CLASSICAL MECHANICS REVIEW

I. NEWTONIAN DYNAMICS

- KINEMATICS (MOTION WHEN $a = a_0 = $ CONSTANT)

$$\ddot{x} = a_0 \Rightarrow v = v_0 + a_0 t \Rightarrow x = x_0 + v_0 t + \tfrac{1}{2} a_0 t^2$$

$$v^2 = v_0^2 + 2a_0(x - x_0)$$

$$x = \frac{v + v_0}{2} t$$

FOR ANGULAR MOTION $x = r\theta$, $v = r\omega$, $a = r\alpha_0$

$$\ddot{\theta} = \alpha \Rightarrow \omega = \omega_0 + \alpha_0 t \Rightarrow \theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha_0 t^2$$

- NEWTON'S LAWS OF DYNAMICS

1. STUFF COASTS (IT TAKES A FORCE TO CHANGE MOMENTUM)
2. $\sum \vec{F} = \frac{d\vec{p}}{dt}$
3. STUFF PUSHES BACK (FORCES ARE INTERACTIONS)

- APPLICATIONS OF NSL
  - SKETCH SITUATION (WITH SEPARATE F.B.D.)
    - SHOW VARIABLES & AXES ON FBD
  - WRITE NSL IN **EACH COORDINATE** DIRECTION
    - **SYMBOLS** (NO VALUES UNTIL VERY END!)
  - FIND **ALGEBRAIC** EXPRESSION FOR UNKNOWN
  - SUBSTITUTE NUMBERS & CALCULATE IF APPROPRIATE

DON'T SKIP THIS!

FOLLOW YOUR HEADLIGHTS!

CIRCULAR DYNAMICS $\vec{\tau} = \vec{r} \times \vec{F}$, $\vec{L} = \vec{r} \times \vec{p}$

$$\vec{r} \times \left(\sum \vec{F} = \frac{d\vec{p}}{dt}\right) \Rightarrow \sum \vec{\tau} = \frac{d\vec{L}}{dt} = I\vec{\alpha}$$

MOMENT OF INERTIA $I = \int r^2 dm$
- LOOK UP IN TABLES
- PARALLEL AXIS THEOREM $I = I_{cm} + mr^2$

CENTRIPETAL ACCELERATION

$a_{cp} = \frac{v^2}{r}$ TOWARD CENTER

- $\perp$ TO VELOCITY $\Rightarrow$ DOES NO WORK
- MUST BE PROVIDED BY A FORCE (STRING, GRAVITY, FRICTION, ETC)

- Applications of NSL

  Drag - Velocity-dependent force

  - Use either

$$F = m\frac{dv}{dt} \quad \text{or} \quad F = mv\frac{dv}{dx}$$

    - Choose the one that works!
    - Integrate to find $v(x)$ or $v(t)$
    - Integrate again to find $x(t)$

- Conservation Laws
  - Work

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{x} = F\Delta x \text{ for constant force}$$

  - Potential Energy

    For conservative forces (work independent of path)

$$W_{cons.} = \int_1^2 \vec{F}_{cons} \cdot d\vec{x} = -(U_2 - U_2) \Rightarrow \vec{\nabla} U = \vec{F}$$

  - Work-Energy Theorem

$$W = \int F\,dx = \int m\frac{dv}{dt}\,dx = \int mv\frac{dv}{dx}\,dx = \frac{1}{2}mv^2$$

$$\Rightarrow \Delta T = W_{NET} \leftarrow \text{Work of all forces}$$

$$\Delta T + \Delta U = W_{NC} \leftarrow \text{Work of non-cons. forces}$$

  - Linear Momentum

$$\Sigma\vec{F} = \frac{d\vec{p}}{dt} \Rightarrow \vec{p} \text{ conserved when } \Sigma F = 0$$

    - Isolated Systems
    - Collisions

  - Angular Momentum

$$\Sigma\vec{\tau} = \frac{d\vec{L}}{dt} \Rightarrow \vec{L} \text{ conserved when } \Sigma\tau = 0$$

    - Systems in dynamic equilibrium
    - Spherically symmetric forces

- EQUILIBRIUM & STABILITY
  - $U(x)$ AT MAXIMUM OR MINIMUM

$$\Rightarrow \left.\frac{dU}{dx}\right|_{x=x_0} = 0 \quad \text{AT EQUILIBRIUM}$$

