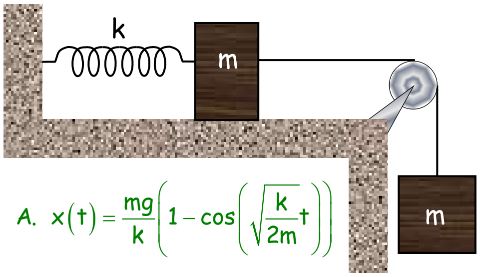
Homework Set 16: Fun with Lagrange
Due Monday, November 7, 2022



A. $x(t) = \frac{mg}{k}\left(1 - \cos\left(\sqrt{\frac{k}{2m}}t\right)\right)$

1) Two equal masses are constrained by the spring-and-pulley system shown. Assume a massless pulley and a frictionless surface. Let x be the extension of the spring from its relaxed length. Derive the equations of motion by Lagrangian methods. Solve for x as a function of time with the boundary conditions $x(t=0) = \dot{x}(t=0) = 0$. Lagrange's Equations will give you a DE, $\ddot{x} + \frac{k}{2m}x = \frac{g}{2}$ for which you should guess a solution $x = A + B\cos(\omega_N t)$.

Problems From TM5

2) *7-2* Work out Example 7.6 showing all the steps, in particular those leading to Equations 7.36 and 7.41. Explain why the sign of the acceleration a cannot affect the frequency ω. Give an argument why the signs of a^2 and g^2 in the solution of ω^2 in Equation 7.42 are the same.

Write down ***every** step* in taking the derivatives of the Lagrangian … no short-cuts or you'll miss something! Be careful about the difference between partial and full derivatives! There's a triangle lurking in tan(θ_e) = (-a/g) and it does ***not*** mean that sin(θ_e) = (-a) … draw the triangle to find sin(θ_e) and cos(θ_e)!

3) *7-3* A sphere of radius ρ is constrained to roll without slipping on the lower half of the inner surface of a hollow cylinder of inside radius R. Here, θ is the angle between the vertical and the position of the center of mass of the small sphere and ϕ is the angle through which the small sphere has rolled. Determine the Lagrangian function, the equation of constraint and show that Lagrange's equations of motion, for the coordinates shown, are

$$-(R-\rho)\, mg\sin\theta - m(R-\rho)^2\ddot{\theta} + \lambda(R-\rho) = 0$$

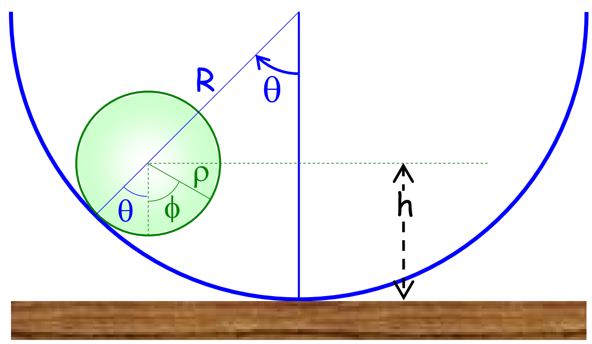
$$-\tfrac{2}{5}m\rho^2\ddot{\phi} - \lambda\rho = 0,$$

where

$$\lambda = -\tfrac{2}{5}m(R-\rho)\ddot{\theta} \;= -\tfrac{2}{5}m\rho\ddot{\phi}$$

Also show that the frequency of small oscillations is

$$\omega = \sqrt{\frac{5g}{7(R-\rho)}}.$$



The energies are due to the height of the CM of the sphere and its translational ($\frac{1}{2}mv^2$) and rotational ($\frac{1}{2}I\omega^2 = \frac{1}{2}I\dot{\phi}^2$) motions. The rolling constraint is that the distance moved by the center of mass is equal to the arc length it rolls along the sphere: $s_{cm} = r_{sphere}\phi_{sphere}$. Describe s_{cm} in terms of θ and R - ρ and use Lagrange's equations with undetermined multipliers (TM5 Equation 7.65). For the last step, recognize that the small oscillations of the CM are expressed by $\omega = \dot{\theta}$. To find it, use the result of Lagrange's equation in θ with the θ expression for λ substituted and get it in the form of $\ddot{\theta} + \omega_N^2\theta = 0$.

A lever and a place to stand, Old Main, Clarkson University, Main St., Potsdam, NY