Homework Set 14: Plane Pendulum

Due: Wednesday, October 31, 2022

Problem From AOD

1) Write out in **explicit detail** the mathematical steps between TM5's equations 4.28 and 4.29. In particular

a) Expand expression 4.28c to show the three elliptical integrals and show the steps in determing their solutions using TM5's Appendix E.3, pp. 615 & 616 (the Γ functions).

b) To go from expression 4.28e to 4.29, expand k = sin(θ/2) as a series using TM5's Appendix D.3 on p. 610. Keep terms with powers up to θ^4 in k, k^2 and k^4. Substitute into 4.28c and show the arithmetic needed to derive 4.29.

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from which

$$\tau = 4\sqrt{\frac{l}{g}}\int_0^1 [(1-z^2)(1-k^2z^2)]^{-1/2}dz \qquad \textbf{(4.28)}$$

Numerical values for integrals of this type can be found in various tables.

For oscillatory motion to result, $|\theta_0| < \pi$, or, equivalently, $\sin(\theta_0/2) = k$, where $-1 < k < +1$. For this case, we can evaluate the integral in Equation 4.28 by expanding $(1 - k^2z^2)^{-1/2}$ in a power series:

$$(1-k^2z^2)^{-1/2} = 1 + \frac{k^2z^2}{2} + \frac{3k^4z^4}{8} + \cdots \qquad \textbf{(4.28b)}$$

Then, the expression for the period becomes

$$\tau = 4\sqrt{\frac{l}{g}}\int_0^1 \frac{dz}{(1-z^2)^{1/2}}\left[1 + \frac{k^2z^2}{2} + \frac{3k^4z^4}{8} + \cdots\right] \qquad \textbf{(4.28c)}$$

$$= 4\sqrt{\frac{l}{g}}\left[\frac{\pi}{2} + \frac{k^2}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{3k^4}{8}\cdot\frac{3}{8}\cdot\frac{\pi}{2} + \cdots\right] \qquad \textbf{(4.28d)}$$

$$= 2\pi\sqrt{\frac{l}{g}}\left[1 + \frac{k^2}{4} + \frac{9k^4}{64} + \cdots\right] \qquad \textbf{(4.28e)}$$

If $|k|$ is large (i.e., near 1), then we need many terms to produce a reasonably accurate result. But for small k, the expansion converges rapidly. And because $k = \sin(\theta_0/2)$, then $k \cong (\theta_0/2) - (\theta_0^3/48)$; the result, correct to the fourth order, is

$$\tau \cong 2\pi\sqrt{\frac{l}{g}}\left[1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4\right] \qquad \textbf{(4.29)}$$