  - TYPE OF EQUILIBRIUM (2^{ND} DERIVATIVE)
    - STABLE $\frac{d^2U}{dx^2} > 0$ CONCAVE UP
    - NEUTRAL $\frac{d^2U}{dx^2} = 0$ FLAT
    - UNSTABLE $\frac{d^2U}{dx^2} < 0$ CONCAVE DOWN

II. HARMONIC OSCILLATIONS

- REQUIRES A LOCAL MINIMUM IN POTENTIAL, $U(x)$

- SHM - LINEAR RESTORING FORCE, NO DAMPING

NSL: $m\ddot{x} = -kx$

DE: $\ddot{x} + \omega_N^2 x = 0, \quad \omega_N = \sqrt{\frac{k}{m}}$

SOLUTION: $x = A\cos(\omega_N t - \phi)$

- DAMPED HARMONIC MOTION - LINEAR RESTORING FORCE

NSL: $m\ddot{x} = -kx - b\dot{x}$

DE: $\ddot{x} + 2\beta\dot{x} + \omega_N^2 x = 0 \quad \beta = \frac{b}{2m}$

SOLUTION: $x = e^{-\beta t}\left[A_1 e^{\sqrt{\beta^2-\omega_N^2}} + A_2 e^{-\sqrt{\beta^2-\omega_N^2}}\right]$

UNDERDAMPED $\omega_N^2 > \beta^2$ (OSCILLATIONS DOMINATE)

$$x(t) = Ae^{-\beta t}\cos(\omega_S t - \phi), \quad \omega_S = \sqrt{\omega_N^2 - \beta^2}$$

CRITICALLY DAMPED $\omega_N^2 = \beta^2$

$$x(t) = (A + Bt)e^{-\beta t}$$

OVERDAMPED $\beta^2 > \omega_N^2$

$$x(t) = e^{-\beta t}\left[A_1 e^{\omega_{OD} t} + A_2 e^{-\omega_{OD} t}\right], \quad \omega_{OD} = \sqrt{\beta^2 - \omega_N^2}$$

- DAMPED HARMONIC MOTION

NOTES

$$\omega = 2\pi\nu, \quad \tau = \frac{1}{\nu} = \frac{2\pi}{\omega} \qquad \nu \sim Hz, \ \omega \sim \frac{rad}{s}, \ \tau \sim s$$

ENERGY

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$

DECREMENT OF MOTION

$$\frac{A(t+\Delta t)}{A(t)} = \frac{A e^{-\beta(t+\Delta t)}}{A e^{-\beta t}} = e^{-\beta \Delta t}$$

- DRIVEN HARMONIC MOTION

NSL: $m\ddot{x} = -kx - b\dot{x} + F_0 \cos \omega_D t$

DE: $\ddot{x} + 2\beta\dot{x} + \omega_N x = \frac{F_0}{m} \cos(\omega_D t)$

SOLUTION: $$x(t) = \underbrace{x_c}_{\text{SOLUTION TO HOMOGENEOUS EQUATION}} + \frac{F_0/m}{\sqrt{(\omega_N^2 - \omega_D^2)^2 + 4\beta^2\omega_D^2}} \cos(\omega_D t - \delta)$$

$$\delta = \tan^{-1}\left(\frac{2\beta\omega_D}{\omega_N^2 - \omega_D^2}\right)$$

RESONANCE:

AMPLITUDE IS MAXIMUM WHEN $\omega_D = \sqrt{\omega_N^2 - 2\beta^2}$

$\Rightarrow$ POTENTIAL ENERGY MAXIMIZED

SPEED IS MAXIMUM WHEN $\omega_D = \omega_0$

$\Rightarrow$ KINETIC ENERGY MAXIMIZED

- MECHANICAL - ELECTRICAL EQUIVALENTS

$$x \to Q, \quad \dot{x} \to I, \quad 1/k \to C, \quad b \to R, \quad m \to L$$

THE DE IS WRITEN

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C} Q = \mathcal{E}_0 \cos \omega_D t$$

SOLUTION IS SAME AS DRIVEN HARMONIC OSCILLATOR

- FOURIER SERIES

$$F(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos(n\omega t) + B_n \sin(n\omega t) \right)$$

WHERE

$$A_0 = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} F(t)\, dt$$

$$A_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} F(t) \cos(m\omega t)\, dt$$

$$B_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} F(t) \sin(m\omega t)\, dt$$

$\Rightarrow$ ANY FUNCTION CAN BE REPRESENTED BY A SERIES OF COSINES AND SINES

Not covered in 2023, but useful!

