NON-COMMUTATIVE CONTINUOUS BERNOULLI SHIFTS

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Abstract: We introduce a non-commutative extension of the Tsirelson-Vershik noises [TV98, Tsi04], called *(non-commutative) continuous Bernoulli shifts*. These shifts encode stochastic independence in terms of commuting squares, familiar in subfactor theory [Pop83, GHJ89]. Such shifts are, in particular, capable of producing Arveson product systems of type I and type II [Arv03]. We investigate the structure of these shifts and prove that the von Neumann algebra of a (scalar-expected) continuous Bernoulli shift is either finite or of type III.

The role of ('classical') G-stationary flows for the Tsirelson-Vershik noises is now played by *cocycles* of continuous Bernoulli shifts. We show that these cocycles provide an operator-algebraic notion of Lévy processes, and that they lead to units and 'logarithms' of units of Arveson product systems [Kösa]. Furthermore, we introduce *(non-commutative) white noises* which are operator-algebraic versions of Tsirelson's 'classical' noises. We give examples coming from probability theory, from quantum probability [AFL82a, Par92] and from Voiculescu's theory of free probability [VDN92].

Our main result is a bijective correspondence between additive and unital shift cocycles. This correspondence is in parallel to the work of Hudson, Journé, Lindsay and Wills on the representability of Fock space Markovian cocycles as solutions of quantum stochastic differential equations [HL85a, Jou87, HL87, LW00]. For the proof of the correspondence we develop tools which are of interest in their own right: non-commutative extensions of Itô integration, stochastic logarithms and exponentials.

Contents

Introduction

Tsirelson and Vershik have recently established the existence of intrinsically non-linear random fields [TV98]. Their surprising result was stimulated by the existence of Arveson-Power product systems which are not of type I [Pow87, Arv89, Arv03]. These random fields or 'noises' go beyond the realm of the Lévy-Khintchine formula and provide a rich probabilistic source of non-type I product systems [Tsi03, Tsi04, Lie03]. Here we are interested in Tsirelson-Vershik noises, as defined in [Tsi98, Tsi04]. We will introduce a non-commutative extension of these noises, called *(non-commutative) continuous Bernoulli shifts*. These shifts incorporate so-called $\mathcal{A}_0$-*independence* which extends amalgamated stochastic independence to an operator algebraic framework, known as commuting squares in subfactor theory [Pop83, GHJ89, JS97]. These shifts may be regarded as two-sided 'time-continuous' analogues of shifts on towers of von Neumann algebras [Rup95, GK] (they are implicitly present in [JS97]). The notion of $\mathcal{A}_0$-independence also includes Voiculescu's amalgamated free independence [VDN92]. Our approach is thus in close contact with free probability theory. Apart from fermionic and bosonic noises, further examples of continuous Bernoulli shifts arise on deformed Fock spaces [BS91, BS94, BG02, BKS97, GM02].

Non-commutative continuous Bernoulli shifts include, in an algebraic form, all 'classical' examples of Tsirelson-Vershik noises. 'Classical noises' are generated by additive (square-integrable adapted) stationary flows, called Lévy processes, and are classified via the Lévy-Khinchin formula (see e.g. [Tsi04, Corollary 6a7]).

What is the operator algebraic counterpart of a 'classical noise'? Here, the situation is much more complex and we are only at the beginning of understanding this complexity (see also [KS]). Let us illustrate this in the case $\mathcal{A}_0 = \mathbb{C}$. Brownian motion is unique (up to stochastic equivalence), but there exist many different non-commutative Brownian motions: the long list begins with q-Brownian motions ($-1 < q < 1$) (including free Brownian motion), bosonic and fermionic Brownian motions (parametrized by 'temperature' or, equivalently, the period of the associated modular automorphism group). What all these diverse examples have in common is that they appear as *additive (adapted) cocycles* in the GNS Hilbert space of a ($\mathbb{C}$-expected) continuous Bernoulli shift. In this paper we will show that such additive cocycles are in bijective correspondence with *multiplicative (adapted) cocycles* in the GNS Hilbert space, called *unital cocycles*. This implies that *unitary* (adapted) cocycles, i.e. multiplicative cocycles in the unitary operators of the von Neumann algebra of a continuous Bernoulli shift, are also in correspondence with additive cocycles. In the terminology of Tsirelson and Vershik this mean that unitary cocycles are 'linearizable'. Thus we have available an operator-algebraic notion of 'classical noise': a continuous Bernoulli shift is called a (non-commutative) *white noise* if it is generated by its set of unitary cocycles.

Why do we *not* say that a continuous Bernoulli shift is a white noise if it is generated by its additive cocycles? The reason is that this would become conceptually cumbersome already for bosonic white noises in Araki-Woods representations: the vector space of additive cocycles does not capture the *type* of the von Neumann algebra (see Example 6.6.3). In all examples this kind of information is normally encoded in the choice of functor or into mixed higher moments which we do not have available in our general setting.

Continuous Bernoulli shifts also include all 'non-classical noises'. But we do not yet know a single example of a 'non-classical quantum noise' which is a non-commutative continuous Bernoulli shift, even though the definition of the latter object is a straightforward extension of a Tsirelson-Vershik noise. Such an example would yield an Arveson product system of Hilbert spaces of type II [Kösb], similarly to how 'non-classical noises' do [Tsi04].

Some clarifying remarks on the terminology are appropriate at this point. The attribute 'non-commutative' will always be used in the sense of 'not necessarily commutative'. Usually, we will drop it altogether and just write, for example, 'continuous Bernoulli shift'. We will avoid the attribute 'classical' and use instead 'commutative'. This convention is motivated by a conflict with the terminology in [Tsi04]. A 'classical noise' there will be a 'commutative white noise' here, a 'white noise' therein will be a 'Gaussian white noise' herein. The attribute 'quantum' will be reserved for situations beyond the realm of commutative probability theory.

We have sought to give a self-contained, comprehensive presentation of the subject, since our approach is not available easily in the published literature. Moreover, we have refrained from treating the most general cases. In so doing we hope that our work is better accessible to the readers coming from probability theory, quantum probability, quantum dynamics, quantum symmetries or operator algebras, who are interested in connecting some of these fields.

The starting point of this work was the operator-algebraic approach to stationary quantum Markov processes [Küm85] and results in [Küm84, Pri89, Küm93, Rup95, Kös00, Hel01]. The present conceptual form has also been stimulated by [Arv03, Tsi04]. In the context of product systems and their classification the following are of relevance [Bha99, Bha01, MS02, Lie03, BBLS04, BS05]. Finally, we also want to bring to the reader's attention the work of Gohm [Goh04, Goh], this is closely related to our approach.

Unital cocycles of continuous Bernoulli shifts are in closest contact to units of product systems of Hilbert spaces or modules. More details about this relation will be provided in [Kösa].

Unitary cocycles of continuous Bernoulli shifts or white noises immediately give rise to Markovian cocycles or stationary quantum Markov processes, as they are relevant in quantum dynamics and quantum probability. There are various approaches to the construction of quantum Markov processes in the literature. An operator algebraic setting is used in [AFL82a, Küm02, Arv03, Kös03]. For accounts on bosonic Fock space and Hudson-Parthasarathy's quantum stochastic calculus we refer the reader to [Par92, Mey93]. Meanwhile, this approach is further developed and a modern account can be found in [Lin05].

Next we give an outline of the major results and contents of each section.

Section 1: A (non-commutative) probability space $(\mathcal{A}, \psi)$ consists of a von Neumann algebra $\mathcal{A}$ together with a faithful normal state ψ. The predual $\mathcal{A}_*$ is *always assumed to be separable.* If $\mathcal{A}_0$ is a subalgebra of $\mathcal{A}$ for which the ψ-invariant conditional expectation from $\mathcal{A}$ onto $\mathcal{A}_0$ exists, then we say that $(\mathcal{A}, \psi; \mathcal{A}_0)$ is an *expected* probability space.

Section 2: Our non-commutative extension of (amalgamated) stochastic independence is intimately connected to *commuting squares*, introduced by Popa in subfactor theory [Pop83, GHJ89, Pop90]. Let $(\mathcal{A}, \psi; \mathcal{A}_0)$ be an expected probability space and denote by E_0 the conditional expectation from $\mathcal{A}$ onto $\mathcal{A}_0$.

Definition 0.0.1. *Let $\mathcal{B}$ and $\mathcal{C}$ be two von Neumann subalgebras of $\mathcal{A}$ such that, respectively, the conditional expectations $E_{\mathcal{B}}$, $E_{\mathcal{C}}$ from $\mathcal{A}$ onto $\mathcal{B}$, $\mathcal{C}$ exist. Then $\mathcal{B}$ and $\mathcal{C}$ are called* $\mathcal{A}_0$-independent *if $\mathcal{B} \cap \mathcal{C} = \mathcal{A}_0$ and $E_{\mathcal{B}} E_{\mathcal{C}} = E_0$.*

In other words, the four von Neumann algebras $\mathcal{A}, \mathcal{A}_0, \mathcal{B}$ and $\mathcal{C}$ form a commuting square:

$$\begin{array}{ccc} \mathcal{C} & \subset & \mathcal{A} \\ \cup & & \cup \\ \mathcal{A}_0 & \subset & \mathcal{B} \end{array}$$

If the von Neumann algebra $\mathcal{A}$ is commutative and $\mathcal{A}_0 \simeq \mathbb{C}$, then one recovers the usual notion of stochastic independence in probability theory. In the general setting, $\mathcal{A}_0$-independence encompases amalgamated stochastic independence, tensor product independence and Voiculescu's amalgamated free independence [VDN92]. But most importantly, it is not restricted to non-commutative notions of stochastic independence with universal product rules [Spe97, BGS02]. Further examples of $\mathcal{A}_0$-independence are accessible by 'white noise functors' [Küm96, BKS97, GM02], applied to von Neumann algebras generated by 'generalized Brownian motions' [BS91, BS94]. Moreover, we expect that Anshelevich's q-Lévy processes will provide further examples of $\mathcal{A}_0$-independence [Ans].

Section 3: In this section we introduce *(non-commutative) continuous Bernoulli shifts* and study their structure. These shifts provide a non-commutative extension of the Tsirelson-Vershik noises or, more precisely, of homogeneous continuous products of probability spaces [TV98, Tsi04]. Their global structure, as stated in Definition 3.1.2, is integral for this paper and we will introduce them informally next.

An *expected (non-commutative) continuous Bernoulli shift* encodes the following structure from an algebraic viewpoint. Consider an expected probability space $(\mathcal{A}, \psi; \mathcal{A}_0)$ together with a (pointwise weak*-continuous) automorphism group $(S_t)_{t\in\mathbb{R}}$ (called *shift*) and a family of von Neumann subalgebras $(\mathcal{A}_I)_I$ (called *filtration*), indexed by ('time'-)intervals $I \subset \mathbb{R}$, such that $S_t\mathcal{A}_I = \mathcal{A}_{I+t}$. Assume that all conditional expectations from $\mathcal{A}$ onto $\mathcal{A}_I$ exist and that ψ is S-invariant. Now we encode $\mathcal{A}_0$-independence by the requirement that the filtration forms a family of *commuting squares* which is moreover shifted covariantly in 'time' by the action of S:

$$\begin{pmatrix} \mathcal{A}_I & \subset & \mathcal{A}_K \\ \cup & & \cup \\ \mathcal{A}_0 & \subset & \mathcal{A}_J \end{pmatrix}_{I,J\subset K:\, I\cap J=\emptyset} \xrightarrow{S_t} \begin{pmatrix} \mathcal{A}_{I+t} & \subset & \mathcal{A}_{K+t} \\ \cup & & \cup \\ \mathcal{A}_0 & \subset & \mathcal{A}_{J+t} \end{pmatrix}_{I,J\subset K:\, I\cap J=\emptyset}$$

The von Neumann algebra $\mathcal{A}_0$ is not shifted; we stipulate the following stronger condition. $\mathcal{A}_0$ is *required to be the fixed point algebra of the shift S*. Moreover, we impose a minimality condition on the system: the family $\{\mathcal{A}_I \mid I \text{ bounded}\}$ generates $\mathcal{A}$ ('minimal filtration'). A priori we do not require 'local minimality', $\mathcal{A}_{[r,s]} \vee \mathcal{A}_{[s,t]} \subsetneqq \mathcal{A}_{[r,t]}$ may occur. Denoting the set of all intervals by $\mathcal{I}$, an object

$$\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$$

enjoying all these properties, will be called an *($\mathcal{A}_0$-)expected (non-commutative) continuous Bernoulli shift* (see Definition 3.1.2). Notice that we *do not assume continuity properties for the filtration itself.* Assuming for a moment the 'local minimality' condition $\mathcal{A}_{[r,s]} \vee \mathcal{A}_{[s,t]} = \mathcal{A}_{[r,t]}$, we observe the following.

(i) Putting $\mathcal{A}_0 \simeq \mathbb{C}$ and requiring that $\mathcal{A}$ be commutative, one obtains an algebraic version of the Tsirelson-Vershik noises (see Subsection 4.4).
(ii) Putting $\mathbb{Z}$ (or $\mathbb{N}_0$) as 'time' instead of $\mathbb{R}$, the above scheme reduces to a (one-sided) Bernoulli shift which finds examples in subfactor theory [Rup95, KM98, GK].

Here we focus on the case of continuous 'time', which includes the Tsirelson-Vershik noises. The 'discrete time' case and its connection to subfactor theory goes beyond the limits of this work and is postponed to future publications.

In Section 3 we will concentrate on properties of expected continuous Bernoulli shifts which follow from Definition 3.1.2 without any further assumptions. Among these are strongly mixing properties of the shift, stability with respect to compositions (such as tensor products and direct sums) and decompositions (such as compressions with conditional expectations). Most importantly, the structure of a continuous Bernoulli shift is already sufficient to give results on the type of its von Neumann algebra (see Subsection 3.2):

Theorem 0.0.2. *Let $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathbb{C}\mathbb{1}\big)$ be an expected continuous Bernoulli shift. Then the von Neumann algebra $\mathcal{A}$ is either finite or of type III, and the state ψ is non-tracial if and only if $\mathcal{A}$ is of type III.*

This result extends to $\mathcal{A}_0$-expected continuous Bernoulli shifts if one passes to 'derived' continuous Bernoulli shifts, in a similar way to towers of von Neumann algebras in subfactor theory.

Section 4: All examples of continuous Bernoulli shifts considered so far enjoy much more algebraic structure and continuity properties than is stipulated in Definition 3.1.2. Among such additional properties, which we study in this section, are *local minimality*, *local maximality*, *enriched $\mathcal{A}_0$-independence* and *commuting past and future*. These are properties which are quite familiar in Arveson's approach to quantum dynamics [Arv03]. We also investigate the relationship between continuous Bernoulli shifts and Tsirelson-Vershik noises (in Subsection 4.4). In Subsection 4.6 we give examples of continuous Bernoulli shifts, coming from both commutative probability and from quantum probability. The former include Gaussian and Poisson white noise and the Tsirelson-Vershik black noise. The latter include fermionic white noises and bosonic white noises in 'finite temperature' representations of Araki-Woods type [AW63], as well as q-white noises, in particular free white noise (see Subsection 4.7). As already stated for $\mathcal{A}_0$-independence, more examples of $\mathcal{A}_0$-expected white noises can easily be constructed from generalized Brownian motions, again using the properties of white noise functors. Moreover, Anshelevich's q-L'evy processes lead to further examples, as soon as it can be proven that the vacuum vector is separating for the von Neumann algebras generated by these processes [Ans].

Section 5: The example of Gaussian white noise already makes it evident that most interesting processes, such as Brownian motion, are not contained in the L^∞-space of the underlying measure space, but are contained in its L^2-space. In consequence, we extend the structure of an $\mathcal{A}_0$-expected continuous Bernoulli shift $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ to the GNS Hilbert bimodule ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$ (see Subsection 5.1). Thus, in analogy with [Tsi04], we obtain a 'homogeneous continuous commuting square system of pointed Hilbert bimodules', or an $\mathcal{A}_0$-*expected (non-commutative) continuous GNS Bernoulli shift*. We develop the representation theory of such shifts only as far as is necessary for this paper. A key result is Proposition 5.3.1 which establishes the existence of the products of two Hilbert bimodule elements, as long as they are $\mathcal{A}_0$-independent. This provides a non-commutative extension of the well-known result in probability theory that the product of two stochastically independent L^2-functions is again an L^2-function. Furthermore, we will show that the shift and all the conditional expectations extend to adjointable operators on the relevant Hilbert bimodules (see Theorem 5.2.2). This will finally put us in a position to introduce (in Definition 5.4.5) an $\mathcal{A}_0$-expected non-commutative continuous GNS Bernoulli shift

$$\big({}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}, \mathbb{1}, \overline{S}, ({}_{\mathcal{A}_I}\mathcal{E}_{\mathcal{A}_0})_{I\in\mathcal{I}}; \mathcal{A}_0\big);$$

that encodes the covariant shift of a commuting square system of pointed Hilbert bimodules:

$$\begin{pmatrix} {}_{\mathcal{A}_I}\mathcal{E}_{\mathcal{A}_0} & \subset & {}_{\mathcal{A}_K}\mathcal{E}_{\mathcal{A}_0} \\ \cup & & \cup \\ {}_{\mathcal{A}_0}\mathcal{E}_{\mathcal{A}_0} & \subset & {}_{\mathcal{A}_J}\mathcal{E}_{\mathcal{A}_0} \end{pmatrix}_{I,J\subset K:\, I\cap J=\emptyset} \xrightarrow{\overline{S}_t} \begin{pmatrix} {}_{\mathcal{A}_{I+t}}\mathcal{E}_{\mathcal{A}_0} & \subset & {}_{\mathcal{A}_{K+t}}\mathcal{E}_{\mathcal{A}_0} \\ \cup & & \cup \\ {}_{\mathcal{A}_0}\mathcal{E}_{\mathcal{A}_0} & \subset & {}_{\mathcal{A}_{J+t}}\mathcal{E}_{\mathcal{A}_0} \end{pmatrix}_{I,J\subset K:\, I\cap J=\emptyset}$$

Here denotes $\overline{S}$ the extension of the shift S to the bounded $\mathcal{A}_0$-linear operators on ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$. The Hilbert modules are 'pointed', because the cyclic and separating vector $\mathbb{1}$ is contained in each Hilbert bimodule and provides a (trivial) multiplicative shift cocycle (see Subsection 6.2). Now it is easy to see that

(iii) if $\mathcal{A}_0 \simeq \mathbb{C}$ and $\mathcal{A}_{[r,t]} \simeq \mathcal{A}_{[r,s]} \otimes \mathcal{A}_{[s,t]}$ for $r < s < t$, then the continuous GNS Bernoulli shift is a homogeneous continuous product system of pointed Hilbert spaces, in the sense of Tsirelson [Tsi04, Kösa].

(iv) under the assumptions of (iii), the family $({}_{\mathcal{A}_{[0,t]}}\mathcal{E}_{\mathcal{A}_0})_{0<t<\infty}$ defines a continuous tensor product system of Hilbert spaces, now in the sense of Arveson [Arv03, Tsi04, Kösa].

The Tsirelson-Vershik noises provide a rich source of Arveson product systems, in particular product systems of type II. Consequently, $\mathbb{C}$-expected commutative continuous Bernoulli shifts do too. The relationship between these three approaches will be further explored in [Kösa].

Section 6: Just as stationary G-flows play a central role for a Tsirelson-Vershik noise, so do cocycles for a continuous Bernoulli shift, or its GNS representation. These cocycles are *adapted to the filtration* of the continuous Bernoulli shift and satisfy either additive or multiplicative cocycle equations. The study of noncommutative stochastic cocycles goes back to the pioneering work of Accardi [Acc80, AFL82b], and in quantum stochastic calculus to [HL85b, HL87](see [LW00]) where 'regular' unitary cocycles are shown to satisfy quantum stochastic differential equations. Multiplicative cocycles come in two kinds: *unitary cocycles* for the continuous Bernoulli shift itself (Definition 6.1.1) and *unital cocycles* for its GNS representation (Definition 6.2.1). Additive cocycles are given in the GNS representation (Definition 6.3.1). These cocycles provide non-commutative versions of Lévy processes; this is proved using the non-commutative martingale inequalities of Junge, Pisier and Xu [PX97, Kös00, JX03, Kös03, Kösb]. Throughout this work we only consider additive cocycles *with a uniformly bounded variance operator*. Also, the unital cocycles are such that their compression to $\mathcal{A}_0$ is a norm-continuous semigroup. These assumptions correspond to the above mentioned regularity.

Let us now present our main results from Section 6. We will assume here for simplicity that $\dim \mathcal{A}_0 < \infty$ (see Theorems 6.4.1 and 6.4.4 for a more general case).

Theorem 0.0.3. *There exists a bijective correspondence between unital cocycles and additive cocycles, where the latter satisfy some structure equation.*

Since every unitary cocycle defines a unital cocycle, we conclude immediately from Theorem 0.0.3 (see also Theorem 6.5.1):

Theorem 0.0.4. *There exists a bijective correspondence between unitary cocycles and additive cocycles, where now the latter satisfy a stronger version of the structure equation.*

To reveal this stronger structure equation will be the topic of future publications (see Subsection 6.5 and [Kös03] for the case of a tracial state). Theorem 0.0.4 (resp. Theorem 6.5.1) can be regarded as a non-commutative version of Tsirelson's result that every stationary $\mathcal{U}(\mathcal{H}_0)$-flow (continuous in probability) is 'classical' (see [Tsi04, Theorem 8a2] and also [Tsi98]). Here $\mathcal{U}(\mathcal{H}_0)$ denotes the group of unitary operators on the (separable) Hilbert space $\mathcal{H}_0$ (corresponding to $\mathcal{A}_0$). (Notice that our result does not cover fully Tsirelson's results, since we stipulate stronger continuity conditions.)

We emphasize that Theorem 0.0.3 is only based on the structure of continuous Bernoulli shifts and cocycles. This puts us in a position to introduce non-commutative white noises, ensuring that they are a non-commutative version (see Definition 6.5.1) of Tsirelson's 'classical' noises:

Definition 0.0.5. *An $\mathcal{A}_0$-expected (non-commutative) continuous Bernoulli shift $(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0)$ is called a* (non-commutative) white noise *if the filtration is generated by adapted unitary S-cocycles (in an adapted manner).*

Now the correspondence ensures that white noises, as defined above, are always 'generated' by additive cocycles. This is in parallel to the well-known fact that, for example, Brownian motion generates the σ-algebras of the filtration of the Gaussian white noise (see e.g. [Tsi98]). We show that the 'non-commutative white noise part' can always be extracted from a continuous Bernoulli shift by compression with a conditional expectation (see Subsection 6.5).

The proof of Theorem 0.0.3 relies on a non-commutative extension of stochastic Itô integration and the construction of non-commutative versions of stochastic exponentials and logarithms, which is treated in Sections 7 and 8.

