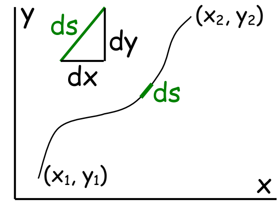
Homework Set 16: Euler's Equation
Due Friday, November 3, 2023

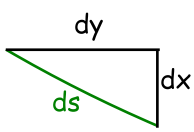
Problems From TM5



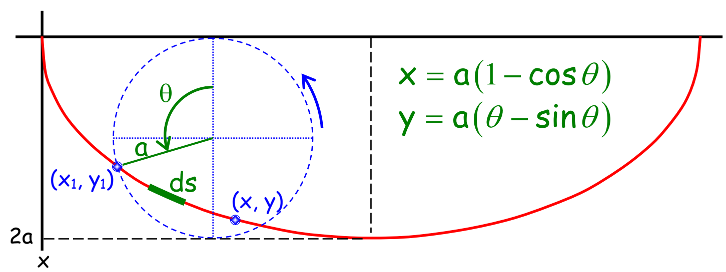
1) *6-2* Show that the shortest distance between two points on a plane is a straight line. HINT: Use the diagram shown to write the segment ds as a function of dx and dy then change to a function of dx and $(dy/dx)^2 = (y')^2$. The length of the line is then the integral of this from x_1 to x_2. The integrand is then f(y, y′, x) that Euler's equation applies to.

2) *6-6* Reexamine the problem of the brachistochrone (Example 6.2) and show that the time required for a particle to move (frictionlessly) to the *minimum* point of the cycloid, *independent* of the starting point, is

$$t_{\text{to minimum}} = \pi\sqrt{\frac{a}{g}}$$



$$ds = \sqrt{dx^2 + dy^2}$$



HINT: Use the geometry shown and the expression for v as in Example 6.2 (with dist = x – x_1 so v = $\sqrt{[2g(x-x_1)]}$) and derive the expression for the travel time from x_1 to an intermediate x (it'll just be an integral very much like 6.20 (below) with limits from x_1 to x = 2a … the upper limit, but the position at the bottom of the curve). You won't apply Euler's equation, you will integrate to find the time. As in Example 6.2, in the integrand, let x = a (1- cosθ) and y = a(θ – sinθ) noting that y′ = (dy/dx) = (dy/dθ)(dθ/dx) where taking (d/dx) of the entire equation x = a(1 – cosθ) gives you dθ/dx = [1/(asinθ)] . Then, after deriving and substituting

$$1+(y')^2 = \frac{2}{1+\cos\theta},\quad x - x_1 = a(\cos\theta_1 - \cos\theta) \text{ and } dx = a\sin\theta\, d\theta$$

into the integrand and re-evaluating the limits, you will get (SHOW THE STEPS, DON'T JUST BELIEVE ME!!)

$$t = \mp\sqrt{\frac{a}{g}}\int_{\theta_1}^{\pi}\sqrt{\frac{1-\cos\theta}{\cos\theta_1 - \cos\theta}}\, d\theta$$

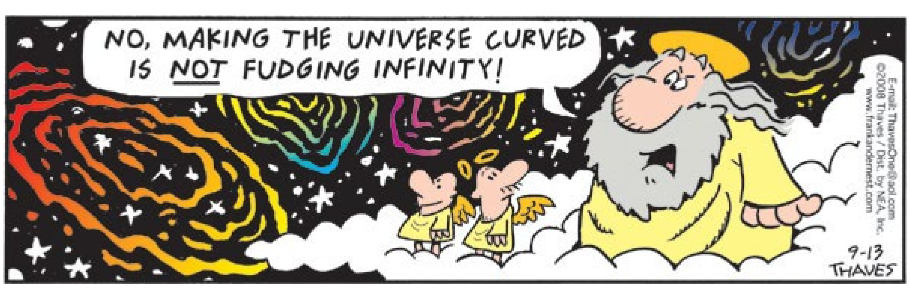
where θ_1 is the starting point (at x_1, y_1) and, thus, a constant. Use a trig substitutions cos(θ) =[$2\cos^2(\frac{1}{2}\theta)$ – 1] and $2\sin^2(\frac{1}{2}\theta)$ = [1-cos(θ)] to get

$$t = \pm\sqrt{\frac{a}{g}}\int_{\theta_1}^{\pi}\sqrt{\frac{\sin^2\left(\frac{\theta}{2}\right)}{\cos^2\left(\frac{\theta_1}{2}\right) - \cos^2\left(\frac{\theta}{2}\right)}}\, d\theta$$

Then let z = $\cos(\frac{1}{2}\theta)$ to get this integral into a form [recognize $\cos^2(\frac{1}{2}\theta_1)$ as c^2 … just the square of a constant] that you can look up in the math tables book (it's not in TM5):

$$\int \frac{dz}{\sqrt{c^2 - z^2}}$$

This is actually a proof of the tautochrone curve … check out the Wikipedia page.

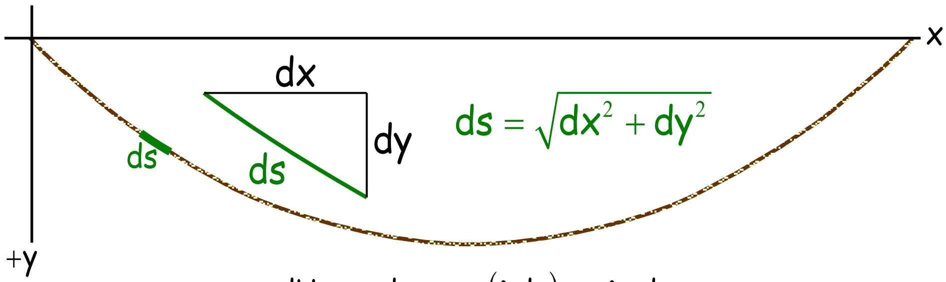


$$t = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{ds}{v} = \int \frac{(dx^2 + dy^2)^{1/2}}{(2gx)^{1/2}}$$

$$= \int_{x_1=0}^{x_2}\left(\frac{1+y'^2}{2gx}\right)^{1/2} dx \qquad \textbf{(6.20)}$$

Extra Credit Problem

1)[30] Derive the catenary curve using Euler's Equation[1] (*show* all the math steps!).



$$dU = gydm = gy(\lambda ds) = g\lambda yds$$

$$U = g\lambda \int yds$$
$$= g\lambda \int y\sqrt{dx^2 + dy^2}$$

Minimize the functional of the potenial energy using Euler's equation

$$U = g\lambda \int y\sqrt{1+(y')^2}dx \qquad \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

After taking derivatives and substituting into Euler's equation, you should get $1+(y')^2 - yy'' = 0$. Multiply through by y′ and rarrange to show this is the diffeence of the derivatives of $\ln(y)$ and $\frac{1}{2}\ln\left[(y')^2+1\right]$.

$$\frac{y'}{y} - \frac{y'y''}{1+(y')^2} = 0 .$$

Integrate these over indefinite limits and write the constant(s) as $\ln(A)$ to derive $A^2y^2 - 1 = (y')^2$. This can then be written as a separable DE. If you recognize the derivative of the inverse cosh and your freedom in writing constants in any form, you can get the equation of the catenary curve shown. Plot this with your favorite software to show it does give the catenary.

$$\frac{dy}{dx} = \frac{1}{\sqrt{A^2y^2 - 1}} \quad \Rightarrow \quad y(x) = C\cosh(Ax) + B$$

https://fractalkitty.com/2022/01/09/catenary-kitties/

[1] In Phys. 333 you'll derive the Catenary curve using a different method.