III GRAVITATION

NEWTON'S LAW OF GRAVITATION

FORCE

$$\vec{F}_G = \underbrace{-}_{\text{ALWAYS}} \frac{GMm}{r^2} \hat{e}_r = -Gm \int \frac{dM}{r^2}$$

FIELD

$$\vec{g} = \frac{\vec{F}}{m} = -\frac{GM}{r^2} \hat{e}_r = -G \int \frac{dM}{r^2}$$

POTENTIAL

$$\Phi = -\frac{\Delta U}{m} = -\frac{GM}{r}$$

GAUSS' GRAVITATIONAL LAW

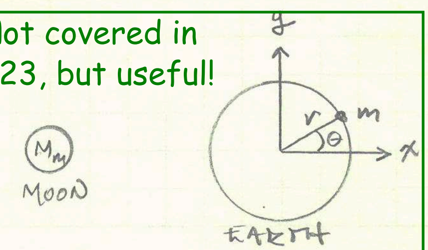
$$\underbrace{\oint \vec{g} \cdot d\vec{A}}_{\text{INTEGRAL OVER SYMMETRIC GAUSSIAN SURFACE ON WHICH } g \text{ IS CONSTANT}} = -4\pi G \underbrace{M_{INSIDE}}_{\text{MASS INSIDE THE GAUSSIAN SURFACE}}$$

OCEAN TIDES

Not covered in 2023, but useful!

$$F_{Tx} = \frac{2GmM_m}{D^3} r\cos\theta, \quad F_{Ty} = \frac{2GmM_m}{D^3} r\sin\theta$$

- STRETCHES IN $\hat{x}$, COMPRESSES IN $\hat{y}$



IV HAMILTONIAN MECHANICS

- CALCULUS OF VARIATIONS

$$J = \int_{x_1}^{x_2} f\{y(x), y'(x); x\}\, dx$$

HAS EXTREMA WHERE

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad \text{EULER'S EQUATION}$$

- THE LAGRANGIAN

$$L(q_j, \dot{q}_j) \equiv T(\dot{q}_j) - U(q_j)$$

FOR PHYSICAL SYSTEMS THE VARIATION IS AN EXTREMUM

$$\Rightarrow \frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0 \qquad \text{LAGRANGE'S EQUATIONS}$$

WHEN THERE IS AN EQUATION OF CONSTRAINT

$$f(q_k, \dot{q}_k, t) = 0 \qquad \text{EQUATION OF CONSTRAINT}$$

UNDETERMINED MULTIPLIERS, λ, ARE USED SO THAT

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_k \lambda_k(t)\frac{\partial f}{\partial q_j} = 0$$

- THE HAMILTONIAN

$$H = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$$

IF $T = T(\dot{q}^2)$ AND $U \neq U(\dot{q})$

$$\Rightarrow H = T + U$$

THE MOMENTA ARE

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

AND HAMILTON'S EQUATIONS ARE

$$\left.\begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k} \\ \dot{p}_k &= -\frac{\partial H}{\partial q_k} \end{aligned}\right\} \text{CANONICAL EQUATIONS OF MOTION}$$

V ROCKETS & ORBITS

- REDUCED MASS
  - ALLOWS ANALYSIS OF ONE PARTICLE ABOUT OTHER (OF REDUCED MASS) INSTEAD OF BOTH ABOUT CM

- CONSERVATION LAWS

  ENERGY: $\frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) = E = \text{CONSTANT}$

  ANGULAR MOMENTUM: $\mu r^2 \dot{\theta} = l = \text{CONSTANT}$

  AREAL VELOCITY: $\frac{1}{2}r^2\dot{\theta}^2 = \frac{1}{2}\frac{l}{\mu} = \text{CONSTANT}$

  EQUATIONS OF MOTION:

  ENERGY EQUATION GIVES

$$\dot{r} = \sqrt{\frac{2}{\mu}(E-U) - \frac{l^2}{\mu r^2}}$$

  ANGULAR MOMENTUM EQUATION GIVES

$$\theta(r) = \int \frac{(l/\mu r^2)\,dr}{\sqrt{2\mu\left(E - \frac{k}{r} + \frac{l^2}{2\mu r^2}\right)}}$$

  LAGRANGE'S EQUATIONS GIVE

$$\left.\begin{aligned} &\mu(\ddot{r} - r\dot{\theta}^2) = F(r) \\ \text{OR}\quad &\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{l^2}F(r) \end{aligned}\right\} \text{FOR } r$$

  AND

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \qquad - \text{FOR } \theta$$

Not covered in 2023, but useful!