Section 7: In this section we develop a theory of non-commutative Itô integration which includes an existence and uniqueness theorem for solutions of non-commutative Itô differential equations. This theory relies only on the structure of non-commutative $\mathcal{A}_0$-continuous (GNS) Bernoulli shifts and its additive cocycles. Crucial for this approach is the notion of $\mathcal{A}_0$-independence which premits the transfer of the famous Itô isometry of Brownian motion to the non-commutative setting. The starting point of this theory is the preliminary results in [Pri89]. These gave valuable evidence that the present general approach was promising.

Our approach to non-commutative Itô integration applies to all examples of non-commutative white noises (in the sense of Definition 0.0.5), in particular fermionic, bosonic, free and q-white noises, including the operator-valued setting. In the case of scalar-expected gauge invariant bosonic white noise one recovers some of the early work on non-Fock/quasi-free quantum stochastic integration of Hudson, Lindsay and Wilde [HL85a, HL85b, Lin85, Lin86] and Barnett, Streater and Wilde [BSW83, LW86]. In contrast to the Hudson-Parthasarathy theory [HP84, Par92, Lin05], the seperating property of the vacuum (making them white noises in our sense) is heavily exploited in this work. The pioneering work on Itô-Clifford integration [BSW82] also comes within our orbit. Of special importance in applications is the Itô integration theory for so-called 'squeezed white noises' which are relevant in modern quantum optics [GZ00]. A treatment of the present approach is given in [HHK$^+$02]. If the underlying non-commutative probability space comes from Voiculescu's free probability theory, one is precisely in the $\mathbb{C}$-expected setting of free stochastic calculus developed by Biane and Speicher [BS98]. In the case of q-commutation relations, one obtains a non-commutative theory of Itô integration, as contained already in [HKK98] and independently developed much further in [DM03].

Section 8: Here we develop the theory of non-commutative logarithms and exponentials for unital cocycles resp. additive cocycles, as far as it is necessary for the proof of Theorem 0.0.3. We introduce the mapping Exp from the set of additive cocycles (with structure equation) to the set of unital cocycles (Subsection 8.1) and the mapping Ln from the set of unital cocycles to the set of additive cocycles (Subsection 8.2). Finally, we show in Subsection 8.3 that the mappings Exp and Ln are mutually inverse. This result completes the proof of the main theorems on the correspondence of additive and unital cocycles, as stated in Subsection 6.4.

Appendix A: For the convenience of the reader, we provide a short survey on Hilbert W*-modules, as far they are needed for the construction of continuous GNS Bernoulli shifts.

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1. Preliminaries

Here we fix the basic mathematical terminology for a non-commutative extension of probability theory, to be used throughout this paper.

1.1. **General terminology.** Throughout, $\mathcal{A}$ is a von Neumann algebra in $\mathcal{B}(\mathcal{H})$, the bounded operators on some fixed Hilbert space $\mathcal{H}$. We require that $\mathcal{A}$ has a separable predual $\mathcal{A}_*$. Beside the norm topology on $\mathcal{A}$, we consider the weak* topology $\sigma(\mathcal{A}, \mathcal{A}_*)$, the strong operator topology (SOT) and the σ-strong operator topology (σ-SOT) induced by the seminorms $\mathrm{d}_\xi(x) := \|x\xi\|$, $\xi \in \mathcal{H}$ resp. $\mathrm{d}_\varphi(x) := |\varphi(x^*x)|^{1/2}$, $\varphi \in \mathcal{A}_*$. The unit of $\mathcal{A}$ is denoted by $\mathbb{1}_\mathcal{A}$, or simply by $\mathbb{1}$, if no confusion can arise.

Since throughout $\mathcal{A}$ is considered in the presence of a fixed faithful state $\psi \in \mathcal{A}_*$, we assume for our convenience that $\mathcal{H}$ is already the GNS Hilbert space $\mathcal{H}_\psi$ corresponding to ψ. Thus we have $\psi = \langle \Omega \,|\, \cdot \Omega \rangle$ for some vector $\Omega \in \mathcal{H}_\psi$ which is cyclic and separating for $\mathcal{A}$. Moreover, since $\mathcal{A}_*$ is separable, the GNS Hilbert space $\mathcal{H}_\psi$ is also separable. Notice that the scalar product is taken to be linear in the second component. Two elements $x, y \in \mathcal{A}$ are called ψ-orthogonal if $\psi(x^*y) = 0$. Finally, the von Neumann algebra generated by a family $(\mathcal{A}_j)_{j\in J} \subset \mathcal{A}$ is denoted by $\bigvee_{j\in J} \mathcal{A}_j$.

As usual, the von Neumann algebra $\mathcal{A}'$ is the commutant of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$ and $\mathcal{Z}(\mathcal{A}) := \mathcal{A} \cap \mathcal{A}'$ is the center of $\mathcal{A}$. The von Neumann algebra $\mathcal{A}$ is called a factor if $\mathcal{Z}(\mathcal{A}) \simeq \mathbb{C}$. For a faithful normal state ψ on $\mathcal{A}$, the associated modular automorphism group is denoted by σ^ψ. The centralizer $\mathcal{A}^\psi := \{x \in \mathcal{A} \,|\, \psi(xy) = \psi(yx) \text{ for all } y \in \mathcal{A}\}$ is the fixed point algebra of σ^ψ.

We use the modulus $|a| := (a^*a)^{1/2}$ and, occasionally, $\operatorname{Re} a := (a + a^*)/2$ and $\operatorname{Im} a := (a - a^*)/2\mathrm{i}$ for $a \in \mathcal{A}$. Finally, for any normed linear space $\mathcal{N}$ we denote by $\mathcal{N}_1 = \{x \in \mathcal{N} \,|\, \|x\| \leq 1\}$ the unit ball of $\mathcal{N}$.

1.2. **Non-commutative probability spaces and their morphisms.** The pair $(\mathcal{A}, \psi)$ will be understood as a *(non-commutative) probability space* consisting of a von Neumann algebra $\mathcal{A}$ which is equipped with a faithful normal state ψ. A probability space $(\mathcal{B}, \varphi)$ is called a *sub-probability space of* $(\mathcal{A}, \psi)$ if $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, $\varphi = \psi|_\mathcal{B}$ and there is a ψ-invariant conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$. We remind the reader that such a conditional expectation exists (uniquely) if and only if the von Neumann algebra $\mathcal{B}$ is globally invariant under the action of the modular automorphism group of $(\mathcal{A}, \psi)$ [Tak71].

The morphisms of $(\mathcal{A}, \psi)$ are completely positive unital maps T on $\mathcal{A}$ such that ψ is T-invariant. They are automatically normal (see Lemma B.1) and we denote them by $\mathrm{Mor}(\mathcal{A}, \psi)$. Similarly, $\mathrm{Aut}(\mathcal{A}, \psi)$ denotes the automorphisms of $(\mathcal{A}, \psi)$. Conditional expectations, as considered throughout this paper, are always morphisms.

An *($\mathcal{A}_0$-)expected (non-commutative) probability space* $(\mathcal{A}, \psi; \mathcal{A}_0)$ consists of a probability space $(\mathcal{A}, \psi)$ with a distinguished sub-probability space $(\mathcal{A}_0, \psi_0)$. Note that, given some von Neumann algebra $\mathcal{B}$, an injective *-homomorphism $\iota\colon \mathcal{B} \to \mathcal{A}$ with $\mathcal{A}_0 := \iota(\mathcal{B})$ is a non-commutative random variable ([AFL82a], [Küm88]). Here we always identify $\mathcal{A}_0$ and $\mathcal{B}$. Finally, two elements $x, y \in \mathcal{A}$ are called $\mathcal{A}_0$-orthogonal if $E_{\mathcal{A}_0}(y^*x) = 0$.

Typical examples of $\mathcal{A}_0$-expected probability spaces are the following: Let $(\mathcal{A}_0, \psi_0)$ and $(\mathcal{B}, \varphi)$ be two probability spaces. Then an expected probability space $(\mathcal{A}, \psi; \mathcal{A}_0 \otimes \mathbb{1}_{\mathcal{B}})$ is defined by the von Neumann algebraic tensor product $\mathcal{A} = \mathcal{A}_0 \otimes \mathcal{B}$ and the tensor product state $\psi = \psi_0 \otimes \varphi$. If $\mathcal{B}$ is commutative, then we are in the context of operator-valued probability theory.

1.3. **Filtrations.** Let $(\mathcal{A}, \psi; \mathcal{A}_0)$ be an expected probability space and let $(\mathcal{A}_I, \psi_I)_{I \in \mathcal{I}}$ be a family of sub-probability spaces of $(\mathcal{A}, \psi)$ satisfying $\mathcal{A}_0 \subset \mathcal{A}_I$, indexed by the set $\mathcal{I}$ of all intervals I of $\mathbb{R}$ (including unbounded, and degenerated intervals and the empty set $\emptyset$). Such a family is called a *filtration of* $(\mathcal{A}, \psi)$ if $I \subseteq J$ implies $\mathcal{A}_I \subseteq \mathcal{A}_J$ for all $I, J \in \mathcal{I}$ (*monotony*). The filtration is *minimal* if $\bigvee\{\mathcal{A}_I \mid I \in \mathcal{I} \text{ bounded}\} = \mathcal{A}$. In particular, a minimal filtration satisfies $\mathcal{A}_{\mathbb{R}} = \mathcal{A}$. Notice that monotony is equivalent to $\mathcal{A}_I \vee \mathcal{A}_J \subseteq \mathcal{A}_K$ whenever $I \cup J = K$. Finally, the sub-filtrations $(\mathcal{A}_{(-\infty,t]})_{t\in\mathbb{R}}$ and $(\mathcal{A}_{[t,\infty)})_{t\in\mathbb{R}}$ are, respectively, called the *past* and *future filtrations*.

The filtration is *continuous downwards* if $\bigcap_{\varepsilon>0} \mathcal{A}_{[s-\varepsilon,t+\varepsilon]} = \mathcal{A}_{[s,t]}$ for $s \leq t$; it is *continuous upwards* if $\bigvee_{\varepsilon>0} \mathcal{A}_{[s+\varepsilon,t-\varepsilon]} = \mathcal{A}_{[s,t]}$ for $s < t$. Since $\bigvee_{\varepsilon>0} \mathcal{A}_{[s+\varepsilon,t-\varepsilon]} \subseteq \mathcal{A}_{(s,t)}$, upward continuity of a filtration implies that $\mathcal{A}_{(s,t)} = \mathcal{A}_{[s,t]}$ for $s < t$. The filtration is called *continuous* if it is both continuous downwards and upwards. Continuity properties of the past or future filtration are understood similarly. Notice that the (downward resp. upward) continuity of the filtration is equivalent to the (downward resp. upward) pointwise SOT-continuity of the associated family of conditional expectations $E_I\colon (\mathcal{A}, \psi) \to (\mathcal{A}_I, \psi_I)$ [Hel01, Kös00]. In particular, the past filtration $(\mathcal{A}_{(-\infty,t]})_{t\in\mathbb{R}}$ is continuous if and only if the family $(E_{(-\infty,t]})_{t\in\mathbb{R}}$ is pointwise SOT-continuous, and similarly for the future filtration.

2. Non-commutative independence

In this section we introduce $\mathcal{A}_0$-*independence* as a non-commutative analogue of (amalgamated) stochastic independence.

2.1. **$\mathcal{A}_0$-independence and Popa's commuting squares.** Recall the definition of sub-probability space from Subsection 1.2.

Definition 2.1.1. *Let $(\mathcal{B}, \varphi)$ and $(\mathcal{C}, \chi)$ be two sub-probability spaces of the expected probability space $(\mathcal{A}, \psi; \mathcal{A}_0)$ such that $\mathcal{A}_0 \subseteq \mathcal{B} \cap \mathcal{C}$. The algebras $\mathcal{B}$*

and $\mathcal{C}$ are called $\mathcal{A}_0$-independent, if for any $x \in \mathcal{B}$ and $y \in \mathcal{C}$

$$E_0(xy) = E_0(x)E_0(y)\,. \tag{2.1.1}$$

Here E_0 denotes the conditional expectation from $(\mathcal{A}, \psi)$ onto $(\mathcal{A}_0, \psi_0)$. Two families $(x_i)_{i\in I}$ and $(y_j)_{j\in J}$ in $\mathcal{A}$ are $\mathcal{A}_0$-independent if $\bigvee_{i\in I}\{x_i\}$ and $\bigvee_{j\in J}\{y_j\}$ are $\mathcal{A}_0$-independent.

Such a structure was introduced by Popa as a 'commuting square' in subfactor theory [Pop83]. Thus the statements '$\mathcal{B}$ and $\mathcal{C}$ are $\mathcal{A}_0$-independent' and '$\mathcal{B}$ and $\mathcal{C}$ form a commuting square over $\mathcal{A}_0$' are essentially the same. They will both be used, depending on whether we want to emphasize the probabilistic or more the algebraic aspect.

If the von Neumann algebra $\mathcal{A}_0$ is one-dimensional, i.e., $\mathcal{A}_0 \sim \mathbb{C}$, then $E_0 = \psi(\cdot)\mathbb{1}_{\mathcal{A}}$ is verified immediately.

$\mathcal{A}_0$-independence is equivalent to other properties for the von Neumann algebras involved, as is well-known for commuting squares in subfactor theory [Pop83, GHJ89, JS97]. We make frequently use of the following fact.

Proposition 2.1.2. *Under the assumptions of Definition 2.1.1 the following conditions are equivalent:*

(i) *$\mathcal{B}$ and $\mathcal{C}$ are $\mathcal{A}_0$-independent;*
(ii) *$E_0(x_1yx_2) = E_0(x_1E_0(y)x_2)$ for any $x_1, x_2 \in \mathcal{B}$, $y \in \mathcal{C}$;*
(iii) *$E_{\mathcal{B}}(\mathcal{C}) = \mathcal{A}_0$;*
(iv) *$E_{\mathcal{B}}E_{\mathcal{C}} = E_0$;*
(v) *$E_{\mathcal{B}}E_{\mathcal{C}} = E_{\mathcal{C}}E_{\mathcal{B}}$ and $\mathcal{B} \cap \mathcal{C} = \mathcal{A}_0$.*

Proof. The equivalences follow from the proof given in [GHJ89, Prop. 4.2.1], after some elementary modifications. □

Remark 2.1.3. (i) In general, Definition 2.1.1 does not incorporate computational rules for expressions like $E_0(xyxy)$ or $E_0(yxyx)$. But expressions like $E_0(xyyx)$ or $E_0(yxxy)$ are pyramidally ordered and can be simplified to $E_0(xE_0(yy)x)$ resp. $E_0(yE_0(xx)y)$ by the module property of conditional expectations. Whether or not enough information for the calculation of non-pyramidally ordered expressions is present depends on the additional algebraic structure of the example.

(ii) $\mathcal{A}_0$-independence applies, in particular, to von Neumann algebras of type III (see Example 2.2.2). Notice also that $\mathcal{B} \vee \mathcal{C}$ may be contained properly in $\mathcal{A}$. If $\mathcal{A}$ is the weak*-closed linear span of $\{xy \,|\, x \in \mathcal{B}, y \in \mathcal{C}\}$, then the corresponding commuting square is said to be 'symmetric' [JS97, Pop83]. Such a situation appears if the von Neumann algebras carry enough (commutation) relations. But already *free* probability leads to an important example of $\mathcal{A}_0$-independence with $\mathcal{A} = \mathcal{B} \vee \mathcal{C}$, where the corresponding commuting squares are 'non-symmetric' (see Example 2.4.2).

2.2. **Commuting subalgebras and $\mathbb{C}$-independence.** A probability space that includes a pair of commuting von Neumann subalgebras produces $\mathbb{C}$-independence of the two subalgebras which is characterized algebraically as tensor product independence. It comprises 'classical independence' and 'bosonic or CCR independence'.

Let $(\mathcal{B}_1, \varphi_1)$ and $(\mathcal{B}_2, \varphi_2)$ be two sub-probability spaces of the probability space $(\mathcal{B}, \varphi)$. Suppose that $\mathcal{B}_1$ and $\mathcal{B}_2$ commute, i.e., $xy = yx$ for $x \in \mathcal{B}_1$, $y \in \mathcal{B}_2$, and, for simplicity, that $\mathcal{B} = \mathcal{B}_1 \vee \mathcal{B}_2$. Then the following are equivalent:

(i) $\mathcal{B}_1$ and $\mathcal{B}_2$ are $\mathbb{C}$-independent.

(ii) $(\mathcal{B}, \varphi)$ is canonically isomorphic to $(\mathcal{B}_1 \otimes \mathcal{B}_2, \varphi_1 \otimes \varphi_2)$ with $\varphi_i = \varphi_{|\mathcal{B}_i}$ for $i = 1, 2$, where we identify $\mathcal{B}_1$ with $\mathcal{B}_1 \otimes \mathbb{1}$ and $\mathcal{B}_2$ with $\mathbb{1} \otimes \mathcal{B}_2$.

Obviously, (ii) implies (i) and we are left to prove the inverse implication. Let $x_i \in \mathcal{B}_1$ and $y_i \in \mathcal{B}_2$ $(i = 1, 2, \ldots, n)$. Since $\varphi(\sum_{i=1}^n x_i y_i) = \varphi_1 \otimes \varphi_2(\sum_{i=1}^n x_i \otimes y_i)$, the map $\sum_{i=1}^n x_i y_i \mapsto \sum_{i=1}^n x_i \otimes y_i$ is well-defined and extends to an isomorphism from $(\mathcal{B}, \varphi)$ onto $(\mathcal{B}_1 \otimes \mathcal{B}_2, \varphi_1 \otimes \varphi_2)$ which is implemented unitarily on the corresponding GNS Hilbert spaces.

Example 2.2.1 (Classical independence)**.** In the case of a commutative von Neumann algebra $\mathcal{B}$, the notion of $\mathbb{C}$-independence is equivalent to the classical notion of independence. Let $\mathcal{B} = L^\infty(\Omega, \Sigma, \mu)$ and $\psi(f) = \int f \,\mathrm{d}\mu$. Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are independent if and only if the sub-σ-algebras of Σ generated by $\mathcal{B}_1$ and $\mathcal{B}_2$ are independent.

Example 2.2.2 (Bosonic or CCR independence)**.** The canonical commutation relations (CCR) lead to the first non-commutative example of $\mathbb{C}$-independence. It is convenient to introduce them in their Weyl form (see [BR81, Pet90] and literature cited therein). These relations are given by

$$\begin{aligned} W(f)W(g) &= \exp\left(-\tfrac{\mathrm{i}}{2}\operatorname{Im}\langle f \,|\, g\rangle\right) W(f+g), \\ W(f)W(f)^* &= W(f)^*W(f) = \mathbb{1}, \end{aligned}$$

where f, g are elements of the Hilbert space $\mathcal{K}$ with scalar product $\langle \cdot \,|\, \cdot \rangle$. They generate the C*-algebra $\mathrm{CCR}(\mathcal{K}, \operatorname{Im}\langle \cdot \,|\, \cdot \rangle)$. Consider on this C*-algebra the quasi-free (gauge invariant) state

$$\psi_\lambda(W(f)) = \exp(-\tfrac{1}{4}(2\lambda + 1)\, \|f\|^2)$$

for some fixed $\lambda > 0$. Let $\mathcal{B}$ denote the von Neumann algebra which is generated by $\{W(f) \,|\, f \in \mathcal{K}\}$ in the GNS representation associated to ψ_λ. Furthermore, let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two orthogonal closed subspaces in $\mathcal{K}$. Then the corresponding von Neumann algebras $\mathcal{B}_i$, generated by $\{W(f) \,|\, f \in \mathcal{K}_i\}$ $(i = 1, 2)$ in the GNS representation, commute and are $\mathbb{C}$-independent.
The above construction works also for more general quasi-free states on a CCR algebra, as we see them in Example 4.7.2. Finally, note that the condition $\lambda > 0$ ensures that ψ_λ extends to a *faithful* normal state on $\mathcal{B}$.

2.3. **From $\mathbb{C}$-independence to $\mathcal{A}_0$-independence.** Examples of $\mathcal{A}_0$-independence are produced canonically from examples of $\mathbb{C}$-independence. Let $(\mathcal{B}_1, \varphi_1)$ and $(\mathcal{B}_2, \varphi_2)$ be two sub-probability spaces of $(\mathcal{B}, \varphi)$. Given in addition the probability space $(\mathcal{B}_0, \varphi_0)$, we let $\mathcal{A} := \mathcal{B}_0 \otimes \mathcal{B}$, $\psi := \varphi_0 \otimes \varphi$ and $\mathcal{A}_i := \mathcal{B}_0 \otimes \mathcal{B}_i$ for $i = 1, 2$, and $\mathcal{A}_0 := \mathcal{B}_0 \otimes \mathbb{1}$. Then $(\mathcal{A}, \psi; \mathcal{A}_0)$ is an expected probability space and $E_0 = \mathrm{id} \otimes \varphi(\cdot)\mathbb{1}$ is the conditional expectation from $(\mathcal{A}, \psi)$ onto $(\mathcal{A}_0, \psi_0)$. Then it holds that $\mathcal{A}_1$ and $\mathcal{A}_2$ are $\mathcal{A}_0$-independent if and only if $\mathcal{B}_1$ and $\mathcal{B}_2$ are $\mathbb{C}$-independent.

Example 2.3.1. Following the above tensor product construction, classical independence (Example 2.2.1) leads immediately to examples of so-called amalgamated (or operator-valued) independence. Similarly, CCR-independence (Example 2.2.2) leads to examples of $\mathcal{A}_0$-independence.

The tensor product construction of an $\mathcal{A}_0$-expected probability space from a $\mathbb{C}$-expected probability space looks very specific. It is worthwhile pointing out that by choosing $\mathcal{A}_0$ isomorphic to the complex $n \times n$-matrices M_n with $2 \leq n \leq \infty$, this construction already captures the general situation:

Proposition 2.3.2. *Let $(\mathcal{A}, \psi; \mathcal{A}_0)$ be an expected probability space with $\mathcal{A}_0 \simeq M_n$. Then there exists a probability space $(\mathcal{B}, \varphi)$ such that $\mathcal{A} \simeq M_n \otimes \mathcal{B}$ and, under this isomorphism, $\psi = \psi_{|\mathcal{A}_0} \otimes \varphi$.*

Proof. We define $\mathcal{B}$ as the relative commutant of $\mathcal{A}_0$ in $\mathcal{A}$. Then $\mathcal{A}$ splits canonically into the tensor product $M_n \otimes \mathcal{B}$, [KR86, 11.4.11] and we may assume $\mathcal{A} = M_n \otimes \mathcal{B}$. Now $\varphi(x) := \psi(\mathbb{1} \otimes x)$ defines a normal state on $\mathcal{B}$. It is checked immediately that the conditional expectation E_0 from $(\mathcal{A}, \psi)$ onto $\mathcal{A}_0$ acts as $E_0(x \otimes y) = x \otimes \varphi(y)$ for any $x \in M_n$ and $y \in \mathcal{B}$. We conclude $\psi(x \otimes y) = \psi(E_0(x \otimes y)) = \psi(x \otimes \mathbb{1})\varphi(y)$ and thus $\psi = \psi_{|\mathcal{A}_0} \otimes \varphi$, [KR86, 11.4.11, 11.2.7]. □

2.4. **Non-commutative examples of $\mathcal{A}_0$-independence.** In the remaining part of this section we present further examples of $\mathcal{A}_0$-independence which, in particular, illustrate that $\mathcal{A}_0$-independent von Neumann algebras may not commute.

