9.22) a) SHOW THE ENERGY SEPARATION OF THE TWO 2p LEVELS OF H DUE TO THE FINE STRUCTURE CAN BE WRITTEN AS

$$\Delta E_{FS} = \frac{m_e (k e^2)^4}{32 \hbar^4 c^2} \qquad (9.38)$$

b) SHOW THIS CAN BE WRITTEN AS

$$\Delta E_{FS} = \frac{\alpha^2 E_R}{16} \quad \text{FOR} \quad \alpha = \frac{k e^2}{\hbar c} \qquad (9.39 \text{ \& } 9.40)$$

c) SHOW THAT $\alpha \times \frac{1}{137}$ SHOWING THE FINE STRUCTURE IS A VERY SMALL EFFECT.

a) THE ENERGY SEPARATION OF THE TWO 2p LEVELS IS

$$\Delta E_{FS} = 2 \mu_B B_{FS} \qquad (\text{BEFORE } 9.32)$$

SUBSTITUTE

$$B = \frac{\mu_0}{4\pi} \frac{e \hbar}{m_e r_p^3} \quad \text{AND} \quad \mu_B = \frac{e \hbar}{2 m_e} \qquad (9.37 \text{ \& } 9.17)$$

$$\Delta E_{FS} = 2 \left(\frac{e \hbar}{2 m_e} \right) \left(\frac{\mu_0}{4\pi} \frac{e \hbar}{m_e r_p^3} \right) = \frac{\mu_0 e^2 \hbar^2}{4\pi m_e^2 r_p^3}$$

TAKING THE APPROXIMATE VALUE $\hbar = 2\hbar$ (SEE PR 9.21), $r_p = 4 a_B$ AND

$$\mu_0 = \frac{1}{\epsilon_0 c^2} = \frac{4\pi k}{c^2}$$

$$\Rightarrow \quad \Delta E_{FS} = \left(\frac{4\pi k}{c^2} \right) \frac{e^2 \hbar^2}{4\pi m_e^2} \left(\frac{1}{4 a_B} \right)^3 (2\hbar) = \frac{k e^2 \hbar^2}{32 m_e^2 c^2 a_B^3}$$

SUBSTITUTING

$$a_B = \frac{\hbar^2}{k e^2 m_e}$$

GIVES

$$\Delta E_{FS} = \frac{k e^2 \hbar^2}{32 m_e^2 c^2} \left(\frac{k^3 e^6 m_e^3}{\hbar^6} \right) = \frac{k^4 e^8 m_e}{32 c^2 \hbar^4}$$

THUS

$$\boxed{\Delta E_{FS} = \frac{m_e (k e^2)^4}{32 \hbar^4 c^2}} \quad \text{QED}$$

NOW LET'S EVALUATE THIS! →

9.22) CONTINUED

b) SUBSTITUTE $\alpha = \frac{ke^2}{\hbar c}$ AND $E_R = \frac{m_e (ke^2)^2}{2\hbar^2}$ (5.22)

$$\Delta E_{FS} = \frac{m_e (ke^2)^2}{2\hbar^2}\left(\frac{(ke^2)^2}{16\hbar^2 c^2}\right) = E_R\left(\frac{(ke^2)^2}{16\hbar^2 c^2}\right)$$

AND

$$\Delta E_{FS} = \frac{E_R}{16}\left[\frac{(ke^2)^2}{\hbar^2 c^2}\right] = \boxed{\frac{\alpha^2 E_R}{16} = \Delta E_{FS}} \quad \text{QED!}$$

∴ SO WHAT'S α?

c) EVALUATING α,

$$\alpha = \frac{ke^2}{\hbar c} = \frac{1.44\ eV\cdot nm}{197\ eV\cdot nm} \simeq \frac{1}{136.805}$$

THUS

$$\boxed{\alpha \approx \frac{1}{137}}$$ AND THE FINE STRUCTURE IS TINY

$$\Rightarrow \Delta E_{FS} \approx \frac{E_R}{16(137)^2} = \frac{E_R}{300{,}304}.$$ ITTY BITTY!