- ROCKETS

  VELOCITY AFTER EJECTING $(m_0 - m)$ MASS AT u

$$v = v_0 + u\ln\left(\frac{m_0}{m}\right)$$

- ORBITAL TRANSFER

$$v_{circ} = \sqrt{\frac{GM}{a}}, \quad v_p = \sqrt{2GM\left(\frac{r_B}{r_B + r_S}\right)}, \quad v_a = \sqrt{2GM\left(\frac{r_S}{r_B + r_S}\right)}$$

  HOHMANN TRANSFER

$$v_p \longleftrightarrow v_{SMALL} \quad \text{AND} \quad v_a \longleftrightarrow v_{BIG}$$

MOTION IN A NON-INERTIAL REFERENCE FRAME

- THE VELOCITY OF A PARTICLE IN A NON-INERTIAL REFERENCE FRAME AS MEASURED IN A FIXED FRAME IS

$$v_f = \vec{V} + \vec{v}_r + \underbrace{\vec{\omega} \times \vec{r}}$$

VELOCITY DUE TO ROTATION OF AXES
ANGULAR VELOCITY OF MOVING AXES
VELOCITY RELATIVE TO MOVING AXES
LINEAR VELOCITY OF MOVING ORIGIN
VELOCITY RELATIVE TO FIXED AXIS

- THE FORCE ON A PARTICLE APPEARS DIFFERENT IN EACH FRAME

$$\vec{F} = m\vec{a}_f = m\ddot{\vec{R}}_f + m\vec{a}_r + 2m(\vec{\omega} \times \vec{v}_r) + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

FORCE IN FIXED FRAME:

F = REAL FORCE

THE MOVING OBSERVER SEES

$$F_{EFF} = m\vec{a}_r$$

$$= \vec{F} - m\ddot{\vec{R}}_f - m\dot{\vec{\omega}} \times \vec{r} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}_r$$

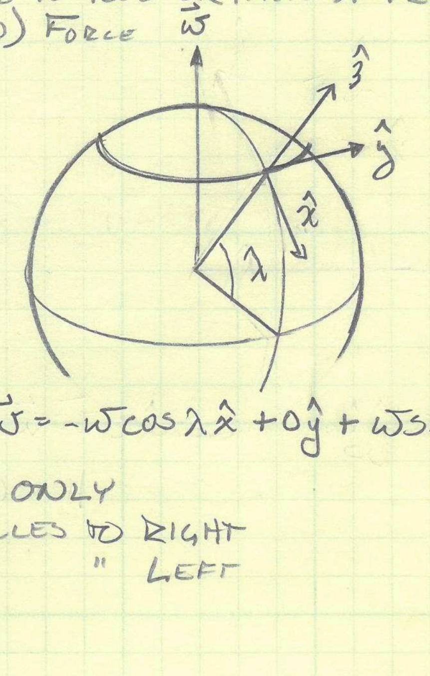
CORIOLIS FORCE
CENTRIFUGAL FORCE
APPARENT FORCE DUE TO ANGULAR ACCELERATION OF FRAME
APPARENT FORCE DUE TO ACCELERATION OF FRAME
EXTERNAL (NEWTONIAN) FORCE

- MOTION RELATIVE TO THE EARTH

CENTRIFUGAL FORCE

- ACTS ON ALL PARTICLES MOVING WITH EARTH
- DEFLECTS THE VERTICAL

$$F_{CF} = m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$



$$\vec{\omega} = -\omega\cos\lambda\,\hat{x} + 0\hat{y} + \omega\sin\lambda\,\hat{z}$$

CORIOLIS FORCE

- ACTS ON MOVING PARTICLES ONLY
- IN NORTH DEFLECTS PARTICLES TO RIGHT
  SOUTH " " " LEFT

$$F_{COR} = -2m\vec{\omega} \times \vec{v}_r$$