Example 2.4.1 (Fermionic or CAR independence)**.** We consider the canonical anticommutation relations (CAR) (see for example [BR81, 5.2.5]). Let $\mathcal{K}$ be a Hilbert space and let CAR($\mathcal{K}$) denote the C*-algebra, generated by the elements $\{a(f) \mid f \in \mathcal{K}\}$, satisfying for all $f, g \in \mathcal{K}$: $f \mapsto a(f)$ is antilinear and

$$a(f)a(g) + a(g)a(f) = 0,$$
$$a(f)a(g)^* + a(g)^*a(f) = \langle f \mid g\rangle \mathbb{1}.$$

Consider on CAR($\mathcal{K}$) the quasi-free (gauge-invariant) state ψ_λ, defined by

$$\psi_\lambda(a^*(f)a(g)) = \lambda\langle g \mid f\rangle$$

for some fixed λ with $0 < \lambda < 1$ [Ara71, Ara87]. Let $(\mathcal{B}, \psi_\lambda)$ be the probability space obtained as the weak closure of $\mathrm{CAR}(\mathcal{K})$ in the GNS representation associated to ψ_λ. Let $\mathcal{K}_i$, $i = 0, 1, 2$, be mutually pairwise orthogonal closed subspaces in $\mathcal{K}$ and denote by $\mathcal{B}_i$ the von Neumann subalgebras generated by $\{a(f) \,|\, f \in \mathcal{K}_i\}$ $(i = 0, 1, 2)$ in the GNS representation. One verifies immediately that $\mathcal{B}_0 \vee \mathcal{B}_1$ and $\mathcal{B}_0 \vee \mathcal{B}_2$ are $\mathcal{B}_0$-independent. In particular, $\mathcal{B}_1$ and $\mathcal{B}_2$ are $\mathbb{C}$-independent.

Example 2.4.2 (Free independence)**.** An important example of $\mathcal{A}_0$-independence is given by Voiculescu's (amalgamated) free independence [VDN92]. Let $\mathcal{B}_1, \mathcal{B}_2$ be two subalgebras of the $\mathcal{B}_0$-expected probability space $(\mathcal{B}, \varphi)$ such that $\mathcal{B}_0 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$. Let $E_0 \in \mathrm{Mor}(\mathcal{B}, \varphi)$ denote the conditional expectation onto $\mathcal{B}_0$. The algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ are $\mathcal{B}_0$-freely independent if

$$E_0(b_1 b_2 \dots b_n) = 0$$

whenever $E(b_i) = 0$, $1 \leq i \leq n$ and $b_i \in \mathcal{B}_{j(i)}$ with $j(i) \neq j(i+1)$, $1 \leq i \leq n-1$. Notice that for $\mathcal{B}_0 \simeq \mathbb{C}$ this definition reduces to free independence with respect to the state φ. It is elementary to check that $\mathcal{B}_0$-free independence implies $\mathcal{B}_0$-independence.

Example 2.4.3 (q-Gaussian processes and $\mathcal{A}_0$-independence)**.** A further example originates from the construction of q-Fock spaces $(-1 < q < 1)$ by Bożejko and Speicher [BS91]. Let $\mathcal{K}_\mathbb{R}$ be a real Hilbert space and $\mathcal{K} = \mathcal{K}_\mathbb{R} \oplus \mathrm{i}\mathcal{K}_\mathbb{R}$ its complexification. Then a family $\{a(f) \,|\, f \in \mathcal{K}_\mathbb{R}\}$, satisfying

$$a(f)a(g) + a(g)a(f) = 0, \qquad a(f)a(g)^* - q a(g)^* a(f) = \langle f \,|\, g \rangle \mathbb{1}$$

for all $f, g \in \mathcal{K}_\mathbb{R}$, is realized as bounded linear operators on the q-Fock space $\mathcal{F}_q(\mathcal{K})$. Let $\mathcal{B}$ denote the von Neumann algebra generated by q-Gaussian processes $\{\Phi(f) := a(f) + a(f)^* \,|\, f \in \mathcal{K}_\mathbb{R}\}$, or equivalently in the case $\mathcal{K} = L^2_\mathbb{R}(\mathbb{R})$, generated by all increments of q-Brownian motions $(\Phi(\chi_{[s,t]}))_{s<t}$ (see [BKS97]). Then the vacuum vector $\Omega \in \mathcal{F}_q(\mathcal{K})$ defines a tracial faithful normal state τ on $\mathcal{B}$. If $\mathcal{K}_1$ and $\mathcal{K}_2$ are two orthogonal closed subspaces in $\mathcal{K}_\mathbb{R}$, then the von Neumann subalgebras $\mathcal{B}_i := \bigvee\{a(f) + a(f)^* \,|\, f \in \mathcal{K}_i\}$ $(i = 1, 2)$ are $\mathbb{C}$-independent. If $\mathcal{K}_0$ is a third closed subspace, orthogonal to $\mathcal{K}_1$ and $\mathcal{K}_2$, which generates the von Neumann subalgebra $\mathcal{B}_0$, then it is again elementary to verify that $\mathcal{B}_0 \vee \mathcal{B}_1$ and $\mathcal{B}_0 \vee \mathcal{B}_2$ are $\mathcal{B}_0$-independent.

This list of examples can be continued easily. More examples of $\mathbb{C}$- or $\mathcal{A}_0$-independence arise from von Neumann algebras generated by so-called generalized Brownian motions on deformed Fock spaces [BS94, BG02, GM02, Kró02], which contain q-Brownian motions as a simple case. All related constructions, necessary to provide these further examples, are captured by so-called functors of white noise, as introduced in [Küm85] and further considered in [GM02]. In view of Anshelevich's results on q-Lévy processes [Ans], the question arises of whether they also provide examples of $\mathbb{C}$-independence.

Aside from these quantum probabilistic approaches to constructing new examples, there is a second rich source for $\mathcal{A}_0$-independence: subfactor theory with all its commuting squares.

3. Continuous Bernoulli shifts I

This section is devoted to the introduction of a *(non-commutative) continuous Bernoulli shift.*

3.1. **Continuous Bernoulli shifts and their basic properties.** We start with some notation and a reminder that our notion of a filtration does not stipulate continuity properties (see Subsection 1.3).

Notation 3.1.1. The set of all intervals I in $\mathbb{R}$ (including the empty set $\emptyset$) is denoted by $\mathcal{I}$. Furthermore we let $I + t := \{s + t \mid s \in I\}$ and $\emptyset + t := \emptyset$. The set $\operatorname{Int} I$ is the interior of I.

Definition 3.1.2. *Let $(\mathcal{A}, \psi; \mathcal{A}_0)$ be an expected probability space, equipped with a pointwise weak*-continuous group of automorphisms $S = (S_t)_{t\in\mathbb{R}} \subset \operatorname{Aut}(\mathcal{A}, \psi)$ and a minimal filtration $(\mathcal{A}_I, \psi_I)_{I\in\mathcal{I}}$. The quintuple $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ is called an* ($\mathcal{A}_0$-)expected (non-commutative) continuous Bernoulli shift *if it enjoys the following properties:*

- (i) *$\mathcal{A}_0$ is the fixed point algebra of S;*
- (ii) *S acts covariantly on the filtration: $S_t\mathcal{A}_I = \mathcal{A}_{I+t}$ for any $t \in \mathbb{R}$, $I \in \mathcal{I}$;*
- (iii) *$\mathcal{A}_I$ and $\mathcal{A}_J$ are $\mathcal{A}_0$-independent whenever $I \cap J = \emptyset$.*

The $\mathcal{A}_0$-expected continuous Bernoulli shift is said to be trivial *if $S = \mathrm{id}$.*

For shortness, when there is no chance of confusion, $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ as well as its automorphism group S will both just be called a shift. If the shift is trivial, then $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ can be identified canonically with the expected non-commutative probability space $(\mathcal{A}, \psi; \mathcal{A}_0)$. If $\mathcal{A} \simeq \mathbb{C}$, a continuous Bernoulli shift will also just be denoted by the quadruple $(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}})$.

Notation 3.1.3. Throughout, E_I denotes the conditional expectation from $(\mathcal{A}, \psi)$ onto $\mathcal{A}_I$, where $I \in \mathcal{I}$.

In Definition 3.1.2 (iii) we required that $\mathcal{A}_I$ and $\mathcal{A}_J$ are $\mathcal{A}_0$-independent if $I \cap J = \emptyset$. But boundary points of such intervals don't matter for $\mathcal{A}_0$-independence. Moreover, we see that $\mathcal{A}_0$ equals $\mathcal{A}_\emptyset$, from which we take advantage occasionally in proofs.

Lemma 3.1.4. *For an expected continuous Bernoulli shift $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ is $\mathcal{A}_0 = \mathcal{A}_\emptyset$. Moreover, the following are equivalent:*

- (iii) *$\mathcal{A}_I$ and $\mathcal{A}_J$ are $\mathcal{A}_0$-independent whenever $I \cap J = \emptyset$.*
- (iii') *$\mathcal{A}_I$ and $\mathcal{A}_J$ are $\mathcal{A}_0$-independent whenever $\operatorname{Int} I \cap \operatorname{Int} J = \emptyset$.*

Proof. We conclude $\mathcal{A}_\emptyset = \mathcal{A}_\emptyset \cap \mathcal{A}_\emptyset = \mathcal{A}_0$ from Definition 3.1.2 (iii) and Proposition 2.1.2 (v).

It is obvious that (iii') implies (iii). We are left to prove the inverse. For $\operatorname{Int} I \cap \operatorname{Int} J = \emptyset$ exists $t \in \mathbb{R}$ such that, without loss of generality, $I \subset (-\infty, t]$ and $J \subset [t, \infty)$. From the continuity of the past and future filtration (see Lemma 3.1.5 (ii) below) we conclude $E_I E_J = E_I E_{(-\infty,t]} E_{[t,\infty)} E_J = E_I E_{(-\infty,t)} E_{(t,\infty)} E_J = E_I E_\emptyset E_J = E_\emptyset$. □

We collect further, frequently used properties of a continuous Bernoulli shift.

Lemma 3.1.5. *A shift, as stated in Definition 3.1.2, enjoys the following properties:*

(i) *The shift acts covariantly on the conditional expectations for any $t \in \mathbb{R}, I \in \mathcal{I}$:*
$$S_t E_I = E_{I+t} S_t;$$
(ii) *the past filtration $(\mathcal{A}_{(-\infty,t]})_{t\in\mathbb{R}}$ and the future filtration $(\mathcal{A}_{[t,\infty)})_{t\in\mathbb{R}}$ are continuous, or equivalently, the families of conditional expectations $(E_{(-\infty,t]})_{t\in\mathbb{R}}$ and $(E_{[t,\infty)})_{t\in\mathbb{R}}$ are pointwise weak*-continuous. In particular, $\mathcal{A}_{(-\infty,t]} = \mathcal{A}_{(-\infty,t)}$ and $\mathcal{A}_{[t,\infty)} = \mathcal{A}_{(t,\infty)}$;*
(iii) *$\mathcal{A}_{[t,t]} = \mathcal{A}_0 = \mathcal{A}_\emptyset$ for any $t \in \mathbb{R}$;*
(iv) *$\mathcal{A}_0 \subset \mathcal{A}_I$ for any $I \in \mathcal{I}$;*
(v) *a shift is* tail trivial*: $\bigcap_{t\in\mathbb{R}} \mathcal{A}_{(-\infty,t]} = \mathcal{A}_0 = \bigcap_{t\in\mathbb{R}} \mathcal{A}_{[t,\infty)}$;*
(vi) *a shift is* locally trivial, *i.e., $\bigcap_{\varepsilon>0} \mathcal{A}_{[t-\varepsilon,t+\varepsilon]} = \mathcal{A}_0$ for any $t \in \mathbb{R}$.*

Notice that the filtration of a shift may not be continuous downwards or upwards. These and additional properties will be discussed in more detail in Section 4.

Proof. From the covariant action of the shift S we get
$$\begin{aligned}\psi(xS_tE_I(y)) &= \psi(E_{I+t}(x)S_tE_I(y)) = \psi(S_{-t}E_{I+t}(x)E_I(y))\\ &= \psi(S_{-t}E_{I+t}(x)y) = \psi(E_{I+t}(x)S_t(y)) = \psi(xE_{I+t}S_t(y))\end{aligned}$$
for any $x, y \in \mathcal{A}$. This implies (i).

(ii) follows from (i) and the pointwise continuity of the shift S in the weak* topology, since $E_{(-\infty,t-\varepsilon]} = S_{-\varepsilon} E_{(-\infty,t]} S_\varepsilon$. (The equivalence of the two formulations is shown by routine arguments.)

(iii) $\mathcal{A}_{[t,t]}$ and $\mathcal{A}_{[t,t]}$ are $\mathcal{A}_0$-independent by Lemma 3.1.4. But this implies $\mathcal{A}_{[t,t]} \cap \mathcal{A}_{[t,t]} = \mathcal{A}_0$ (see Proposition 2.1.2 (v)). The equality $\mathcal{A}_0 = \mathcal{A}_\emptyset$ is already shown in Lemma 3.1.4.

(iv) is part of the definition of $\mathcal{A}_0$-independence.

The first equality of (v) follows from the observation that if $x \in \bigcap_{t\in\mathbb{R}} \mathcal{A}_{(-\infty,t]}$ and $y \in \bigcup_{n\in\mathbb{N}} \mathcal{A}_{[-n,\infty)}$, then x and y are $\mathcal{A}_0$-independent elements. It follows that $\psi((x - E_0(x))y) = \psi((x - E_0(x)))\psi(y) = 0$. From the weak* density of $\bigcup_{n\in\mathbb{N}} \mathcal{A}_{[-n,\infty)}$ in $\mathcal{A}$, we conclude that $x = E_0(x)$, hence $\bigcap_{t\in\mathbb{R}} \mathcal{A}_{(-\infty,t]} = \mathcal{A}_0$. The second equality of (v) is shown by the same arguments.

We are left to prove (vi). From

$$\bigcap_{\varepsilon>0} \mathcal{A}_{[t-\varepsilon,t+\varepsilon]} \subseteq \bigcap_{\varepsilon>0} (\mathcal{A}_{(-\infty,t+\varepsilon]} \cap \mathcal{A}_{[t-\varepsilon,\infty)}) = \Big[\bigcap_{\varepsilon>0} \mathcal{A}_{(-\infty,t+\varepsilon]}\Big] \cap \Big[\bigcap_{\varepsilon>0} \mathcal{A}_{[t-\varepsilon,\infty)}\Big]$$

we conclude, using the continuity of the past and future filtration and finally the $\mathcal{A}_0$-independence,

$$\mathcal{A}_0 \subseteq \bigcap_{\varepsilon>0} \mathcal{A}_{[t-\varepsilon,t+\varepsilon]} \subseteq \mathcal{A}_{(-\infty,t]} \cap \mathcal{A}_{[t,\infty)} = \mathcal{A}_0 \,. \qquad \square$$

The following result states that the shift S is strongly mixing. It will be crucial in the proof of Theorem 3.2.1.

Lemma 3.1.6. *Let $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ be an expected continuous Bernoulli shift. For any $x \in \mathcal{A}$,*

$$\lim_{|t|\to\infty} S_t(x) = E_0(x),$$

in the weak topology.*

Proof. For bounded intervals I and J and $x \in \mathcal{A}_I$, $y \in \mathcal{A}_J$ one calculates

$$\lim_{|t|\to\infty} \psi(yS_t(x)) = \lim_{|t|\to\infty} \psi(E_0(yS_t(x))) = \psi(E_0(y)E_0(x)) = \psi(yE_0(x))\,.$$

Since the filtration is minimal, these identities extend to arbitrary $x, y \in \mathcal{A}$ by standard arguments. Now the assertion follows from the norm density of the functionals $\{\psi(y\,\cdot) \mid y \in \mathcal{A}\}$ in $\mathcal{A}_*$ and the boundedness of the set $\{S_t(x) \mid t \in \mathbb{R}\}$. $\square$

Let us mention a subtle fact in the above proof: the mixing property of the shift hinges on the notion of the minimality of a filtration, as introduced in Subsection 1.3. It includes that $\mathcal{A}$ is approximated by the 'local' net $(\mathcal{A}_{[s,t]})_{-\infty<s\leq t<\infty}$. Thus the 'global' structure of a continuous Bernoulli shift is already determined by its 'local' structure.

3.2. **The type of a $\mathbb{C}$-expected continuous Bernoulli shift.** From a probabilistic viewpoint our next result may be viewed as the application of Kolmogorov's zero-one law to central projections.

Theorem 3.2.1. *Let $(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}})$ be a non-trivial $\mathbb{C}$-expected continuous Bernoulli shift. Then the von Neumann algebra $\mathcal{A}$ is either finite or of type III. Moreover, $\mathcal{A}$ is finite if and only if ψ is a trace.*

This result generalizes immediately to a factor-expected continuous Bernoulli shift if one considers, instead of $\mathcal{A}$, the relative commutant ${\mathcal{A}_0}' \cap \mathcal{A}$.

Before we start the proof of the theorem, we note an immediate consequence.

Corollary 3.2.2. *Let the $\mathbb{C}$-expected shift be given as stated in Theorem 3.2.1. If $\mathcal{A}$ is a factor, then $\mathcal{A}$ is either of type II_1 or of type III.*

Proof. Choose $x \in \mathcal{A}_{[0,1]}$ with $E_0(x) \neq 0$ and $x_n := S_n(x)$, $n \in \mathbb{N}$. Then, by $\mathbb{C}$-independence, $\{x_n - E_0(x_n) \,|\, n \in \mathbb{N}\}$ is a mutually ψ-orthogonal family in $\mathcal{A}$, thus $\mathcal{A}$ is infinite-dimensional. Consequently, the factor $\mathcal{A}$ is not of finite type I, which proves the corollary. □

Proof of Theorem 3.2.1. Let $z \in \mathcal{Z}(\mathcal{A})$ be the maximal central semi-finite projection. Since for any automorphism α on $\mathcal{A}$, the projection $\alpha(z)$ is again a central semi-finite projection, we have $\alpha(z) \leq z$. Interchanging α with α^{-1} yields $\alpha(z) = z$ and thus in particular $S_t(z) = z$ for any $t \in \mathbb{R}$. By the mixing property of the shift S, as stated in Proposition 3.1.6, it follows $z \in \mathbb{C} \cdot \mathbb{1}$. If $z = 0$ then $\mathcal{A}$ is of type III and we are done. Therefore we may assume in the following that $\mathcal{A}$ is semi-finite. It follows that there exists a SOT-continuous unitary group $(u_s)_{s\in\mathbb{R}} \subset \mathcal{A}$ with the property $\sigma_s^\psi(x) = u_s^* x u_s$ for any $x \in \mathcal{A}$ (see [Ped79, Prop. 8.14.13]). Since S_t and σ_s^ψ commute, one concludes $S_t(u_s^*) x S_t(u_s) = u_s^* x u_s$ for all $x \in \mathcal{A}$ and $s, t \in \mathbb{R}$. Thus $S_t(u_s) u_s^* \in \mathcal{Z}(\mathcal{A})$ for any $t \in \mathbb{R}$. From this follows, again by Proposition 3.1.6, that the weak* limit $E_0(u_s) u_s^*$ of this sequence is an element of $\mathcal{Z}(\mathcal{A})$. Moreover, one has $E_0(u_s) = \psi(u_s)\mathbb{1}$. Since $s \mapsto u_s$ is SOT-continuous and $u_0 = \mathbb{1}$, it follows that $\psi(u_s) \neq 0$ for all s in some small 0-neighborhood. Consequently, $u_s^* \in \mathcal{Z}(\mathcal{A})$ for any s in this 0-neighborhood. But this implies $\sigma_s^\psi = \mathrm{id}$ for any $s \in \mathbb{R}$ by the group property of σ^ψ. Thus ψ is a normal trace on $\mathcal{A}$ [Ped79, Lem. 8.14.6] and, since ψ is faithful, we conclude that $\mathcal{A}$ is a finite von Neumann algebra, [Tak03a, Thm. V.2.4].
By the above arguments, we have in particular proven that the finiteness of $\mathcal{A}$ implies that ψ is a trace. The converse is obvious. □

Remark 3.2.3. (i) Up to the present we know examples of $\mathbb{C}$-expected continuous Bernoulli shifts with a von Neumann algebra $\mathcal{A}$ of type I_1, type II_1 and type III_λ $(0 < \lambda \leq 1)$. We conjecture that for a $\mathbb{C}$-expected continuous Bernoulli shift $\mathcal{A}$ cannot be of type I_n for $n \neq 1$. It is of interest to investigate further the case of type III_0.

(ii) A general result for the type of $\mathcal{A}_0{}' \cap \mathcal{A}$ is obtained by disintegration theory and will be presented elsewhere. It relies on the fact that the structure of continuous Bernoulli shifts is stable with respect to relative commutants. In particular, we introduce the notion of a 'derived' continuous Bernoulli shift and investigate its properties, following subfactor theory [HK].

3.3. **Composition of continuous Bernoulli shifts.** It is easy to see that shifts, as introduced in Definition 3.1.2, are closed under tensor product and direct sum compositions.

Proposition 3.3.1. *Let $(\mathcal{A}^{(i)}, \psi^{(i)}, S^{(i)}, (\mathcal{A}_I^{(i)})_{I\in\mathcal{I}}; \mathcal{A}_0^{(i)})$ be expected continuous Bernoulli shifts $(i = 1, 2)$. Then so are the following:*

(i) $\big(\mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}, \psi^{(1)} \otimes \psi^{(2)}, S^{(1)} \otimes S^{(2)}, (\mathcal{A}_I^{(1)} \otimes \mathcal{A}_I^{(2)})_{I\in\mathcal{I}}; \mathcal{A}_0^{(1)} \otimes \mathcal{A}_0^{(2)}\big)$, *and*

(ii) $\big(\mathcal{A}^{(1)}\oplus\mathcal{A}^{(2)}, \mu\psi^{(1)}\oplus(1-\mu)\psi^{(2)}, S^{(1)}\oplus S^{(2)}, (\mathcal{A}_I^{(1)}\oplus\mathcal{A}_I^{(2)})_{I\in\mathcal{I}}; \mathcal{A}_0^{(1)}\oplus\mathcal{A}_0^{(2)}\big)$, *for* $0<\mu<1$.

The tensor product of a $\mathbb{C}$-expected and a trivial $\mathcal{A}_0$-expected shift gives rise to an $\mathcal{A}_0$-expected shift.

