2} - \frac{2}{a_B r} \right] A r \sin \theta e^{-\frac{r}{2a_B}} e^{-i\phi} \end{aligned}$$

MULTIPLY THROUGH BY $\frac{r}{\sin \theta}$

$$2 - \frac{2r}{a_B} + \frac{r^2}{4a_B^2} + \frac{(1 - \sin^2 \theta) - \sin^2 \theta}{\sin^2 \theta} - \frac{1}{\sin^2 \theta} = \frac{r^2}{4a_B^2} - \frac{2r}{a_B}$$

$$2 - \frac{(1 - \sin^2 \theta) - \sin^2 \theta}{\sin^2 \theta} - \frac{1}{\sin^2 \theta} = 2 - \frac{1}{\sin^2 \theta} - \frac{2 \sin^2 \theta}{\sin^2 \theta} - \frac{1}{\sin^2 \theta}$$

$$\boxed{2 - 2 = 0} \quad \text{QED!}$$

So $\Psi_{2,1,-1}(r, \theta, \phi) = \frac{1}{8} \sqrt{\frac{1}{a_B^5 \pi}} r \sin \theta e^{-\frac{r}{2a_B}} e^{-i\phi}$ IS A SOLUTION!