Definition 3.3.2. *The tensor product of an expected continuous Bernoulli shift* $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ *and a trivial* $\mathcal{B}$*-expected continuous Bernoulli shift is called the* amplification of $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ by $\mathcal{B}$.

Remark 3.3.3. The composition by tensor product extends easily to infinite tensor products. With a bit more work it may be seen that the procedure of direct sum compositions carries over to direct integrals of $\mathcal{A}_0^{(\gamma)}$-expected shifts, $\gamma\in\Gamma$, with respect to a standard probability space (Γ,μ).

3.4. **Decomposition of continuous Bernoulli shifts.** In the following we focus on decompositions of continuous Bernoulli shifts. They are closed under compression by conditional expectations and by orthogonal projections, subject to some further conditions. These compressions provide, roughly speaking, the inverse procedures to the compositions as stated in Proposition 3.3.1.

Proposition 3.4.1. *Let* $E\colon (\mathcal{A},\psi)\to(\mathcal{A}_0,\psi_0)$ *be a conditional expectation and let* $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ *be an expected continuous Bernoulli shift. If* $ES_t = S_tE$ *and* $EE_I = E_IE$ *for all* $t\in\mathbb{R}$, $I\in\mathcal{I}$, *then* $\big(E(\mathcal{A}), \psi_{|E(\mathcal{A})}, S_{|E(\mathcal{A})}, (E(\mathcal{A}_I))_{I\in\mathcal{I}}; E(\mathcal{A}_0)\big)$ *is an expected continuous Bernoulli shift.*

Definition 3.4.2. *The shift* $\big(E(\mathcal{A}), \psi_{|E(\mathcal{A})}, S_{|E(\mathcal{A})}, (E(\mathcal{A}_I))_{I\in\mathcal{I}}; E(\mathcal{A}_0)\big)$ *is called the* compression *of* $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ *by the conditional expectation* E.

If $E=E_0$ then the compression leads to a trivial shift.

Proof of Proposition 3.4.1. The restriction $\psi_{|E(\mathcal{A})}$ is a faithful normal state on $E(\mathcal{A})$. Since the conditional expectation E and the shift S commute, $E(\mathcal{A})$ is globally invariant under the action of S. Thus the restriction $S_{|E(\mathcal{A})}$ is well-defined and has $E(\mathcal{A}_0)$ as fixed point algebra. Moreover, it is $S_tE(\mathcal{A}_I) = ES_t(\mathcal{A}_I) = E(\mathcal{A}_{I+t})$ for any $t\in\mathbb{R}$, $I\in\mathcal{I}$. Since $E(\mathcal{A}_I)\subseteq E(\mathcal{A}_J)$ whenever $I\subseteq J$, the family $(E(\mathcal{A}_I))_{I\in\mathcal{I}}$ defines a filtration of $(E(\mathcal{A}),\psi_{|E(\mathcal{A})})$. From the minimality of $(\mathcal{A}_I)_{I\in\mathcal{I}}$ and the normality of E we conclude that $\bigcup\{E(\mathcal{A}_I)\mid I\in\mathcal{I}\text{ is bounded}\}$ is weak*-dense in $E(\mathcal{A})$. The minimality of the filtration $(E(\mathcal{A}_I))_{I\in\mathcal{I}}$ follows now by the double commutation theorem. Finally, the independence of $E(\mathcal{A}_I)$ and $\mathrm{E}(\mathcal{A}_J)$ for $I\cap J=\emptyset$ is an immediate consequence of $EE_IEE_J = EE_IE_J = EE_0$. $\square$

The following compression will be needed in Subsection 4.5. Recall that $\mathcal{A}^\psi$ is the centralizer of $(\mathcal{A},\psi)$.

Corollary 3.4.3. $\big(\mathcal{A}^\psi, \psi_{|\mathcal{A}^\psi}, S_{|\mathcal{A}^\psi}, (\mathcal{A}^\psi \cap A_I)_{I\in\mathcal{I}}; \mathcal{A}^\psi \cap \mathcal{A}_0\big)$ *is an expected continuous Bernoulli shift.*

Proof. We shall show that the conditional expectation E from $\mathcal{A}$ onto the centralizer $\mathcal{A}^\psi$ commutes with the shift S and the conditional expectations E_I. Since $\mathcal{A}^\psi$ is the fixed point algebra of the modular automorphism group σ^ψ, we have $E(x) = \text{SOT-}\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sigma_t^\psi(x)\,\mathrm{d}t$ for any $x \in \mathcal{A}$. Since the modular automorphism group σ^ψ commutes with morphisms of $(\mathcal{A},\psi)$, we conclude that E commutes with S_t and E_I for any $t \in \mathbb{R}$, $I \in \mathcal{I}$. □

In Proposition 3.4.1 it is required that the conditional expectation E commutes with the conditional expectations E_I of the filtration. Dropping this condition leads to more general compressions.

Proposition 3.4.4. *Let* $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ *be an expected shift and* $E \in \mathrm{Mor}(\mathcal{A},\psi)$ *a conditional expectation with* $S_tE = ES_t$ *and* $EE_0 = E_0E$ *for all* $t \in \mathbb{R}$. *Then* $(\mathcal{B}, \psi_{|E(\mathcal{A})}, S_{|E(\mathcal{A})}, (E(\mathcal{A}) \cap \mathcal{A}_I)_{I\in\mathcal{I}}; E(\mathcal{A}) \cap \mathcal{A}_0)$ *is an expected continuous Bernoulli shift, where* $\mathcal{B} := \bigvee_{I\in\mathcal{I}} E(\mathcal{A}) \cap \mathcal{A}_I$.

Notice that $\bigvee_{I\in\mathcal{I}} E(\mathcal{A}) \cap \mathcal{A}_I$ can be much smaller than $E(\mathcal{A})$.

Proof. All arguments in the proof of Proposition 3.4.1 are valid, aside from those for the covariant action of the shift, for the minimality and the independence. We alter them as follows. From $S_t(E(\mathcal{A}) \cap \mathcal{A}_I) = ES_t(\mathcal{A}) \cap S_t(\mathcal{A}_I) = E(\mathcal{A}) \cap \mathcal{A}_{I+t}$ it follows the covariant action of the shift. The minimality of the filtration $(E(\mathcal{A}) \cap \mathcal{A}_I))_{I\in\mathcal{I}}$ is evident. Finally, let $\widetilde{E}_I$ denote the conditional expectation from $(\mathcal{A},\psi)$ onto $E(\mathcal{A}) \cap \mathcal{A}_I$. Notice that $\widetilde{E}_I = \widetilde{E}_I E_I = E_I \widetilde{E}_I$, $\widetilde{E}_I = \widetilde{E}_I E = E\widetilde{E}_I$, since conditional expectations are ψ-selfadjoint (compare Theorem B.2), and $\widetilde{E}_0 = EE_0 = E_0E$, since E and E_0 commute. Consequently, we obtain $\widetilde{E}_I\widetilde{E}_J = \widetilde{E}_I E_I E_J \widetilde{E}_J = \widetilde{E}_I E_0 \widetilde{E}_J = \widetilde{E}_I E E_0 \widetilde{E}_J = \widetilde{E}_I \widetilde{E}_\emptyset \widetilde{E}_J = \widetilde{E}_0$ for all $I, J \in \mathcal{I}$. Since ψ and S restrict to $E(\mathcal{A})$, the compression defines a continuous Bernoulli shift as spelt out in the proposition. □

Proposition 3.3.1 states that shifts can be composed by direct sums. In the following we present compressions which, roughly speaking, provide the reverse procedure. For a non-zero orthogonal projection $e \in \mathcal{A}$ we define $\psi_e(x) = \psi(exe)/\psi(e)$ for $x \in \mathcal{A}$.

Proposition 3.4.5. *Let* $e \in \mathcal{A}_0 \cap \mathcal{A}^\psi$ *be a non-zero orthogonal projection. Then* $(e\mathcal{A}e, \psi_e, S_{|e\mathcal{A}e}, (e\mathcal{A}_Ie)_{I\in\mathcal{I}}; e\mathcal{A}_0e)$ *is an expected continuous Bernoulli shift.*

Proof. Clearly, ψ_e is a faithful normal state on the von Neumann algebra $e\mathcal{A}e$. Moreover, we observe $\sigma_t^{\psi_e}(e\mathcal{A}_Ie) = e\sigma_t^\psi(\mathcal{A}_I)e = e\mathcal{A}_Ie$. Thus, there exist uniquely conditional expectations Q_I from $(e\mathcal{A}e, \psi_e)$ onto $e\mathcal{A}_Ie$ for $I \in \mathcal{I}$ such that $Q_I(exe) = eE_I(x)e$ for any $x \in \mathcal{A}$. Since e is a fixed point of the shift S, the restriction of $S_{|e\mathcal{A}e}$ is well-defined. It leaves the state ψ_e invariant and has the fixed point algebra $e\mathcal{A}_0 e$. Since $S_t(e\mathcal{A}_I e) = eS_t(\mathcal{A}_I)e = e\mathcal{A}_{I+t}e$ for any $t \in \mathbb{R}$, $I \in \mathcal{I}$, the shift acts covariantly on $(e\mathcal{A}_I e)_{I\in\mathcal{I}}$. The family $(e\mathcal{A}_I e)_{I\in\mathcal{I}}$ defines a filtration on $e\mathcal{A}e$ since $e\mathcal{A}_I e \subseteq e\mathcal{A}_J e$ whenever $I \subseteq J$. From the minimality of $(\mathcal{A}_I)_{I\in\mathcal{I}}$ we conclude that $\bigcup_I\{e\mathcal{A}_I e \mid I \in \mathcal{I} \text{ is bounded}\}$ is weak*-dense in $e\mathcal{A}e$. This ensures the minimality of $(e\mathcal{A}_I e)_{I\in\mathcal{I}}$. Finally, the $(e\mathcal{A}_0 e)$-independence of $e\mathcal{A}_I e$ and $e\mathcal{A}_J e$ for $I \cap J = \emptyset$ follows from $Q_I Q_J(exe) = eE_J E_I(x)e = eE_0(x)e = eE_\emptyset(x)e = Q_\emptyset(x)$. □

4. Continuous Bernoulli shifts II

A continuous Bernoulli shift is a family of von Neumann subalgebras which enjoys essentially two further structures: $\mathcal{A}_0$-independence and shift covariance. We did not stipulate further algebraic structure elements in Definition 3.1.2. Here we approach systematically some of these additional features which a continuous Bernoulli shift may carry. Large parts of the terminology will be in analogy to [Arv03] and/or [Tsi04], since we seek to highlight what is common and what is different.

4.1. Local minimality and local maximality.

The following two algebraic properties ensure the continuity of a filtration.

Definition 4.1.1. *The filtration $(\mathcal{A}_I)_{I\in\mathcal{I}}$ of an $\mathcal{A}_0$-expected continuous Bernoulli shift is*

(i) locally minimal *if $\mathcal{A}_I \vee \mathcal{A}_J = \mathcal{A}_K$ for any $I, J, K \in \mathcal{I}$ with $I \cup J = K$;*
(ii) locally maximal *if $\mathcal{A}_{(-\infty,t]} \cap \mathcal{A}_{[s,\infty)} = \mathcal{A}_{[s,t]}$ for any $s \leq t$.*

Lemma 4.1.2. *Let $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ be an expected continuous Bernoulli shift. If the filtration $(\mathcal{A}_I)_{I\in\mathcal{I}}$ is locally minimal (maximal), then it is continuous upwards (downwards).*

Consequently, such a filtration is continuous if it is locally minimal and locally maximal.

Proof. We first prove that local minimality implies upward continuity. For a closed interval K with non-empty interior we choose two intervals I, J and some $\varepsilon > 0$ with $I \cup J = K$ and $I + \varepsilon, J - \varepsilon \subset K$. The local minimality guarantees $\mathcal{A}_I \vee \mathcal{A}_J = \mathcal{A}_K$. If any $x \in \mathcal{A}_I$ and $y \in \mathcal{A}_J$ can be approximated by sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subset \bigcup\{\mathcal{A}_{K_0} \mid K_0 = \overline{K_0} \subset \operatorname{Int} K\}$, then $\mathcal{A}_I \vee \mathcal{A}_J \subseteq \bigvee\{\mathcal{A}_{K_0} \mid K_0 = \overline{K_0} \subset \operatorname{Int} K\} \subseteq \mathcal{A}_K$ implies the upward continuity. But such sequences are given by $x_n := S_{\varepsilon/n}(x)$ resp. $y_n := S_{-\varepsilon/n}(y)$, since they approximate x resp. y due to the pointwise weak* continuity of S.

For a locally maximal filtration $(\mathcal{A}_I)_{I\in\mathcal{I}}$ holds

$$\bigcap_{\varepsilon>0}\mathcal{A}_{[s-\varepsilon,t+\varepsilon]} = \bigcap_{\varepsilon>0}(\mathcal{A}_{(-\infty,t+\varepsilon]} \cap \mathcal{A}_{[s-\varepsilon,\infty)}) = \Big[\bigcap_{\varepsilon>0}\mathcal{A}_{(-\infty,t+\varepsilon]}\Big] \cap \Big[\bigcap_{\varepsilon>0}\mathcal{A}_{[s-\varepsilon,\infty)}\Big].$$

Now, by Lemma 3.1.5 (ii), the continuity of the past filtration and future filtration establishes the downward continuity. □

Corollary 4.1.3. *A locally minimal filtration $(\mathcal{A}_I)_{I\in\mathcal{I}}$ of a shift enjoys $\mathcal{A}_I = \mathcal{A}_{\overline{I}}$ for any $I \in \mathcal{I}$.*

Proof. Lemma 4.1.2 insures the upward continuity. Since $\bigvee_{\varepsilon>0}\mathcal{A}_{[s+\varepsilon,t-\varepsilon]} \subseteq \mathcal{A}_{(s,t)}$, the upward continuity of a filtration implies immediately $\mathcal{A}_{(s,t)} = \mathcal{A}_{[s,t]}$ for any $s < t$. Similar arguments prove the two remaining cases of unbounded intervals. □

Remark 4.1.4. It is tempting to stipulate local minimality and/or local maximality in Definition 3.1.2 of an $\mathcal{A}_0$-expected continuous Bernoulli shift. But local minimality is obstructed by compressions, as will be shown in Subsection 4.5. On the other hand, it is elementary to see that local maximality is stable under the compression with conditional expectations. Local maximality is always present for continuous Bernoulli shifts, which are non-commutative white noises (see Definition 6.5.2). But we do not know whether every continuous Bernoulli shift is locally maximal.

4.2. **Enriched independence.** All examples of $\mathcal{A}_0$-expected shifts enjoy a richer independence structure than $\mathcal{A}_0$-independence, at least as known presently by the authors. Many of these examples come in particular from 'functors of white noise' (see [Küm85, GM02]).

Definition 4.2.1. *An $\mathcal{A}_0$-expected continuous Bernoulli shift or its filtration $(\mathcal{A}_I)_{I\in\mathcal{I}}$ has an* enriched independence *if $\mathcal{A}_I$ and $\mathcal{A}_J$ are $\mathcal{A}_{I\cap J}$-independent for any $I, J \in \mathcal{I}$.*

Proposition 4.2.2. *For an expected shift $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ are equivalent:*

(i) *$\mathcal{A}_I$ and $\mathcal{A}_J$ are $\mathcal{A}_{I\cap J}$-independent for any $I, J \in \mathcal{I}$;*
(ii) *$\mathcal{A}_I \cap \mathcal{A}_J = \mathcal{A}_{I\cap J}$ and $E_I E_J = E_J E_I$ for any $I, J \in \mathcal{I}$;*
(iii) *$(\mathcal{A}_I)_{I\in\mathcal{I}}$ is locally maximal and $E_{(-\infty,t]}E_{[s,\infty)} = E_{[s,\infty)}E_{(-\infty,t]}$ for any $s, t \in \mathbb{R}$.*

A filtration $(\mathcal{A}_I)_{I\in\mathcal{I}}$ with enriched independence is continuous and enjoys $\mathcal{A}_I = \mathcal{A}_{\overline{I}}$ for all $I \in \mathcal{I}$.

Notice that the enriched independence of a filtration does not imply its local minimality (see Proposition 4.5.1 for a counterexample).

Proof. The equivalence of (i) and (ii) is evident by Proposition 2.1.2 (v). Also it is clear that that (ii) implies (iii). We are left to prove the converse. Local maximality guarantees $\mathcal{A}_{(-\infty,t]} \cap \mathcal{A}_{[s,\infty)} = \mathcal{A}_{[s,t]}$ for $s \leq t$ and consequently, by the equivalence of (v) and (iv) in Proposition 2.1.2, $E_{(-\infty,t]}E_{[s,\infty)} = E_{[s,t]}$. This yields $E_{(r,t]}E_{[s,u)} = E_{(r,t]}E_{(-\infty,t]}E_{[s,\infty)}E_{[s,u)} = E_{(r,t]}E_{[s,t]}E_{[s,u)} = E_{[s,t]} = E_{[s,u)}E_{[s,t]}E_{(r,t]} = E_{[s,u)}E_{(r,t]}$ for any $-\infty \leq r < s < t < u \leq \infty$. Aside of the cases $I \cap J = \emptyset$ or $I \subseteq I \cap J$ (which follow easily from the $\mathcal{A}_0$-independence resp. the monotony of the filtration), the proof (iii) $\Rightarrow$ (ii) is completed if the filtration $(\mathcal{A}_I)_{I\in\mathcal{I}}$ is continuous upwards, entailing $\mathcal{A}_I = \mathcal{A}_{\overline{I}}$ for any $I \in$

$\mathcal{I}$. But $E_{[s+\varepsilon,t-\varepsilon]} = E_{(-\infty,t-\varepsilon]}E_{[s+\varepsilon,\infty)} \xrightarrow{\varepsilon\searrow 0} E_{(-\infty,t]}E_{[s,\infty)} = E_{[s,t]}$ from the poinwise SOT-continuity of $t \mapsto E_{(-\infty,t]}$ resp. $s \mapsto E_{[s,\infty)}$. Thus the filtration is continuous upwards. Finally, the local maximality implies the downward continuity. Consequently, $(\mathcal{A}_I)_{I\in\mathcal{I}}$ is continuous. $\square$

Corollary 4.2.3. *If a continuous Bernoulli shift has an enriched independence, then so does its compression by any conditional expectation E, satisfying the conditions of Proposition 3.4.1.*

Proof. This follows immediately from Proposition 4.2.2 (ii) and $EE_I = E_IE$ for any interval $I \subset \mathbb{R}$, since $\big(E(\mathcal{A}) \cap \mathcal{A}_I\big) \cap \big(E(\mathcal{A}) \cap \mathcal{A}_J\big) = E(\mathcal{A}) \cap \mathcal{A}_{I\cap J}$ and $E_IEE_JE = E_JEE_IE$. $\square$

The stability of shifts with enriched independence under compressions suggests stipulating this structure in Definition 3.1.2. Nevertheless, it is not needed for the proofs of our main results in this paper. Moreover, for more general compressions (see Proposition 3.4.4) the proof of Corollary 4.2.3 breaks down, since the conditional expectations E and E_I may no longer commute.

Remark 4.2.4. (i) It is an open problem to give examples of continuous Bernoulli shifts without enriched independence.

(ii) Further structure can be added to the index set $\mathcal{I}$ on $\mathbb{R}$ to provide other enriched forms of $\mathcal{A}_0$-independence. Here we focus on (possibly degenerate and unbounded) intervals as set $\mathcal{I}$. Stipulating a Boolean algebra structure for $\mathcal{I}$ (as is done for example in [TV98]) leads to even more enriched forms of $\mathcal{A}_0$-independence.

4.3. **Commuting past and future.** Shifts with a commuting past/future form an interesting class on their own, as they stem from tensor product independence (see Subsection 2.2). Roughly speaking, such commuting structures are present in the Hudson-Parthasarathy approach to quantum stochastic calculus [Par92], in the Arveson approach to continuous product systems of Hilbert spaces [Arv03] and in the Tsirelson-Vershik approach to continuous product systems of probability spaces [Tsi04].

Definition 4.3.1. *A $\mathbb{C}$-expected shift $(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}})$ (or its filtration) is said to have a* commuting past/future *if $\mathcal{A}_{(-\infty,0]}$ and $\mathcal{A}_{[0,\infty)}$ commute.*

Lemma 4.3.2. *Suppose that a $\mathbb{C}$-expected shift $(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}})$ satisfies one of the following (equivalent) additional conditions:*

(i) *the past $\mathcal{A}_{(-\infty,0]}$ and the future $\mathcal{A}_{[0,\infty)}$ commute;*
(ii) *$\mathcal{A}_{(-\infty,t]}$ and $\mathcal{A}_{[t,\infty)}$ commute for any $t \in \mathbb{R}$;*
(iii) *$\mathcal{A}_I$ and $\mathcal{A}_J$ commute for all $I, J \in \mathcal{I}$ with $\operatorname{Int} I \cap \operatorname{Int} J = \emptyset$.*

If in addition the filtration is locally minimal, then it is locally maximal, continuous and enjoys an enriched independence.

It is well-known that local maximality does not imply local minimality, even when adding commutativity of the von Neumann algebras to the assumptions of this lemma (see for example [Tsi03, Rem. 3.9]).

Proof. The equivalence of (i) and (ii) is obvious by stationarity, (iii) clearly implies (ii), the inverse implication follows from the monotony of the filtration and the $\mathbb{C}$-independence. For a locally minimal filtration with commuting past and future $\mathcal{A}_{(-\infty,t]} = \mathcal{A}_{(-\infty,s]} \vee \mathcal{A}_{[s,t]}$ is isomorphic to $\mathcal{A}_{(-\infty,s]} \otimes \mathcal{A}_{[s,t]} \otimes \mathbb{1}_{\mathcal{A}_{[t,\infty)}}$ for $s < t$. Similarly, one decomposes $\mathcal{A}_{[s,\infty)}$ and concludes that $\mathcal{A}_{(-\infty,t]} \cap \mathcal{A}_{[s,\infty)} \simeq (\mathcal{A}_{(-\infty,s]} \otimes \mathcal{A}_{[s,t]} \otimes \mathbb{1}_{\mathcal{A}_{[t,\infty)}}) \cap (\mathbb{1}_{\mathcal{A}_{(-\infty,s]}} \otimes \mathcal{A}_{[s,t]} \otimes \mathcal{A}_{[t,\infty)}) \simeq \mathcal{A}_{[s,t]}$. This shows the local maximality of the filtration. The continuity of the filtration follows directly from Lemma 4.1.2. Finally, using ideas from Subsection 2.3, it is easy to see that $E_{(-\infty,t]}$ and $E_{[s,\infty)}$ commute. This ensures, by Lemma 4.2.2, the enriched independence structure. □

Corollary 4.3.3. *If the $\mathbb{C}$-expected shift $(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}})$ has a commuting past/future, then its compression by any conditional expectation E, satisfying the conditions of Proposition 3.4.1, does too.*

Proof. This is an elementary conclusion from Definitions 3.4.2 and 4.3.1. □

Remark 4.3.4. $\mathbb{C}$-expected continuous Bernoulli shift with a locally minimal filtration and a commuting past/future lead to examples of continuous tensor product systems of W*-algebras (see [Lie03, Sec. 7.3, Def. 7.1]).

4.4. **Commutative von Neumann algebras.** A rich source for $\mathbb{C}$-expected continuous Bernoulli systems with a commutative von Neumann algebra is provided by probability theory. Here we focus on the connection to noises, as they appear in the work of Tsirelson and Vershik [TV98, Tsi98, Tsi04].

Let us start with the measure theoretic notion of a continuous Bernoulli shift. We assume throughout that the probability spaces are Lebesgue spaces and that the σ-algebras are complete.

Definition 4.4.1. *A* continuous Bernoulli shift (on a probability space) *consists of a probability space (Ω, Σ, μ), a measure preserving Borel-measurable group $(s_t)_{t\in\mathbb{R}}$ on Ω and a family of sub-σ-algebras $(\Sigma_I)_{I\in\mathcal{I}} \subset \Sigma$, such that for all $I, J, K \in \mathcal{I}$*

- (o) *Σ is generated by the family $\{\Sigma_I \mid I \in \mathcal{I}$ bounded$\}$;*
- (i) *s_t maps Σ_I onto Σ_{I+t} for any $t \in \mathbb{R}$;*
- (ii) *Σ_I and Σ_J are independent whenever $\operatorname{Int} I \cap \operatorname{Int} J = \emptyset$;*
- (iii) *Σ_K contains the sub-σ-algebra generated by the union of Σ_I and Σ_J whenever $K = I \cup J$.*

$(\Sigma_I)_{I\in\mathcal{I}}$ (or the continuous Bernoulli shift) is called locally minimal *if Σ_K is generated by the union of Σ_I and Σ_J whenever $K = I \cup J$.*

A locally minimal *continuous Bernoulli shift on a probability space is also called a* Tsirelson-Vershik noise *or a* homogeneous continuous product (system) of probability spaces *(see [Tsi98] or [Tsi04, Definition 2d1]).*

It is elementary to turn a measure theoretic shift into an algebraic shift in the sense of Definition 3.1.2. The other direction is less elementary, but also a very familiar fact. Let us state the following result without proof and in our terminology. Actually, it is an immediate corollary of Mackey's paper [Mac62].

Definition 4.4.2. *A $\mathbb{C}$-expected continuous Bernoulli shift $(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}})$ is called* commutative *if the von Neumann algebra $\mathcal{A}$ is commutative.*

Theorem 4.4.3. *There is one-to-one correspondence between (isomorphism classes of)*

(i) *(locally minimal) continuous Bernoulli shifts (Ω, Σ, μ), $(s_r)_{r\in\mathbb{R}}$, $(\Sigma_I)_{I\in\mathcal{I}}$;*

(ii) *(locally minimal) $\mathbb{C}$-expected commutative continuous Bernoulli shifts $(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}})$.*

The correspondence is given by $\mathcal{A} = L^\infty(\Omega, \Sigma, \mu)$, $\psi = \int \cdot \, \mathrm{d}\mu$, $S_r(f) = f \circ s_r$ for all $f \in L^\infty(\Omega, \Sigma, \mu)$ and $\mathcal{A}_I = L^\infty(\Omega, \Sigma_I, \mu_{|\Sigma_I})$ for all $I \in \mathcal{I}$.

The terminology 'isomorphism' means in the present context 'isomorphism between the dynamical systems which preserves the filtration structure'. Since we use this equivalence only occasionally, we have omitted the (evident) definition of such an isomorphism between algebraic and measure theoretic continuous Bernoulli shifts.

Corollary 4.4.4. *A Tsirelson-Vershik noise corresponds to a $\mathbb{C}$-expected continuous Bernoulli shift which is commutative, locally minimal, locally maximal and has a continuous filtration. Moreover it enjoys an enriched independence and a commuting past/future.*

Proof. The isomorphism of Theorem 4.4.3 respects local minimality, consequently Lemma 4.3.2 applies. □

It is clear that all these properties can be translated back into properties of the underlying probability spaces, for example upwards and downwards continuity in [Tsi04, 2d2, 2d4]. Let us close this section with a remark on the notation. Since the Tsirelson-Vershik noises are continuous, one always has $\mathcal{A}_{(s,t)} = \mathcal{A}_{[s,t]} =: \mathcal{A}_{s,t}$. The latter notation is used for example in [Tsi04].

4.5. **Local minimality and compressions.** Local minimality of a shift is not included in Definition 3.1.2, since this property is not stable with respect to compressions. This is already a well-known phenomenon in probability theory (see Remark 4.5.2). But the situation is even more dramatic for probability spaces based on properly infinite von Neumann algebras. We describe a class of shifts, for which a quite natural compression by a conditional expectation destroys the local minimality. The von Neumann algebra of these shifts is of type III_λ $(0 < \lambda < 1)$, equipped with a periodic state, and its compression onto the centralizer yields a von Neumann algebra of type II_1. The example also shows that this obstruction cannot be removed or controlled by stipulating algebraic structures as an enriched $\mathcal{A}_0$-independence and/or a commuting past/future.

Recall that a state ψ on $\mathcal{A}$ is called periodic if there exists $T > 0$ such that $\sigma_T^\psi = \mathrm{id}$. The smallest such T is called the period of ψ [Tak73].

Lemma 4.5.1. *Let $(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}})$ be a non-trivial $\mathbb{C}$-expected continuous Bernoulli shift with a locally minimal filtration and a periodic state ψ. If the filtration carries a commuting past/future, then the compressed filtration $(\mathcal{A}_I \cap \mathcal{A}^\psi)_{I\in\mathcal{I}}$ is not locally minimal.*

Proof. Since the shift is $\mathbb{C}$-expected, $S \subset \mathrm{Aut}(\mathcal{A}, \psi)$ acts ergodically on $\mathcal{A}$. Thus ψ is a homogeneous periodic state with period $T > 0$ [Tak73]. Put $\kappa := e^{-2\pi/T}$ and let $\mathcal{B}_n := \{x \in \mathcal{A} \,|\, \sigma_t^\psi(x) = \kappa^{\mathrm{i}nt}x\}$ for $n \in \mathbb{Z}$. Note that $\mathcal{B}_0 = \mathcal{A}^\psi$. Furthermore, $\mathcal{B}_n$ and $\mathcal{B}_m$ are ψ-orthogonal whenever $n \neq m$. Recall also $\mathcal{B}_n\mathcal{B}_m \subseteq \mathcal{B}_{n+m}$ for any $n, m \in \mathbb{Z}$. Finally, we need from [Tak73] that $\varepsilon_n(x) := \frac{1}{T}\int_0^T \kappa^{-\mathrm{i}nt}\sigma_t^\psi(x)\,\mathrm{d}t$ is a projection from $\mathcal{A}$ onto $\mathcal{B}_n$ and $x\Omega = \sum_{n\in\mathbb{Z}} \varepsilon_n(x)\Omega$ for any $x \in \mathcal{A}$.

Since σ_t^ψ commutes with S_t and E_I, also ε_n does so. Thus, one has $S_t(\mathcal{B}_n) = \mathcal{B}_n$ and $E_I(\mathcal{B}_n) \subseteq \mathcal{B}_n$ for any $t \in \mathbb{R}$, $I \subseteq \mathbb{R}$ and $n \in \mathbb{Z}$. Now, let $I := [0, 1]$, $J := [1, 2] = S_1(I)$ and $K := [0, 2]$. First, we show that there exists a non-zero $x \in \mathcal{A}_I \cap \mathcal{B}_n$ for some $n \neq 0$. If $\mathcal{A}_I \cap \mathcal{B}_n = \{0\}$ for all $n \neq 0$, then $\mathcal{A}_I \subset \mathcal{A}^\psi$ and furthermore $\mathcal{A} = \bigvee_{n\in\mathbb{Z}} S_n\mathcal{A}_I \subseteq \mathcal{A}^\psi$ by local minimality. But this contradicts the periodicity of ψ. Hence there exists a non-zero $x \in \mathcal{A}_I \cap \mathcal{B}_n$ for some $n \neq 0$. Since $S_1(x^*) \in \mathcal{A}_J \cap \mathcal{B}_{-n}$ is also non-zero, the $\mathbb{C}$-independence of x and $S_1(x^*)$ implies that $xS_1(x^*) \in \mathcal{A}_K \cap \mathcal{B}_0$ is non-zero: $\psi(S_1(x)x^*xS_1(x^*)) = \psi(xx^*)\psi(x^*x) \neq 0$. We are left to prove that $xS_1(x^*) \in \mathcal{A}_K \cap \mathcal{B}_0$ is ψ-orthogonal to $(\mathcal{A}_I \cap \mathcal{B}_0) \vee (\mathcal{A}_J \cap \mathcal{B}_0)$. It holds

$$\psi(S_1(b)axS_1(x^*)) = \psi(ax)\psi(S_1(b)S_1(x)) = 0 \tag{4.5.1}$$

for any $a, b \in \mathcal{A}_I \cap \mathcal{B}_0$, since $a \in \mathcal{B}_0$ and $x \in \mathcal{B}_n$ are ψ-orthogonal. Now, the commuting past/future structure ensures that $c \in (\mathcal{A}_I \cap \mathcal{B}_0) \vee (\mathcal{A}_J \cap \mathcal{B}_0)$ is approximated in the weak* topology by some sequence $(c_i)_{i\in\mathbb{N}}$ with terms of the form $c_i = \sum_l S_1(b_{i,l})a_{i,l}$, where $b_{i,l}, a_{i,l} \in \mathcal{A}_I \cap \mathcal{A}^\psi$. Thus, (4.5.1) extends to $\psi(cxS_1(x^*)) = 0$ for any $c \in (\mathcal{A}_I \cap \mathcal{A}^\psi) \vee (\mathcal{A}_J \cap \mathcal{A}^\psi)$. This shows that $(\mathcal{A}_I \cap \mathcal{A}^\psi)_I$ is not locally minimal. □

A concrete example for such a continuous Bernoulli shift is provided by the CCR white noise in Example 4.7.1.

Remark 4.5.2. It is well-known in probability theory that multi-dimensional stochastic processes may lead to filtrations without local minimality. Let us next sketch the construction of such an example, starting from Gaussian white noise, as introduced in Example 4.6.1. Consider two independent generalized stochastic processes $(X_f^{(1)})_{f\in\mathscr{S}}$ and $(X_g^{(2)})_{g\in\mathscr{S}}$, realized on

$(\mathscr{S}' \times \mathscr{S}', \Sigma \times \Sigma, \mu \times \mu)$ with characteristic functional $C(f,g) := \mathrm{e}^{-1/2(\|f\|^2+\|g\|^2)}$. A slight modification of the construction in Example 4.6.1 shows that the corresponding $\mathbb{C}$-expected continuous Bernoulli shift is given by $(\mathcal{C}\otimes\mathcal{C}, \psi_\mu\otimes\psi_\mu, (S_t\otimes S_t)_{t\in\mathbb{R}}, (\mathcal{C}_I\otimes\mathcal{C}_I)_{I\in\mathcal{I}})$, the tensor product of two Gaussian white noises. Now let $\widetilde{E}$ be the conditional expectation associated to the sub-σ-algebra $\widetilde{\Sigma}$ generated by $\{X_f^{(1)} \cdot X_f^{(2)} \mid f \in \mathscr{S}\}$. Then it can be checked that $(\widetilde{E}(\mathcal{C}\otimes\mathcal{C}) \cap (\mathcal{C}_I\otimes\mathcal{C}_I))_{I\in\mathcal{I}}$ is a filtration without local minimality. From this follows that the tensor product of two Gaussian white noises is compressed by the conditional expectation $\tilde{E}$ to a $\mathbb{C}$-centred continuous Bernoulli shift which fails to be locally minimal.

4.6. **Examples from probability theory.** The Tsirelson-Vershik noises [Tsi04, Definition 2d1] correspond to $\mathbb{C}$-expected continuous Bernoulli shifts which are locally minimal and commutative (see Subsection 4.4). These noises divide into two classes: *classical* and *non-classical* (in the sense of [Tsi04, Definition 5c4]), or alternatively phrased: *type I* and *non-type I* (following the analogy with Arveson's product systems [Arv03]).

'Type I' examples in probability theory correspond to Lévy processes described by the Lévy-Khinchin formula. Essentially, these processes are combinations of Brownian motion and Poisson processes. We address the corresponding $\mathbb{C}$-expected continuous Bernoulli shifts as white noises. The attribute 'white' emphasizes that the spectral density of Lévy processes is constant (see also Subsection 6.5 for the general case). Notice that our usage of 'white noise' differs from [Tsi04] where it is reserved for 'Gaussian white noise'(as stated in Example 4.6.1). Thus a 'classical noise' therein corresponds to a 'white noise' herein.

Example 4.6.1 (Gaussian white noise)**.** Let $\mathscr{S} \subset L^2_{\mathbb{R}}(\mathbb{R})$ denote the space of all smooth rapidly decreasing real valued functions and $\mathscr{S}'$ its dual, the space of tempered distributions. Consider the generalized stochastic process $(X_f)_{f\in\mathscr{S}}$ with $X_f(x') := \langle f, x'\rangle$, $x' \in \mathscr{S}'$, on the probability space $(\mathscr{S}', \Sigma, \mu)$, where the measure μ is determined by the characteristic functional $C(f) := \mathrm{e}^{-\frac{1}{2}\|f\|^2} = \int_{\mathscr{S}'} \mathrm{e}^{\mathrm{i}X_f}\,\mathrm{d}\mu$ [Hid80, GV64]. Let Σ_I be the σ-algebra generated by the functions $\mathrm{e}^{\mathrm{i}X_f}$ with $\operatorname{supp} f \subseteq I$, where $I \in \mathcal{I}$. From $C(f+g) = C(f)C(g)$ for all functions $f, g \in \mathscr{S}$ with support in the disjoint intervals I and J we obtain the independence of the random variables X_f and X_g and hence the independence of Σ_I and Σ_J. The characteristic functional is invariant under the right shift σ_t on $\mathscr{S}$. Consequently, μ is invariant under the dual action $s_t := \sigma_t^*$ on $\mathscr{S}'$. Now properties (o) to (iii) of Definition 4.4.1 can be checked. Hence, by Theorem 4.4.3, we get a locally minimal $\mathbb{C}$-expected shift $(\mathcal{C}, \psi_\mu, S, (\mathcal{C}_I)_{I\in\mathcal{I}})$ which is called *Gaussian white noise.* Notice that Brownian motion $B_t \in L^2(\mathscr{S}', \Sigma, \mu)$ is approximated by $(X_{f_n})_{n\in\mathbb{N}}$ with $f_n \to \chi_{[0,t]}$ in the L^2-norm and generates the sub-σ-algebra $\Sigma_{[0,t]}$ for any $t > 0$.

Example 4.6.2 (Poisson white noises)**.** Let $N := (N_t)_{t\geq 0}$ be the Poisson process with intensity $\lambda > 0$. Then N can be realized on a probability space

(Ω, Σ, μ), where Ω is the set of paths $\omega\colon \mathbb{R} \to \mathbb{Z}$, $\omega(0) = 0$, which are increasing and right continuous, and with left limits [Pro95]. We extend N to negative times by $N_{-t}(\omega) := \omega(-t)$. Thus $\omega(t)$ and $-\omega(-t)$ count the jumps of ω in $[0, t]$ resp. $(-t, 0]$. The σ-algebras Σ and $\Sigma_{[s,t]}$ are generated by the sets $\Omega_n((s, t]) := \{N_t - N_s = n\}$, $s < t \in \mathbb{R}$, $n \in \mathbb{N}_0$ resp. $\Omega_n((s', t'])$ with $s \leq s'$ and $t' \leq t$, $n \in \mathbb{N}_0$. The measure μ is given by $\mu(\Omega_n((s, t])) := \lambda^n (t - s)^n \mathrm{e}^{-\lambda(t-s)}/n!$. The σ-algebras $\Sigma_{[s,t]}$ and $\Sigma_{[u,v]}$ are independent for disjoint intervals $[s, t]$ and $[u, v]$. Finally, a measure preserving shift $(s_r)_{r\in\mathbb{R}}$ on Ω is defined by $(s_r(\omega))(t) := \omega(t + r) - \omega(r)$. Now, by Theorem 4.4.3, one can associate canonically a locally minimal $\mathbb{C}$-expected shift $(\mathcal{P}, \psi_\mu, S, (\mathcal{P}_I)_{I\in\mathcal{I}})$ to the Poisson process. It is called *Poisson white noise*. Notice that the sub-σ-algebras of the filtration are generated by increments of the Poisson process.

Remark 4.6.3. Tensor products of Poisson white noises and Gaussian white noises, compressed to the von Neumann subalgebra generated by a specified linear combination of the underlying Brownian motion and Poisson processes, give first examples of white noises coming from Lévy processes. Moreover, (countable many) tensor products of Gaussian white noise lead again to Gaussian white noises, now with multiplicities. Furthermore, the amplification with a non-commutative probability space gives operator-expected white noises.

Tsirelson-Vershik noises with a 'non-classical' or 'non-type I' part have no representation in Fock spaces. The existence of such intrinsically non-linear random fields was revealed by A. Vershik and B. Tsirelson [TV98]. For a detailed survey on recent developments and examples, as well as the close connection to Arveson's non-type I product systems, we refer the interested reader to [Tsi04]. Prominent examples among these 'non-classical' noises are 'black noises', as named by B. Tsirelson in [Tsi98]. These 'black noises' lead to Arveson's continuous product systems of Hilbert spaces of type II_0 [Arv03].

Example 4.6.4 (Black noises)**.** Examples of 'black' noises, as stated in [Tsi04], are locally minimal continuous Bernoulli shifts on probability spaces. They correspond to $\mathbb{C}$-expected locally minimal continuous Bernoulli shifts according to Corollary 4.4.4 and will also be addressed as $\mathbb{C}$-expected black noises. They are called 'black' because they have only trivial additive shift cocycles (see Definition 6.3.1). No 'linear sensors' exist to detect their color, as Tsirelson would say. See also the discussion of 'whiteness' at the end of Subsection 6.5. For further details on black noises and their construction we refer to [Tsi04] and the literature cited therein.

4.7. **Examples from quantum probability theory.** We continue with examples of continuous Bernoulli shifts coming from quantum probability. In analogy to probability theory, we term the 'type I' examples 'quantum white noises'. In a heuristic sense (justified up to now by all known examples) these examples come from quantum Lévy processes which generate the filtration of the quantum white noise (see Definition 6.5.2). These quantum Lévy processes are provided by additive shift cocycles (see Definition 6.3.1). Already a rich source for the construction of quantum white noises is provided by generalized Brownian motions which are realized on (deformed) Fock spaces [BS91, BS94, BKS97, GM02, BG02]. Moreover, promising candidates for further examples of quantum white noises appear in the work of Anshelevich on q-Lévy processes [Ans].

Example 4.7.1 (CCR white noises)**.** We continue the discussion of Example 2.2.2. Let $\mathcal{K} := L^2(\mathbb{R})$ and let $\mathcal{B}_I$ be the von Neumann algebra generated by functions $f \in \mathcal{K}$ with support in the interval I. Second quantization of the right shift on $L^2(\mathbb{R})$ provides the shift S and it is easily seen that $(\mathcal{B}, \psi_\lambda, S, (\mathcal{B}_I)_{I\in\mathcal{I}})$ is a locally minimal $\mathbb{C}$-expected continuous Bernoulli shift, called CCR white noise. It is well-known that the von Neumann algebra of such a shift is a factor of type $\mathrm{III}_{\lambda/(1+\lambda)}$. Note that this shift has a commuting past/future and an enriched independence.

Multi-dimensional CCR white noises are just tensor products of $\mathbb{C}$-expected CCR white noises.

Example 4.7.2 (Squeezed CCR white noises)**.** More generally, consider the quasi-free state $\psi_{\lambda,c}$ on $\mathrm{CCR}(L^2(\mathbb{R}), \mathrm{Im}\langle\cdot \mid \cdot\rangle)$, given by $\psi_{\lambda,c}(W(f)) = \exp\big(-\frac{1}{4}q_{\lambda,c}(f)\big)$ with

$$q_{\lambda,c}(f) := (2\lambda+1)\|f\|^2 + 2\,\mathrm{Re}(c\langle f \mid Jf\rangle),$$

where J is the complex conjugation in $L^2(\mathbb{R})$ and $\lambda > \sqrt{|c|^2 + 1/4} - 1/2$ for some $c \in \mathbb{C}$. This state is non-gauge invariant for $c \neq 0$ and the corresponding Araki-Woods representation leads again to a $\mathbb{C}$-expected quantum white noise. This noise has important applications in quantum optics, where it is referred to as 'squeezed white noise'. We refer the reader to [HHK$^+$02] for its construction and further references, and to [GZ00] for its applications.

Example 4.7.3 (CAR white noise)**.** Let $(\mathcal{B}, \psi_\lambda)$ be the probability space introduced in Example 2.4.1 with $\mathcal{K} := L^2(\mathbb{R})$. For any interval $I \subseteq \mathbb{R}$ let $\mathcal{B}_I$ be the subalgebra generated by the functions in $L^2(\mathbb{R})$ with support in I. Second quantization of the right shift on $L^2(\mathbb{R})$ provides a shift S, which fulfills obviously $S_t\mathcal{B}_I = \mathcal{B}_{I+t}$. This gives the $\mathbb{C}$-expected locally minimal continuous Bernoulli shift $(\mathcal{B}, \psi_\lambda, S, (\mathcal{B}_I)_{I\in\mathcal{I}})$, called CAR *white noise.* Note that these shifts have an enriched independence, but they do not have a commuting past and future. Moreover, the von Neumann algebra of such shifts is a type $\mathrm{III}_{\lambda/(1-\lambda)}$ factor for $0 < \lambda < 1/2$ and a type II_1 factor in the case $\lambda = 1/2$.

Besides of amplification with a non-commutative probability space, the above construction can also be promoted to a $\mathcal{B}_0$-expected shift as follows. Let $\mathcal{K} := \mathcal{K}_0 \oplus L^2(\mathbb{R})$ ($\mathcal{K}_0$ separable) and let $\mathcal{B}_I$ be the von Neumann algebra generated by the closed subspace $\mathcal{K}_I := \mathcal{K}_0 \oplus L^2(I) \subseteq \mathcal{K}$. Define the shift S by second quantization of $(\mathrm{id} \oplus s_t)_{t\in\mathbb{R}}$. Here s denotes the right shift on $L^2(\mathbb{R})$. It is again elementary to verify that $(\mathcal{B}, \psi_\lambda, S, (\mathcal{B}_I)_{I\in\mathcal{I}}; \mathcal{B}_0)$ is an expected continuous Bernoulli shift.

Remark 4.7.4. Let two $\mathbb{C}$-expected continuous Bernoulli shifts have von Neumann algebras of type III_κ resp. $\mathrm{III}_{\tilde\kappa}$ $(0 < \kappa, \tilde\kappa < 1)$. If $\ln(\kappa)/\ln(\tilde\kappa)$ is irrational then the tensor product of these two shifts leads to a $\mathbb{C}$-expected white noise with a von Neumann algebra of type III_1 (see also [Tak03b, Theorem 4.16] for approximately finite dimensional von Neumann algebras). Such examples are in particular provided by the CAR white noises. The authors doubt that examples of $\mathbb{C}$-expected continuous Bernoulli shifts with a von Neumann algebra of type III_0 exist.

The following class of examples contains quantum white noise from free probability theory, including a special case of amalgamated free independence.

Example 4.7.5 (q-Gaussian white noises)**.** We continue the discussion of Example 2.4.3. Let $\mathcal{K}_\mathbb{R} := L^2_\mathbb{R}(\mathbb{R})$ and let $\mathcal{B}_I$ be the von Neumann algebra, generated by the q-Gaussian processes $\{\Phi(f) \mid f \in \mathcal{K}_I\}$ with $\mathcal{K}_I := L^2_\mathbb{R}(I) \subseteq \mathcal{K}_\mathbb{R}$. The shift S is again obtained by second quantization of the right shift on $L^2(\mathbb{R})$. From this one verifies that $(\mathcal{B}, \tau, S, (\mathcal{B}_I)_{I\in\mathcal{I}})$ is a locally minimal $\mathbb{C}$-expected shift with enriched independence, but without commuting past and future. Such shifts are called *q-Gaussian white noises* and their von Neumann algebras are factors of type II_1 [BKS97].

Operator-expected shifts are produced by amplification. Similar to Example 4.7.3, here is an alternative approach to the multi-dimensional case. Put $\mathcal{K}_I := \mathcal{K}_0 \oplus L^2_\mathbb{R}(I, \mathcal{K}_1)$, where $\mathcal{K}_0, \mathcal{K}_1$ are two real separable Hilbert spaces. The dimension of $\mathcal{K}_1$ will give the multiplicity of the shift. As usual, $\mathcal{K}_I$ is identified canonically as a subspace of $\mathcal{K}_\mathbb{R}$. Let $\mathcal{C}_I := \bigvee\{\Phi(f) \mid f \in \mathcal{K}_I\}$. The shift S arises again through the second quantization of $(\mathrm{id}_{\mathcal{K}_0} \oplus s_t)_{t\in\mathbb{R}}$, where $(s_t)_{t\in\mathbb{R}}$ is the right shift on $L^2(\mathbb{R}, \mathcal{K}_1)$. Now one obtains the $\mathcal{C}_0$-expected shift $(\mathcal{C}_\mathbb{R}, \psi, S, (\mathcal{C}_I)_{I\in\mathcal{I}})$. Notice in the case $q = 0$ that $\mathcal{C}_\mathbb{R}$ is isomorphic to the free product of $\mathcal{B}_0$ with the $\dim(\mathcal{K}_1)$-fold free product of $\mathcal{B}$, as stated at the beginning of the present example.

Remark 4.7.6. Further examples of $\mathcal{A}_0$-expected shifts arise from generalized Brownian motions by functors of white noise, as indicated already at the end of Section 2.

Remark 4.7.7. We close this section with a digression on 'non-classical' (or 'non-type I') examples of continuous Bernoulli shifts which come from quantum probability. Presently, the existence of such examples is an open problem, if one insists on the local minimality of the filtration (see Subsection 4.5 for an example lacking local minimality). The analogy between Tsirelson-Vershik noises and continuous Bernoulli shifts is evident, particularly for the 'classical or type I' parts. Thus it is tempting to conjecture the existence of 'quantum black noises', in the sense of continuous Bernoulli shifts. Such objects would enjoy a probabilistic interpretation and could provide the notion of an 'intrinsically non-linear random quantum field'. Moreover, such examples would serve as a quantum probabilistic source of Arveson product systems of type II [Arv03].

5. Continuous GNS Bernoulli shifts

In this section we develop some of the GNS representation theory of an $\mathcal{A}_0$-expected continuous Bernoulli shift. The goal of this section is Definition 5.4.5, which provides our notion of an $\mathcal{A}_0$-expected continuous GNS Bernoulli shift. It is integral for the remaining Sections 6 to 8.

5.1. **Hilbert bimodules of $\mathcal{A}_0$-expected probability spaces.** In the following we present a concrete construction of the Hilbert $\mathcal{A}$-$\mathcal{A}_0$ bimodule ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$, starting from the expected probability space $(\mathcal{A}, \psi; \mathcal{A}_0)$. It will be realized in $\mathcal{B}(\mathcal{H}_\psi)$, where $\mathcal{H}_\psi$ is the GNS Hilbert space. We may assume that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}_\psi)$ (see Section 1). Notice that the constructed Hilbert bimodule is (isomorphic to) the GNS Hilbert space $\mathcal{H}_\psi$ if $\mathcal{A}_0 \simeq \mathbb{C}$.

Let E_0 be the conditional expectation from $(\mathcal{A}, \psi)$ onto $\mathcal{A}_0$. Its GNS representation defines an orthogonal projection e_0 such that $E_0(x)e_0 = e_0 x e_0$ for any $x \in \mathcal{A}$. Furthermore, let $\mathcal{H}_0 := e_0\mathcal{H}_\psi$. The vector space $\mathcal{A}e_0 \subset \mathcal{B}(\mathcal{H}_\psi)$ is an $\mathcal{A}$-$\mathcal{A}_0$-bimodule with left multiplication

$$\mathcal{A} \times \mathcal{A}e_0 \ni (y, xe_0) \mapsto yxe_0 \in \mathcal{A}e_0 \tag{5.1.1}$$

and right multiplication

$$\mathcal{A}e_0 \times \mathcal{A}_0 \ni (xe_0, a) \mapsto xe_0 a = xae_0 \in \mathcal{A}e_0\,. \tag{5.1.2}$$

The $\mathcal{A}_0$-valued inner product

$$\mathcal{A}e_0 \times \mathcal{A}e_0 \ni (x, y) \mapsto \langle x \,|\, y\rangle_0 := x^* y \in \mathcal{A}_0 \tag{5.1.3}$$

turns $\mathcal{A}e_0$ into a pre-Hilbert $\mathcal{A}_0$-module. From now on, $\mathcal{A}_0 e_0$ and $\mathcal{A}_0$ are identified canonically.

Definition 5.1.1. *${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$ is the closure of $\mathcal{A}e_0$ in the strong operator topology of $\mathcal{B}(\mathcal{H}_\psi)$.*

Then ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0} = \{\mathcal{A}, e_0\}'' e_0$, and so it is easily seen that the left and right multiplication (5.1.1) resp. (5.1.2), and the inner product (5.1.3), all extend to ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$. The $\mathcal{A}_0$-valued 'norm' and its induced norm on ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$ are denoted by

$$|x|_0 := \langle x \,|\, x\rangle_0^{1/2} \quad \text{resp.} \quad \|x\|_0 := \||x|_0\|.$$

In order to see the connection with the definition of a Hilbert W*-module in Appendix A, we note, that $\mathcal{B}(\mathcal{H}_\psi)e_0$ is canonically isomorphic to $\mathcal{B}(\mathcal{H}_0, \mathcal{H}_\psi) \subset \mathcal{B}(\mathcal{H}_0 \oplus \mathcal{H}_\psi)$. Under this isomorphism ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$ becomes a Hilbert W*-module in the sense of Definition A.1. Identifying ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$ with its isomorphic image, we arrive at the following Lemma.

Lemma 5.1.2. *${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$ is an $\mathcal{A}$-$\mathcal{A}_0$-bimodule and a Hilbert W*-module (over $\mathcal{A}_0$).*

Note that left multiplication by an element in $\mathcal{A}$ defines a bounded linear operator on ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$ which is adjointable.

The strong operator topology (SOT) on ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$ is generated by the seminorms

$$\mathrm{d}_\xi(x) := \| |x|_0 \xi \|, \qquad \xi \in \mathcal{H}_0$$

and the σ-strong operator topology σ-SOT by

$$\mathrm{d}_\varphi(x) := \varphi(\langle x \,|\, x \rangle_0)^{1/2}, \quad \varphi \in \mathcal{A}_{0*}^{+}.$$

Moreover the following continuity properties are valid:

(i) ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0} \ni x \mapsto \langle x \,|\, y \rangle_0$ is weak*-weak* continuous for all $y \in {}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$;

(ii) ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0} \times {}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0} \ni (x, y) \mapsto \langle x \,|\, y \rangle_0$ is jointly continuous on bounded sets in the strong operator topology on ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$ and in the weak* topology on $\mathcal{A}_0$.

Note that $\mathcal{A}$ is a SOT-dense subspace of ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$, since the algebra $\mathcal{A}$ embeds contractively into ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$ by the strongly continuous mapping $\mathcal{A} \ni x \mapsto xe_0$. From the separability of $\mathcal{A}_*$ it follows that ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$ has a separable predual (see also Theorem A.2). Finally, we note that Kaplansky's density Theorem A.5 ensures that elements in ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$ can be approximated in the strong operator topology by bounded sequences in $\mathcal{A}$.

5.2. **GNS representation of morphisms.** We characterize the elements of $\mathrm{Mor}(\mathcal{A}, \psi)$ which extend to adjointable bounded linear operators on ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$. The following definition is needed for the formulation of Theorem 5.2.2.

Definition 5.2.1. *A morphism $T^* \in \mathrm{Mor}(\mathcal{A}, \psi)$ is called the* ψ-adjoint *of $T \in \mathrm{Mor}(\mathcal{A}, \psi)$ if $\psi(T^*(x)y) = \psi(xT(y))$ for all $x, y \in \mathcal{A}$.*

Note that T^* exists (uniquely) if and only if T commutes with the modular automorphism group σ^ψ (see Theorem B.2).

Theorem 5.2.2. *Let $T \in \mathrm{Mor}(\mathcal{A}, \psi)$ commute with the modular automorphism group σ^ψ and leave $\mathcal{A}_0$ pointwise fixed. Then the morphism T has a unique extension to an adjointable bounded linear operator $\overline{T}$ on ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$ such that $\overline{T^*} = \overline{T}^*$. Moreover, $T \mapsto \overline{T}$ is pointwise weak*-weak*-continuous and pointwise strongly-strongly continuous.*

Proof of Theorem 5.2.2. Putting $\overline{T} x\Omega := T(x)\Omega$, $x \in \mathcal{A}$, the morphism T defines a contraction which extends to the GNS Hilbert space $\mathcal{H}_\psi$. Moreover, we know from Theorem B.2 that its ψ-adjoint $T^* \in \mathrm{Mor}(\mathcal{A}, \psi)$ exists uniquely and thus also extends to a contraction $\overline{T^*}$ such that $\overline{T^*} = \overline{T}^*$.

We need that $\mathcal{A}_0$ is contained in the fixed point algebra of T^*: Since $\mathcal{A}_0$ is contained in the fixed point algebra of T, we conclude $T(xa) = T(x)a$ and $T(ax) = aT(x)$ for any $x \in \mathcal{A}$ and $a \in \mathcal{A}_0$ (see also [Küm84, Rob82]). Furthermore is $\psi(T^*(a)x) = \psi(aT(x)) = \psi(T(ax)) = \psi(ax)$ for any $x \in \mathcal{A}$,

$a \in \mathcal{A}_0$. Thus $\mathcal{A}_0$ is also contained in the fixed point algebra of T^*. We notice for later arguments that T^* satisfies also the conditions of the theorem.

We conclude next for any $x, y \in \mathcal{A}$

$$\overline{T} x e_0 y \Omega = \overline{T} x E_0(y) \Omega = T(x E_0(y)) \Omega = T(x) E_0(y) \Omega = T(x) e_0 y \Omega \,,$$

and consequently $\overline{T} x e_0 = T(x) e_0$. But this implies $\overline{T}\, {}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0} \subseteq {}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$ and

$$|\overline{T} x e_0|_0^2 = e_0 (x^* \overline{T}^* \overline{T} x) e_0 \leq \|\overline{T}\|^2 |x|_0^2$$

for any $x \in \mathcal{A}$. From inequality (A.2) follows now $\overline{T} \in \mathcal{B}({}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0})$. Furthermore we conclude with Corollary A.3 that $\overline{T}$ is adjointable. The morphism T^* satisfies again all conditions of the theorem and therefore extends also to an adjointable operator $\overline{T^*}$. Thus we can conclude

$$\begin{aligned} \psi(a \langle x \,|\, \overline{T} y \rangle_0) &= \psi(a E_0(x^* T(y))) = \psi(a x^* T(y)) \\ &= \psi(T^*(a x^*) y) = \psi(a E_0(T^*(x^*) y)) = \psi(a \langle \overline{T^*} x \,|\, y \rangle_0) \end{aligned}$$

for any $a \in \mathcal{A}_0$, $x, y \in \mathcal{A}$. But this implies $\overline{T^*} = \overline{T}^*$.

Assume that $(T_\alpha)_{\alpha \in I} \in \mathrm{Mor}(\mathcal{A}, \psi)$ converges pointwise to $T \in \mathrm{Mor}(\mathcal{A}, \psi)$ in the weak* topology. Moreover, assume that each T_α and T satisfy the assumptions of the theorem. Then it follows that $(\overline{T}_\alpha)_{\alpha \in I}$ maps B into itself, where B is the unit ball of ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$.

The map $T \mapsto T^*$ is pointwise weak*-continuous, since the weak* topology on $\mathcal{A}_1$ is induced by the family of seminorms $\{|\psi(y\,\cdot)| \,|\, y \in \mathcal{A}\}$. It follows

$$\varphi(\langle y \,|(\overline{T}_\alpha - \overline{T}) x \rangle_0) = \varphi(\langle (T_\alpha^* - T^*)(y) \,|\, x \rangle_0) \xrightarrow{\alpha} 0$$

for any $\varphi \in \mathcal{A}_{0*}^+$, $y \in \mathcal{A}_1$ and $x \in B$. This implies the pointwise weak*-weak* continuity of the extension $\overline{T}$, since the weak* topology on B is induced by the family of seminorms $\{|\varphi(\langle y \,|\, \cdot \rangle_0)| \,|\, \varphi \in \mathcal{A}_{0*}^+, y \in \mathcal{A}_1\}$.

Now, let us assume that $F_\alpha := T_\alpha - T$ converges to 0 in the pointwise SOT sense. Then it follows for any $x, y \in \mathcal{A}$ that

$$\psi(y (T_\alpha^* \circ T_\alpha - T^* \circ T)(x)) = \psi(F_\alpha(y) T_\alpha(x)) + \psi(T(y) F_\alpha(x)).$$

In other words, $T_\alpha^* \circ T_\alpha$ converges to $T^* \circ T$ in the pointwise weak* topology. Since T^* is weak*-weak*-continuous, we conclude that $F_\alpha^* \circ F_\alpha$ converges to 0 in the pointwise weak* topology on $\mathcal{A}$. With $x \in {}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$ and $\varphi \in \mathcal{A}_{0*}^+$, we conclude further for any $y \in \mathcal{A}$ that

$$\varphi(\langle y \,|\, \overline{F}_\alpha^* \overline{F}_\alpha x \rangle_0) = \varphi(\langle F_\alpha^* \circ F_\alpha(y) \,|\, x \rangle_0) \xrightarrow{\alpha} 0.$$

Thus one has $\lim_\alpha \overline{F}_\alpha^* \overline{F}_\alpha x = 0$ in the weak* topology. The pointwise strong convergence of $\overline{T}_\alpha$ to $\overline{T}$ is an immediate consequence. □

Convention 5.2.3. The argument x of a morphism $T \in \mathrm{Mor}(\mathcal{A}, \psi)$ will always be put in parenthesis, in contrast to the argument of $\overline{T}$, its extension to an adjointable bounded linear operator on ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$. This distinguishes morphisms acting on $\mathcal{A}$ and bounded linear operators on ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$. Thus we denote them by the same symbol T from now on, to lighten the notation.

Notice finally that a conditional expectation $E \in \mathrm{Mor}(\mathcal{A}, \psi)$ extends to an orthogonal projection on ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$. In particular, the extension of the conditional expectation E_0 onto $\mathcal{A}_0$ satisfies $E_0 x = \langle \mathbb{1} \,|\, x\rangle_0$ $(= e_0 x e_0)$, where $x \in {}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$.

5.3. **The product of $\mathcal{A}_0$-independent elements.** Let B be a subalgebra of $\mathcal{A}$ containing $\mathcal{A}_0$ such that the conditional expectation $E_{\mathcal{B}} \in \mathrm{Mor}(\mathcal{A}, \psi)$ exists. Since $E_{\mathcal{B}}$ is a morphism which commutes with the modular automorphism group and which leaves $\mathcal{A}_0$ pointwise fixed, it extends to an orthogonal projection on ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$. One verifies easily that $E_{\mathcal{B}}\,{}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0} = {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}_0}$, where ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}_0}$ corresponds to $(\mathcal{B}, \psi; \mathcal{A}_0)$.

Proposition 5.3.1. *Let $\mathcal{B}$ and $\mathcal{C}$ be two von Neumann subalgebras of the expected probability space $(\mathcal{A}, \psi; \mathcal{A}_0)$ such that the conditional expectations $E_{\mathcal{B}}$ resp. $E_{\mathcal{C}}$ exist.*

If $\mathcal{B}$ and $\mathcal{C}$ are $\mathcal{A}_0$-independent, then

$$ {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}_0} \times {}_{\mathcal{C}}\mathcal{E}_{\mathcal{A}_0} \ni (x, y) \mapsto xy \in {}_{\mathcal{B}\vee\mathcal{C}}\mathcal{E}_{\mathcal{A}_0} \tag{5.3.1} $$

defines a product which extends the left multiplication

$$ \mathcal{B} \times {}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0} \ni (x, y) \mapsto xy \in {}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}. $$

This product is jointly $\|\,\|_0$-continuous. If its first component is $\|\,\|_0$-bounded, it is also jointly σ-SOT- σ-SOT-continuous. The product satisfies

$$ \langle x_1 y_1 \,|\, x_2 y_2 \rangle_0 = \langle y_1 \,|\langle x_1 \,|\, x_2\rangle_0\, y_2\rangle_0 \,, \tag{5.3.2} $$

for any $x_1, x_2 \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}_0}$ and $y_1, y_2 \in {}_{\mathcal{C}}\mathcal{E}_{\mathcal{A}_0}$. Moreover, the module property of the conditional expectations $E_{\mathcal{B}}$ resp. $E_{\mathcal{C}}$ extends to

$$ E_{\mathcal{B}}\, xy = x\, E_0 y \qquad \text{and} \qquad E_{\mathcal{C}}\, xy = (E_0 x) y \,, \tag{5.3.3} $$

for any $x \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}_0}$ and $y \in {}_{\mathcal{C}}\mathcal{E}_{\mathcal{A}_0}$.

We emphasize that in general the product (5.3.1) is not minimal: The $\|\,\|_0$-closure of the product ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}_0}\,{}_{\mathcal{C}}\mathcal{E}_{\mathcal{A}_0}$, where $\mathcal{B}$ and $\mathcal{C}$ are $\mathcal{A}_0$-independent, may be contained properly in ${}_{\mathcal{B}\vee\mathcal{C}}\mathcal{E}_{\mathcal{A}_0}$. Consider for example $\mathbb{C}$-free independence as stated in Example 2.4.2.

Proof. The equation (5.3.2) reduces for $x_i \in \mathcal{B}$ and $y_i \in \mathcal{C}$ $(i = 1, 2)$ simply to the equation in Proposition 2.1.2 (ii). It is still valid for $y_i \in {}_{\mathcal{C}}\mathcal{E}_{\mathcal{A}_0}$. This is concluded immediately from the approximation of y_i by bounded sequences in $\mathcal{C}$, from the continuity of the left multiplication by elements in $\mathcal{B}$, and finally the continuity of the inner product. Next, we approximate $x \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}_0}$ by a sequence $(x_n)_{n\in\mathbb{N}}$, with $x_n \in \mathcal{B}$, in the strong (operator) topology on ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}_0}$ and show that for $y \in {}_{\mathcal{C}}\mathcal{E}_{\mathcal{A}_0}$ the element

$$ xy := \lim_{n\to\infty} x_n y \tag{5.3.4} $$

is well-defined. We verify the claimed convergence in the strong topology, and independence of the choice of approximating sequence $(x_n)_{n\in\mathbb{N}}$, as follows: for another approximating sequence $(\tilde{x}_n)_{n\in\mathbb{N}} \subset \mathcal{B}$ of x in the strong topology on ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$, and for any $\xi \in \mathcal{H}_0$:

$$\begin{aligned}\| |(x_n - \tilde{x}_m)y|_0\xi\|^2 &= \big\langle (x_n - \tilde{x}_m)y\xi \,\big|\, (x_n - \tilde{x}_m)y\xi \big\rangle \\ &= \big\langle y\xi \,\big|\, |x_n - \tilde{x}_m|_0^2 y\xi \big\rangle = \mathrm{d}_\varphi(x_n - \tilde{x}_m)^2 \xrightarrow{m,n\to\infty} 0\,.\end{aligned}$$

with $\varphi := \langle y\xi \,|\, \cdot y\xi\rangle \in {\mathcal{A}_0}_*^+$. Here we used that SOT-convergent sequences are also σ-SOT convergent. One observes that xy is just the usual left multiplication whenever x is in $\mathcal{B}$. Consequently, the equation (5.3.2) follows, since the inner product is continuous.

We are left to prove the continuity of the product. Let $x = \text{SOT-}\lim_\alpha x_\alpha$, where $x_\alpha \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}_0}$ and $\|x_\alpha\|_0 \le M$. Moreover, let $y = \text{SOT-}\lim_\beta y_\beta$, where $y_\beta \in {}_{\mathcal{C}}\mathcal{E}_{\mathcal{A}_0}$. We observe

$$x_\alpha y_\beta - xy = (x_\alpha - x)(y_\beta - y) + (x_\alpha - x)y + x(y_\beta - y)\,.$$

For the first summand we conclude for any $\varphi \in \mathcal{A}_{0*}^+$

$$\begin{aligned}\mathrm{d}_\varphi((x_\alpha - x)(y_\beta - y))^2 &= \varphi(\langle y_\beta - y \,|\, |x_\alpha - x|_0^2\,(y_\alpha - y)\rangle_0) \\ &\le \|x_\alpha - x\|_0^2\, \mathrm{d}_\varphi(y_\beta - y)^2 \le 4M^2 \mathrm{d}_\varphi(y_\beta - y)^2 \xrightarrow{\beta} 0\,.\end{aligned}$$

Similarly, we proceed with the third summand. The second summand delivers the expression $\varphi(\langle y \,|\, |x_\alpha - x|_0^2\, y\rangle_0)$ which converges to zero, since the map ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0} \ni z \mapsto \varphi(\langle y \,|\, |z|_0^2\, y\rangle_0)$ is SOT-continuous. The $\| \|_0$-continuity is shown easily. Finally, we conclude (5.3.3) from the defining equation (5.3.4) of the product xy and the strong continuity of the conditional expectations $E_{\mathcal{B}}$ resp. $E_{\mathcal{C}}$. □

Remark 5.3.2. If $\mathcal{B}$ and $\mathcal{C}$ are $\mathcal{A}_0$-independent, we obtain for $x \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}_0}$ and $y \in {}_{\mathcal{C}}\mathcal{E}_{\mathcal{A}_0}$

$$\langle x \,|\, y\rangle_0 = \langle x \,|\, \mathbb{1}\rangle_0 \langle \mathbb{1} \,|\, y\rangle_0\,. \tag{5.3.5}$$

This is evident by choosing $x_1 = y_2 = \mathbb{1}$ and $y_1 = x$, $x_2 = y$ in equation (5.3.2). Similarly,

$$E_0 xy = \langle \mathbb{1} \,|\, xy\rangle_0 = \langle \mathbb{1} \,|\, x\rangle_0 \langle \mathbb{1} \,|\, y\rangle_0 = E_0 x E_0 y. \tag{5.3.6}$$

Notice that the module property of E_0 ensures that $E_0(xE_0y) = (E_0x)(E_0y)$. The identities (5.3.5) and (5.3.6) feature the factorization property (2.1.1) in the language of Hilbert modules.

Remark 5.3.3. An n-fold product $x_1x_2\cdots x_n$ of elements $x_i \in {}_{\mathcal{B}_i}\mathcal{E}_{\mathcal{A}_0} \subset {}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$ $(i = 1, 2, \ldots, n)$ is well-defined and associative whenever $\mathcal{B}_j$ and $\bigvee_{i=1}^{j-1} \mathcal{B}_i$ are $\mathcal{A}_0$-independent for all $j = 2, \ldots, n$. Moreover, $E_0x_1x_2x_3\cdots x_n = E_0x_1E_0x_2E_0x_3\cdots E_0x_n$.

The following notion of $\mathcal{A}_0$-independence will be used for ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$.

Definition 5.3.4. ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}_0}$ *and* ${}_{\mathcal{C}}\mathcal{E}_{\mathcal{A}_0}$ *in* ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$ *are called* $\mathcal{A}_0$-independent *if* $\mathcal{B}$ *and* $\mathcal{C}$ *are* $\mathcal{A}_0$*-independent. Two elements* $x, y \in {}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$ *are* $\mathcal{A}_0$*-independent if they are, respectively, elements of two* $\mathcal{A}_0$*-independent* ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}_0}$ *and* ${}_{\mathcal{C}}\mathcal{E}_{\mathcal{A}_0}$.

All properties of commuting squares, as stated in Proposition 2.1.2, carry over to ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$.

Proposition 5.3.5. *Under the assumptions of Proposition 5.3.1, the following conditions are equivalent:*

(i) ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}_0}$ *and* ${}_{\mathcal{C}}\mathcal{E}_{\mathcal{A}_0}$ *are* $\mathcal{A}_0$*-independent;*
(ii) $\langle x_1 y_1 \,|\, x_2 y_2 \rangle_0 = \langle y_1 \,|\langle x_1 \,|\, x_2\rangle_0\, y_2\rangle_0$ *for any* $x_1, x_2 \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}_0}$ *and* $y_1, y_2 \in {}_{\mathcal{C}}\mathcal{E}_{\mathcal{A}_0}$*;*
(iii) $E_{\mathcal{B}\mathcal{C}}\mathcal{E}_{\mathcal{A}_0} = \mathcal{A}_0$*;*
(iv) $E_{\mathcal{B}}E_{\mathcal{C}} = E_0$.

Proof. The stated equivalences follow from Proposition 2.1.2 and (5.3.4), if one approximates elements of ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$ using sequences in the underlying von Neumann algebra $\mathcal{A}$. □

5.4. **Continuous GNS Bernoulli shifts.** We use the results of the previous subsections to extend the structure of an $\mathcal{A}_0$-expected continuous Bernoulli shift $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$.

Notation 5.4.1. To lighten the notation let $\mathcal{E} := {}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_0}$ and $\mathcal{E}_I := {}_{\mathcal{A}_I}\mathcal{E}_{\mathcal{A}_0}$.

Proposition 5.4.2. *Let the expected continuous Bernoulli shift* $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ *be given. The family of Hilbert bimodules* $(\mathcal{E}_I)_{I\in\mathcal{I}}$ *induced by the filtration* $(\mathcal{A}_I)_{I\in\mathcal{I}}$ *satisfies*

(o) $\mathcal{E}$ *is the* SOT*-closure of* $\{\mathcal{E}_I \,|\, I \in \mathcal{I} \text{ bounded}\}$.
(i) $\mathcal{E}_I \subseteq \mathcal{E}_J$ *whenever* $I \subseteq J$*;*
(ii) $\bigcup\{\mathcal{E}_I \,|\, I \in \mathcal{I} \text{ bounded}\}$ *is strongly dense in* $\mathcal{E}$.
(iii) $\mathcal{E}_I$ *and* $\mathcal{E}_J$ *are* $\mathcal{A}_0$*-independent whenever* $I \cap J = \emptyset$*;*

Proof. This follows from the results in the present section. □

Notice that due to Proposition 5.3.1, the product xy of independent elements x and y is well-defined in $\mathcal{E}$.

Each shift S_t satisfies the assumptions of Theorem 5.2.2; we let S_t also stand for its extension to $\mathcal{E}$.

Proposition 5.4.3. *Let the expected continuous Bernoulli shift* $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ *be given. Then the shift* S *on the Hilbert bimodule* $\mathcal{E}$ *enjoys the following properties:*

(i) $S = (S_t)_{t\in\mathbb{R}}$ *is a unitary group on* $\mathcal{E}$*, which is pointwise* SOT*-continuous;*
(ii) *the fixed point space of* S *in* $\mathcal{E}$ *is* $\mathcal{A}_0$*, in particular* $S_t E_0 = E_0$ *for each* t*;*
(iii) S_t *maps* $\mathcal{E}_I$ *onto* $\mathcal{E}_{I+t}$ *for all* I *and* t.

Proof. This is clear. □

We use frequently the following result.

Corollary 5.4.4. *Let an $\mathcal{A}_0$-expected continuous Bernoulli shift be given. $(E_{(-\infty,t]})_{t\in\mathbb{R}}$ and $(E_{[t,\infty)})_{t\in\mathbb{R}}$ are pointwise* SOT*-continuous on $\mathcal{E}$.*

Proof. This follows from Proposition 5.4.2 (iii) and 5.4.3 (i), similar to the proof of Lemma 3.1.5. □

In this paper we refrain from axiomatizing these structures in the language of Hilbert bimodules, and work instead with the following definition.

Definition 5.4.5. *The tuple $(\mathcal{E}, \mathbb{1}, S, (\mathcal{E}_I)_{I\in\mathcal{I}}; \mathcal{A}_0)$ is called an* ($\mathcal{A}_0$-)expected continuous GNS Bernoulli shift*, whenever it is constructed from an expected continuous Bernoulli shift $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ as stated above.*

Remark 5.4.6. (i) Since a $\mathbb{C}$-expected locally minimal commutative continuous Bernoulli shift corresponds to a Tsirelson-Vershik noise, it is evident that both cover the same class of of continuous product systems of pointed Hilbert spaces (see [Tsi04, Definitions 3c1 and 6d6] for a definition).

(ii) $\mathcal{A}_0$-expected locally minimal continuous GNS Bernoulli shifts with a commuting past/future are closely related to product systems of Hilbert modules, as they are considered in [MS02] and [BS00, BBLS04].

6. Cocycles of continuous (GNS) Bernoulli shifts

In this section we introduce and investigate additive and multiplicative cocycles for $\mathcal{A}_0$-expected (non-commutative) continuous Bernoulli shifts.

Throughout we assume that a fixed expected continuous Bernoulli shift $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ and its continuous GNS Bernoulli shift $(\mathcal{E}, \mathbb{1}, S, (\mathcal{E}_I)_{I\in\mathcal{I}}; \mathcal{A}_0)$ are given.

6.1. Multiplicative cocycles of continuous Bernoulli shifts.

We introduce unitary cocycles. They are a non-commutative version of Lévy processes, taking values in unitary operators.

Definition 6.1.1. *Let $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ be an expected continuous Bernoulli shift. A weak*-continuous family $u = (u_t)_{t\geq 0}$ in $\mathcal{A}$ is called a* unitary cocycle *if for any $s, t \geq 0$*

(i) *u_t is unitary;*
(ii) *$u_t \in \mathcal{A}_{[0,t]}$;*
(iii) *$u_{s+t} = S_t(u_s)u_t$.*

The unitary cocycle u is called trivial *if $u \subseteq \mathcal{A}_0$. The set of all ($\|\,\|_0$-continuous) unitary cocycles is denoted by $\mathscr{C}_0(\mathcal{A}, \cdot\,)$ (resp. $\mathscr{C}_0^0(\mathcal{A}, \cdot\,)$).*

Putting $u_t = S_t(u_{-t}^*)$ for $t \leq 0$, the unitary cocycle u extends to $\mathbb{R}$. For a given $\mathcal{A}_0$-expected shift, a unitary cocycle u defines via $T_t(x) = u_t^* S_t(x) u_t$, $x \in \mathcal{A}$ a pointwise weak*-continuous group of automorphisms $T = (T_t)_{t\in\mathbb{R}}$ on

$\mathcal{A}$. The compression $R = E_0 T E_0$ thus defines a pointwise weak*-continuous unital semigroup of completely positive contractions, also called a *Markovian or CP_0-semigroup on $\mathcal{A}_0$*.

Note that $T \subset \mathrm{Aut}(\mathcal{A}, \psi)$ if and only if $u \subset \mathcal{A}^\psi$. In such a situation, the quadruple $(\mathcal{A}, \psi, T; \mathcal{A}_0)$ defines an $\mathcal{A}_0$-valued stationary quantum Markov process in the sense of [Küm85]. The Markovian semigroup $R = (R_t)_{t\geq 0}$ is again obtained by the compression $R_t = E_0 T_t E_0$ and leaves ψ invariant.

6.2. **Multiplicative cocycles of continuous GNS Bernoulli shifts.** In the framework of continuous GNS Bernoulli shifts, the role of a unitary cocycle is filled by a *unital* cocycle.

Definition 6.2.1. *Let $(\mathcal{E}, \mathbb{1}, S, (\mathcal{E}_I)_{I\in\mathcal{I}}; \mathcal{A}_0)$ be an expected continuous GNS Bernoulli shift. A* unital cocycle *is a weak*-continuous family $u := (u_t)_{t\geq 0} \subset \mathcal{E}$ such that for any $s, t \geq 0$*

(i) $|u_t|_0 = \mathbb{1}$; *(unitality)*
(ii) $u_t \in \mathcal{E}_{[0,t]}$; *(adaptedness)*
(iii) $u_{t+s} = (S_t u_s) u_t$. *(cocycle identity)*

The unital cocycle u is called trivial *if $u \subseteq \mathcal{A}_0$.*

The set of all ($\|\,\|_0$-continuous) unital cocycles is denoted by $\mathscr{C}_0(\mathcal{E}, \cdot\,)$ (resp. $\mathscr{C}_0^0(\mathcal{E}, \cdot\,)$).

Notice that every unitary cocycle gives a unital cocycle, since $\mathcal{A} \subset \mathcal{E}$. Note also that the cocycle identity relies on the fact that the product of the independent elements $S_t u_s \in \mathcal{E}_{[t,t+s]}$ and $u_t \in \mathcal{E}_{[0,t]}$ is well-defined by Proposition 5.3.1.

A unital cocycle u is trivial if and only if u is a strongly continuous semigroup of isometries in $\mathcal{A}_0$. Notice moreover that a $\|\,\|_0$-continuous unital cocycle is trivial if and only if u is a uniformly continuous semigroup of unitaries in $\mathcal{A}_0$.

If $\mathcal{A}_0 \simeq \mathbb{C}$, then a unital cocycle is a family of unit vectors in the GNS Hilbert space $\mathcal{H}_\psi$.

From the definition of a unital cocycle it follows that $u_0 = \mathbb{1}$. Indeed, $u_0 \in \mathcal{A}_0$ by adaptedness, $u_0 = u_0^2$ by the cocycle identity, and $u_0^* u_0 = \mathbb{1}$ by unitality. From this we conclude $u_0 - \mathbb{1} = u_0^* u_0 (u_0 - \mathbb{1}) = 0$. As in the case of unitary cocycles, $\mathcal{A}_0$-independence ensures that the compression of a unital cocycle yields contraction semigroups as follows.

Proposition 6.2.2. *Let u be a unital cocycle.*

(i) *The compression $A := (E_0 u_t)_{t\geq 0}$ is a strongly continuous semigroup of contractions in $\mathcal{A}_0$. Moreover, the following are equivalent:*
  (a) *the unital cocycle u is $\|\,\|_0$-continuous;*
  (b) *the semigroup A is $\|\,\|$-continuous;*
  (c) *A has a bounded generator K lying in $\mathcal{A}_0$, i.e., $A_t = \mathrm{e}^{tK}$.*

(ii) *The compression* $R_t(a) := \langle u_t \,|\, a\, u_t\rangle_0$ *defines a pointwise weak*-continuous Markovian semigroup* R *on* $\mathcal{A}_0$*. If* u *is* $\|\,\|_0$*-continuous then* R *is norm-continuous.*

Notice that the state ψ of the continuous Bernoulli shift may not be R-invariant.

Proof. (i): From the cocycle identity, $A_{s+t} = E_0 u_{s+t} = E_0(S_t u_s)u_t$. Since $S_t u_s$ and u_t are $\mathcal{A}_0$-independent and $E_0 S_t = E_0$, we conclude $A_{s+t} = A_s A_t$ from equation (5.3.6). The equivalence of (b) and (c) is evident from the theory of semigroups on Banach spaces (see e.g. [Dav80]). The equivalence of (a) and (b) comes essentially from the Cauchy-Schwarz inequality (A.1):

$$\|A_t - \mathbb{1}\|^2 = \|\langle \mathbb{1} \,|\, u_t - \mathbb{1}\rangle_0\|^2 \leq \|u_t - \mathbb{1}\|_0^2 = \|\mathbb{1} - A_t^* - A_t + \mathbb{1}\|\,. \qquad (6.2.1)$$

(ii): The semigroup property is a consequence of 6.2.1 (iii) and (5.3.2):

$$R_{s+t}(a) = \big\langle (S_t u_s)u_t \,\big|\, a(S_t u_s)u_t\big\rangle_0 = \big\langle u_t \,\big|\, \langle u_s \,|\, a\, u_s\rangle_0 u_t\big\rangle_0 = R_t(R_s(a)).$$

Using (A.1) again we obtain the $\|\,\|$-continuity of R from the $\|\,\|_0$-continuity of u:

$$\begin{aligned}\|R_t(a) - a\| &\leq \|\langle u_t - \mathbb{1} \,|\, a(u_t - \mathbb{1})\rangle_0\| + \|\langle u_t - \mathbb{1} \,|\, a\mathbb{1}\rangle_0\| + \|\langle a^*\mathbb{1} \,|\, u_t - \mathbb{1}\rangle_0\| \\ &\leq \|u_t - \mathbb{1}\|_0 \|\langle u_t - \mathbb{1} \,|\, a^* a(u_t - \mathbb{1})\rangle_0\|^{1/2} + 2\|a\|\|u_t - \mathbb{1}\|_0 \\ &\leq 4\|a\|\|u_t - \mathbb{1}\|_0\,. \qquad (6.2.2)\end{aligned}$$

All other properties of the semigroup are clear by construction. □

Throughout this paper we only consider $\|\,\|$-continuous semigroups on $\mathcal{A}_0$.

6.3. **Additive cocycles of continuous GNS Bernoulli shifts.** We turn our attention to additive cocycles which are non-commutative versions of Lévy processes with values in an operator algebra.

Definition 6.3.1. *Let* $(\mathcal{E}, \mathbb{1}, S, (\mathcal{E}_I)_{I\in\mathcal{I}}; \mathcal{A}_0)$ *be an expected continuous GNS Bernoulli shift. An* additive cocycle *is a family* $b := (b_t)_{t\geq 0} \subset \mathcal{E}$ *such that for all* $s, t \geq 0$

(i) $t \mapsto b_t$ *is* $\|\,\|_0$*-continuous.*
(ii) $b_t \in \mathcal{E}_{[0,t]}$, *(adaptedness)*
(iii) $b_{t+s} = b_t + S_t b_s$. *(cocycle identity)*

The additive cocycle b *is* centred *if in addition* $E_0 b_t = 0$ *for all* $t \geq 0$*. The operators* $\langle b_t \,|\, b_t\rangle_0$ *and* $E_0 b_t$ *are called the* variance *and the* drift parts *of* b*, respectively.*

$\mathscr{C}_0(\mathcal{E}, +)$ *denotes the set of all additive cocycles and* $\mathscr{C}_0^0(\mathcal{E}, +)$ *denotes the set of all additive cocycles* b *satisfying the structure equation*

$$\langle b - E_0 b \,|\, b - E_0 b\rangle_0 + E_0 b + (E_0 b)^* = 0.$$

Notation 6.3.2. Whenever it is convenient, we use $E_0 b := (E_0 b_t)_{t\geq 0}$ and $\langle b \,|\, c\rangle_0 := (\langle b_t \,|\, c_t\rangle_0)_{t\geq 0}$ for additive cocycles b and c. We have already used this convention in the above structure equation.

Standard arguments show that $\| \, \|_0$-continuity is equivalent to weak*-continuity (or measurability). Note also that $\| \, \|_0$-continuity is a redundant requirement for a centred additive cocycle b; this continuity property follows also from the martingale property

$$E_{(-\infty,t]} b_{t+s} = b_t \quad (s \geq 0), \tag{6.3.1}$$

and the continuity of the past filtration (see Corollary 5.4.4). Thus the continuity/measurability condition is only needed to ensure that $E_0 b_t = t\, E_0 b_1$ holds (cf. the proof of the next proposition).

Proposition 6.3.3. *The variance of a centred additive cocycle b satisfies*

$$|b_t|_0^2 = t|b_1|_0^2 \,. \tag{6.3.2}$$

More generally, the covariance operator $\langle b_t \,|\, \cdot\, c_t\rangle_0$ of two centred additive cocycles b and c satisfies

$$\langle b_t \,|\, a c_t\rangle_0 = t\langle b_1 \,|\, a c_1\rangle_0 \,, \tag{6.3.3}$$

for any $a \in \mathcal{A}_0$.

Proof. Equation (6.3.2) is obtained from equation (6.3.3) by putting $b = c$ and $a = \mathbb{1}$. For the proof of the latter we define the linear map $\Gamma_t : \mathcal{A}_0 \ni a \mapsto \langle b_t \,|\, a c_t\rangle_0$. By the continuity of b and c mentioned above, Γ is pointwise weak*-continuous. From $\langle b_t \,|\, a S_t c_s\rangle_0 = \langle b_t \,|\, \mathbb{1}\rangle_0 \langle \mathbb{1} \,|\, a c_s\rangle_0 = 0$ we conclude $\Gamma_{t+s}(a) = \langle b_t \,|\, a c_t\rangle_0 + \langle S_t b_s \,|\, S_t a c_s\rangle_0 = \Gamma_t(a) + \Gamma_s(a)$. Thus we obtain for any $\varphi \in \mathcal{A}_{0*}$ the functional equation $\varphi(\Gamma_{t+s}(a)) = \varphi(\Gamma_t(a)) + \varphi(\Gamma_s(a))$. Since $t \mapsto \varphi(\Gamma_t(a))$ is continuous, this functional equation has the unique solution $\varphi(\Gamma_t(a)) = t\varphi(\Gamma_1(a))$. Equation (6.3.3) follows now, since the predual of $\mathcal{A}_0$ is separating for $\mathcal{A}_0$. □

Remark 6.3.4. In the case of q-Gaussian white noises and the CAR-white noises, it is well-known that their additive cocycles are bounded in the operator norm and thus contained in the von Neumann algebra $\mathcal{A}$ itself. Nevertheless, we have refrained here from explicitely defining such additive cocycles for continuous Bernoulli shifts. This additional feature can be exploited in applications, for example, when it is necessary or helpful on the computational side.

6.4. **The correspondence.** Recall that $\mathscr{C}_0^0(\mathcal{E}, \cdot\,)$ and $\mathscr{C}_0^0(\mathcal{E}, +)$ are, respctively, sets of $\| \, \|_0$-continuous unital cocycles and additive cocycles with structure equation, as introduced in Definitions 6.2.1 and 6.3.1. The abstract version of our main result is as follows.

Theorem 6.4.1. *Let $(\mathcal{E}, \mathbb{1}, S, (\mathcal{E}_I)_{I\in\mathcal{I}}; \mathcal{A}_0)$ be an expected continuous GNS Bernoulli shift. Then there is a bijective mapping*

$$\mathrm{Ln}\colon \mathscr{C}_0^0(\mathcal{E}, \cdot\,) \to \mathscr{C}_0^0(\mathcal{E}, +),$$

whose inverse we denote Exp.

Notice that one has $\mathscr{C}_0^0(\mathcal{E}, \cdot\,) = \mathscr{C}_0(\mathcal{E}, \cdot\,)$ for $\dim \mathcal{A}_0 < \infty$, since in this case every semigroup A is uniformly continuous and thus Proposition 6.2.2 (i) applies.

Notation 6.4.2. The cocycles $(\mathrm{Exp}(b)_t)_{t\geq 0}$ and $(\mathrm{Ln}(u)_t)_{t\geq 0}$ will also be written as $(\mathrm{Exp}(b_t))_{t\geq 0}$ resp. $(\mathrm{Ln}(u_t))_{t\geq 0}$. Since these two mappings will be constructed 'pointwise', this slight abuse of notation will vanish anyway.

The above result can be viewed as an abstract corollary of Theorem 6.4.4, which is formulated from a (quantum) stochastic perspective and brings much more structure to the surface. This will allow us to establish the mappings Ln and Exp in a constructive manner. The related proof is based on non-commutative Itô integration, non-commutative exponentials and logarithms. All these tools will be developed in Section 7 and 8, where we will also finish the proof of Theorem 6.4.1.

Notation 6.4.3. $\mathcal{Z}(t)$ denotes an arbitrary net of partitions Z of the interval $[0, t]$, which is partially ordered by inclusion, and such that its mesh size tends to zero.

Theorem 6.4.4. *Let $(\mathcal{E}, \mathbb{1}, S, (\mathcal{E}_I)_{I\in\mathcal{I}}; \mathcal{A}_0)$ be an expected continuous GNS Bernoulli shift. The following are in a bijective correspondence:*

(i) *$\| \, \|_0$-continuous unital cocycles u in $\mathcal{E}$;*

(ii) *pairs (c, K), where $c \subset \mathcal{E}$ is a centred additive cocycle and $K \in \mathcal{A}_0$, satisfy the structure equation*

$$|c_t|_0^2 + t(K + K^*) = 0.$$

The unital cocycle u is obtained from the pair (c, K) as the solution of the non-commutative Itô differential equation (IDE)

$$u_t = \mathbb{1} + \int_0^t \mathrm{d}c_s u_s + \int_0^t \mathrm{d}s\, K u_s\,. \tag{6.4.1}$$

Conversely, the centred additive cocycle c is obtained from the unital cocycle u as the non-commutative logarithm

$$c_t = \| \, \|_0\text{-}\lim_{Z\in\mathcal{Z}(t)} \sum_{t_i\in Z} S_{t_i}(u_{t_{i+1}-t_i} - A_{t_{i+1}-t_i})\,, \tag{6.4.2}$$

and A is the semigroup $A := (E_0 u_t)_{t\geq 0}$ and K is its generator.

We first give some idea of the strategy of the proof. In doing so, the concrete form of the mappings Ln and Exp will appear.

Outlined proof of Theorem 6.4.1 and Theorem 6.4.4. In Section 7 we develop a theory of non-commutative Itô integration that is based solely on an expected continuous GNS Bernoulli shift $(\mathcal{E}, \mathbb{1}, S, (\mathcal{E}_I)_{I\in\mathcal{I}}; \mathcal{A}_0)$ (see Definition 5.4.5) and its additive cocycles $\mathscr{C}_0(\mathcal{E}, +)$ (see Definition 6.3.1). Then Proposition 7.2.1 ensures that the non-commutative Itô integral $\int_0^t \mathrm{d}c_s u_s$ is well-defined. In Theorem 7.3.1 we establish that the IDE (6.4.1) has a unique solution. Now, all terms are introduced and well-defined as they appear in the formulation of Theorem 6.4.4.

In Section 8 we develop the notion of non-commutative exponentials and logarithms for a given $\mathcal{A}_0$-expected continuous Bernoulli shift. We start in Subsection 8.1 to investigate the IDE (6.4.1) for an arbitrary additive cocycle in $\mathscr{C}_0(\mathcal{E}, +)$. Theorem 8.1.1 ensures that an additive cocycle in $\mathscr{C}_0^0(\mathcal{E}, +)$ gives a $\| \|_0$-continuous unital cocycle as the solution of the IDE (6.4.1). Then we are in position to introduce non-commutative exponentials (see Definition 8.1.2). From the uniqueness of the solution we conclude that the mapping $\mathrm{Exp}\colon \mathscr{C}_0^0(\mathcal{E}, +) \to \mathscr{C}_0^0(\mathcal{E}, \cdot\,)$ is well-defined by the family of IDEs

$$\mathrm{Exp}(b_t) = \mathbb{1} + \int_0^t \mathrm{d}b_s \, \mathrm{Exp}(b_s) \quad (b \in \mathscr{C}_0^0(\mathcal{E}, +), t \geq 0).$$

Here we make use of the convention $\int \mathrm{d}b_s \cdot = \int \mathrm{d}c_s \cdot + \int \mathrm{d}sK \cdot$.

We proceed in Subsection 8.2 with the proof that

$$\mathrm{Ln}_0(u_t) := \| \|_0\text{-}\lim_{Z\in\mathcal{Z}(t)} \sum_{t_i\in Z} S_{t_i} u_{t_{i+1}-t_i} - A_{t_{i+1}-t_i} \qquad (t \geq 0)$$

and

$$\mathrm{Ln}(u_t) := \| \|_0\text{-}\lim_{Z\in\mathcal{Z}(t)} \sum_{t_i\in Z} S_{t_i}(u_{t_{i+1}-t_i} - \mathbb{1}) \qquad (t \geq 0)$$

are well-defined in $\mathcal{E}$ for every unital cocycle u in $\mathscr{C}_0^0(\mathcal{E}, \cdot\,)$. In particular, we show that $\mathrm{Ln}(u_t) = \mathrm{Ln}_0(u_t) + Kt$, where K is the generator of the semigroup A. Moreover in Theorem 8.2.2 we verify that $\mathrm{Ln}(u_t)$ is an additive cocycle in $\mathscr{C}_0^0(\mathcal{E}, +)$. We thereby obtain the mapping $\mathrm{Ln}\colon \mathscr{C}_0^0(\mathcal{E}, \cdot\,) \to \mathscr{C}_0^0(\mathcal{E}, +)$, (see Definition 8.2.3). At this stage all terms, appearing in the formulation of Theorem 6.4.1, are well-defined.

It then remains only to prove that the correspondence in Theorem 6.4.4 is bijective, respectively that the mappings Exp and Ln in Theorem 6.4.1 are injective. But this is ensured by Proposition 8.3.1 and Proposition 8.3.3. We complete the proof of Theorem 6.4.1 and Theorem 6.4.4 at the end of Subsection 8.3. □

If the shift $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ is trivial, i.e. if $\mathcal{A}_0 = \mathcal{A}$ and thus $S = \mathrm{id}_{\mathcal{A}}$, then the correspondence reduces to Stone's theorem on uniformly continuous unitary groups. The following result emphasizes the stochastic character of the additive and unital cocycles.

Recall that a function $\mathbb{R} \ni t \mapsto x_t \in \mathcal{E}$ is weak*-differentiable if $\frac{\mathrm{d}}{\mathrm{d}t}\varphi(x_t)$ exists for any φ in the predual of $\mathcal{E}$ (see Theorem A.2).

Corollary 6.4.5. *If either the unital cocycle u, or the additive cocycle c, respectively as stated in (i) or (ii) of Theorem 6.4.4, is weak*-differentiable, then u is a semigroup of unitaries in $\mathcal{A}_0$ with generator $K = \mathrm{i}H$ for some selfadjoint operator $H \in \mathcal{A}_0$ and $c = 0$.*

Consequently, $\| \, \|_0$-continuous cocycles are weak*-differentiable if and only if they are trivial.

Proof. We show that a weak*-differentiable unital cocycle u lies in $\mathcal{A}_0$, the fixed point algebra of the shift S. Since $u_t - \mathbb{1} \in \mathcal{E}_{(-\infty,t]} \cap \mathcal{E}_{[0,\infty)}$ for any $t > 0$, and since the filtration $(\mathcal{A}_{(-\infty,t]})_{t\in\mathbb{R}}$ is continuous, we conclude that the weak* limit $u_0' := \lim_{t\to 0} \frac{1}{t}(u_t - \mathbb{1})$ is in $\bigcap_{t>0} \mathcal{E}_{(-\infty,t]} \cap \mathcal{E}_{[0,\infty)} = \mathcal{E}_{(-\infty,0]} \cap \mathcal{E}_{[0,\infty)} = \mathcal{A}_0$. Furthermore, the cocycle identity implies that $u_t' = (S_t u_0')u_t = u_0' u_t$, where u_t' is the weak* derivative of u_t. Consequently, $u_t = \mathrm{e}^{Kt} \in \mathcal{A}_0$ with $K := u_0'$. Since $u_t = E_0 u_t$, the correspondence of Theorem 6.4.4 implies that $c_t = 0$, hence $K + K^* = 0$ and consequently $K = \mathrm{i}H$ for some selfadjoint operator $H \in \mathcal{A}_0$.

Conversely, the centred additive cocycle c is weak*-differentiable if and only if it is weak*-differentiable at $t = 0$. Indeed, by the cocycle property, it is $c_t' = S_t c_0'$. Similar arguments as for the unital cocycle show that $c_0' \in \mathcal{A}_0$ and thus $c_t' = c_0'$. Hence, since the cocycle is centred, $c_t = c_0' t = 0$. Again $K = \mathrm{i}H$ for some selfadjoint operator $H \in \mathcal{A}_0$ and (6.4.1) implies that $u_t = \mathrm{e}^{\mathrm{i}Ht}$, which is weak*-differentiable. □

The correspondence allows to identify the generator of the semigroup R, introduced in Proposition 6.2.2 (ii).

Corollary 6.4.6. *The generator of the norm-continuous semigroup of completely positive contractions R on $\mathcal{A}_0$ has the Christensen-Evans form [CE79]*

$$\mathcal{L}(a) = \Lambda(a) + K^* a + aK, \qquad a \in \mathcal{A}_0,$$

where $\Lambda := \langle c_1 \,|\, \cdot \, c_1 \rangle_0$.

Proof. Since the convergence in (6.4.2) is independent of the chosen net, we consider the sequence of equidistant partitions $Z_n := \{i\delta_n \,|\, \delta_n := 2^{-n}t, i = 0, \ldots, 2^n\} \in \mathcal{Z}(t)$ of $[0,t]$. Together with (6.3.3) we calculate

$$\begin{aligned}
\Lambda(a)t = \langle c_t \,|\, a c_t \rangle_0 &= \lim_{n\to\infty} \sum_{i,j=0}^{2^n-1} \langle S_{i\delta_n}(u_{\delta_n} - A_{\delta_n}) \,|\, a S_{j\delta_n}(u_{\delta_n} - A_{\delta_n}) \rangle_0 \\
&= \lim_{n\to\infty} \sum_{i=0}^{2^n-1} \langle u_{\delta_n} - A_{\delta_n} \,|\, a(u_{\delta_n} - A_{\delta_n}) \rangle_0 \\
&= \lim_{n\to\infty} \frac{1}{\delta_n} (R_{\delta_n}(a) - a + a - A_{\delta_n}^* a A_{\delta_n}) t
\end{aligned}$$

$$= (\mathcal{L}(a) - K^*a - aK)t\,.$$

Thus, $\mathcal{L}$ is given by $\mathcal{L}(a) = \Lambda(a) + K^*a + aK$. □

The structure equation of the additive cocycle immediately gives conditions, on how additive cocycles should be composed for the construction of unital cocycles.

Lemma 6.4.7. *Let $b, c \in \mathscr{C}_0^0(\mathcal{E}, +)$. Then,*

$$b + c - \mathrm{Re}\langle b - E_0 b \,|\, c - E_0 c\rangle_0 \in \mathscr{C}_0^0(\mathcal{E}, +).$$

If the centred parts of b and c are, in addition, $\mathcal{A}_0$-independent or $\mathcal{A}_0$-orthogonal then $b + c \in \mathscr{C}_0^0(\mathcal{E}, +)$.

Proof. This is an elementary consequence of Definition 6.3.1. □

Notice that $\mathcal{A}_0$-independence of two centred additive cocycles implies their $\mathcal{A}_0$-orthogonality.

Lemma 6.4.7 allows to generalize Corollary 6.4.6 to a countable family of mutually $\mathcal{A}_0$-orthogonal additive centred cocycles $(c^i)_{i\in\mathbb{N}}$ with drifts $(K^i)_{i\in\mathbb{N}}$, satisfying the structure equation. If the sequences, defined by $\sum_{i=1}^n c_t^i + K^i t$, are SOT-convergent for $n \to \infty$ for any $t > 0$, then their limits define an additive cocycle in $\mathscr{C}_0^0(\mathcal{E}, +)$. In this case, the Christensen-Evans form of the generator of the semigroup R, associated to $\sum_{i\in\mathbb{N}} c_1^i + K^i$, is given by

$$\mathcal{L}(a) = \Big\langle \sum_{i\in\mathbb{N}} c_1^i \,\Big|\, a \sum_{i\in\mathbb{N}} c_1^i \Big\rangle_0 + \sum_{i\in\mathbb{N}} K^{i*}a + aK^i\,.$$

The more familiar form $\sum_{i\in\mathbb{N}}\langle c_1^i \,|\, a c_1^i\rangle_0 + K^{i*}a + aK^i$ of the generator is obtained if the considered family of additive cocycles is mutually $\mathcal{A}_0$-independent. Notice that the sum over infinitely many terms is meant as limit in the weak* topology on $\mathcal{A}_0$.

Remark 6.4.8. (i) The correspondence in Theorem 6.4.4 provides a concrete Stinespring decomposition of norm continuous CP_0-semigroups on a von Neumann algebra. Further details are postponed to sequel publications.

(ii) There are many other independent approaches to the dilation of norm continuous CP_0-semigroups on von Neumann algebras. For further information on this huge subject we refer the reader to [Arv03, BS00, BBLS04, GLW01, GLSW03, GS99, LW00, Lin05, MS02] and the references therein. In this context we wish to also mention the approach of [Sau86, CS03a, CS03b].

6.5. **Non-commutative white noises.** Our main result, stated in Theorem 6.4.4, establishes a bijective correspondence between unital and additive shift cocycles. Actually, it already contains an (abstract) result for unitary shift cocycles, as introduced in Definition 6.1.1. Recall from Subsection 6.4 that Ln maps a $\| \|_0$-continuous cocycle to an additive cocycle.

Theorem 6.5.1. *Let* $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ *be an expected continuous Bernoulli shift. Then there exists a bijective correspondence between the set of* $\|\,\|_0$*-continuous unitary cocycles* $\mathscr{C}_0^0(\mathcal{A}, \boldsymbol{\cdot})$ *and the set of additive cocycles* $\mathrm{Ln}\,\big(\mathscr{C}_0^0(\mathcal{A}, \boldsymbol{\cdot})\big) \subset \mathscr{C}_0^0(\mathcal{E}, +)$.

Proof. Every unitary cocycle defines a unital cocycle. According to Theorem 6.4.4, the latter corresponds uniquely to an additive cocycle. This ensures the existence of the claimed bijection. □

What is the structure of the set $\mathrm{Ln}(\mathscr{C}_0^0(\mathcal{A}, \boldsymbol{\cdot}))$? The development of the necessary material for a satisfactory answer goes beyond the limits of the present paper. Due to the importance of this question, let us at least outline an answer for $\mathbb{C}$-expected shifts with a tracial state ψ (see also the survey [Kös03]). Notice that $E_0 = \psi(\cdot)\mathbb{1}$ in this case.

Up to now we have not further specified the GNS Hilbert space of a continuous Bernoulli shift. It turns out that the non-commutative L^2-space provides all the further structure which is sufficient to reveal the structure of $\mathrm{Ln}(\mathscr{C}_0^0(\mathcal{A}, \boldsymbol{\cdot}))$. We have available the whole scale of non-commutative L^p-spaces and in particular the adjoint b_t^* of an additive cocycle b_t. More importantly, the sesquilinear quadratic variation

$$[\![b, b]\!]_t := \lim_{n\to\infty} \sum_{j=0}^{n-1} S_{jt/n} |b_{t/n}|^2,$$

is well-defined as an L^1-norm limit [Kösb]. Now it turns out that a $\|\,\|_0$-continuous unital cocycle is unitary if and only if the corresponding additive cocycle b satisfies

$$[\![b, b]\!]_t + b_t^* + b_t = 0.$$

Notice that the compression of this structure equation by the conditional expectation E_0 gives back the structure equation appearing in the correspondence between unital and additive cocycles. This conceptual approach works also in the more general setting of an $\mathcal{A}_0$-expected continuous Bernoulli shift, at least in the case of a tracial state.

We return to the discussion of the structure of continuous Bernoulli shifts and present an operator algebraic notion of 'white noise'.

Definition 6.5.2. *An expected continuous Bernoulli shift* $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ *is called an* ($\mathcal{A}_0$-)expected (non-commutative) white noise *if for each* $t > 0$

$$\mathcal{A}_{[0,t]} = \bigvee \{u_s \,|\, u \in \mathscr{C}_0(\mathcal{A}, \boldsymbol{\cdot}), 0 \le s \le t\}. \tag{6.5.1}$$

The $\mathcal{A}_0$*-expected white noise is said to be* generated by $\mathscr{C}_0(\mathcal{A}, \boldsymbol{\cdot})$.

Notice that $\mathcal{A}_0 \subseteq \bigvee\{u_s \,|\, u \in \mathscr{C}_0(\mathcal{A}, \boldsymbol{\cdot}), 0 \le s \le t\}$ for any $t > 0$, because each unitary in $\mathcal{A}_0$ is a trivial unitary cocycle and a von Neumann algebra is generated by its unitaries. A plausible explanation of 'whiteness' is provided at the end of this subsection.

It follows easily from the cocycle equation that an $\mathcal{A}_0$-expected white noise is always locally minimal. This implies upward continuity and in particular $\mathcal{A}_I = \mathcal{A}_{\overline{I}}$ for its filtration (see Subsection 4.1).

Proposition 6.5.3. *An $\mathcal{A}_0$-expected white noise with enriched independence is locally minimal, locally maximal and has a continuous filtration.*

Proof. Enriched independence implies the local maximality of the filtration and thus downward continuity (see Subsection 4.2). □

We expect that the enriched independence condition can be removed, due to the continuity of the unitary cocycles. It is evident that all continuity properties of the filtration pass to the corresponding continuous GNS Bernoulli shift $(\mathcal{E}, \mathbb{1}, S, (\mathcal{E}_I)_{I\in\mathcal{I}}; \mathcal{A}_0)$.

At this point it is worthwhile to recall our convention. We use the attribute 'non-commutative' always in the sense of 'not necessarily commutative'. If an $\mathcal{A}_0$-expected non-commutative white noise does not come from (operator-expected) probability theory, we call it a *quantum white noise*.

Remark 6.5.4. (i) If the von Neumann algebra $\mathcal{A}$ of a $\mathbb{C}$-expected white noise is commutative then its measure theoretic version is identified as a 'classical noise' in [Tsi04]. Note also that unitary cocycles play a generating role similar to the units of Arveson product systems. Thus white noises may also be called continuous Bernoulli shifts of 'type I'. The 'classical or type I' part is well-understood for continuous product systems of probability spaces or Hilbert spaces. Stressing the analogy, we hope to gain a better understanding of 'continuous commuting square systems of operator algebras', starting in the (time-)homogeneous 'type I' setting.

(ii) There are other notions of 'quantum white noise' in the literature, e.g. the bialgebra approach of Schürmann [Sch93]. Essentially, these approaches have in common that they start with generalized or quantum Brownian motions (or Lévy processes). These processes are given explicitly in (deformed) Fock spaces and generate their 'quantum white noises'. From our work the question arises of whether every $\mathbb{C}$-expected non-commutative white noise induces a deformed Fock space such that it can be generated from quantum Lévy processes on this Fock space. Notice also in this context that such a Fock space structure is anticipated by multiple non-commutative Itô integrals which can be formulated easily, starting from the results in Section 7.

The following result states that one can always extract the 'classical' or 'type I part' of a shift.

Proposition 6.5.5. *Let $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ be an $\mathcal{A}_0$-expected continuous Bernoulli shift. Then there exists a conditional expectation E such that the* *compression of the shift by E is an $\mathcal{A}_0$-expected non-commutative white noise generated by $\mathscr{C}_0(\mathcal{A}, \cdot\,)$.*

Notice that a single unitary cocycle $u \in \mathscr{C}_0(\mathcal{A}, \cdot\,)$ (together with all trivial unitary cocycles) may not generate a non-commutative white noise. However this is guaranteed if $u \subset \mathcal{A}^{\psi}$.

Proof. Let $\mathcal{B}_{[r,r+t]} := \bigvee\{S_r(u_s) \,|\, u \in \mathscr{C}_0(\mathcal{A}, \cdot\,), r \in \mathbb{R}, 0 \leq s \leq t\}$ and define similarly $\mathcal{B}_I$ for more general intervals $I \subseteq \mathbb{R}$. Note that if u is a unitary cocycle then $\sigma_s^{\psi}(u)$ is so, for any $s \in \mathbb{R}$, since the modular automorphism group and the shift commute (see Theorem B.2). We conclude from the σ^{ψ}-invariance of $\mathcal{B}_I$ that the conditional expectation $E_{[r,r+t]}$ from $(\mathcal{A}, \psi)$ onto $\mathcal{B}_{[r,r+t]}$ exists. Furthermore, $\mathcal{B}_{\mathbb{R}}$ is S-invariant and $S_t(\mathcal{B}_I) = \mathcal{B}_{I+t}$. The $\mathcal{A}_0$-independence of $\mathcal{B}_I$ and $\mathcal{B}_J$ for disjoint I and J follows immediately from the inclusions $\mathcal{A}_0 \subseteq \mathcal{B}_K \subseteq \mathcal{A}_K$ for any $K \in \mathcal{I}$. Thus $(\mathcal{B}, \psi_{|\mathcal{B}}, S_{|\mathcal{B}}, (\mathcal{B}_I)_{I\in\mathcal{I}}; \mathcal{A}_0)$ is an expected continuous Bernoulli shift and, by construction, an $\mathcal{A}_0$-expected non-commutative white noise. □

Proposition 6.5.6. *Any expected non-commutative white noise $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ is generated already by a countable set of unitary cocycles.*

Proof. Since $\mathcal{A}$ has a separable predual and is represented with respect to the faithful normal state ψ, there exists a SOT-dense sequence $(x_n)_{n\in\mathbb{N}}$ [Ped79, Proposition 3.8.4]. Kaplansky's density theorem (and the separability of the predual) ensures that each x_n can be approximated by sequence $(x_{n,k})_{k\in\mathbb{N}}$ in the algebraic hull of the set $\{S_t u_s \,|\, u \in \mathscr{C}_0(\mathcal{A}, \cdot\,), t \in \mathbb{R}, s \geq 0\}$. Clearly there are at most countable many cocycles involved to generate all elements $(x_{n,k})_{n,k\in\mathbb{N}}$. □

An immediate consequence of Theorem 6.5.1 is the following result.

Corollary 6.5.7. *If its only centred additive cocycles is 0, then the expected continuous Bernoulli shift $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ restricts to a trivial non-commutative white noise.*

Proof. It follows from Theorem 6.5.1 that $\mathscr{C}_0(\mathcal{A}, \cdot\,)$ is a subset of $\mathcal{A}_0$. But this implies the triviality of the compressed shift stated in Proposition 6.5.5. □

From the cocycle identity it follows that a continuous Bernoulli shift without local minimality (see Subsection 4.5 for an example) cannot be a white noise. We emphasize that a locally minimal *commutative* $\mathbb{C}$-expected continuous Bernoulli shift may not be a white noise. The surprising existence of such examples emerges from work of Tsirelson and Vershik on the construction of intrinsically non-linear random fields [TV98]. We summarize their result, phrased in our terminology, as follows (see also Example 4.6.4).

Theorem 6.5.8 (Tsirelson-Vershik)**.** *There are non-trivial locally minimal commutative $\mathbb{C}$-expected continuous Bernoulli shifts with no non-zero centred additive cocycles.*

We close this subsection with a (common) explanation of 'whiteness' which also captures the non-commutative case. By Theorem 6.5.1, a $\|\,\|_0$-continuous unitary cocycle u uniquely determines a centred additive cocycle b. Let $\mathcal{K}$ be the Hilbert space given by the closed $\mathbb{C}$-linear span of $\{S_t b_s \,|\, t \in \mathbb{R}, s \geq 0\}$. The shift S defines a strongly continuous unitary group on $\mathcal{K}$, denoted by the same symbol. Thus we obtain a Hilbert space cocycle system $(\mathcal{K}, S, b)$ which captures the linear theory of a white noise (up to multiplicity). Its theory is equivalent to that of the triple $(L^2(\mathbb{R}), (\sigma_t)_{t\in\mathbb{R}}, (\chi_{[0,t]})_{t\geq 0})$ (see e.g. [Gui71, Gui72]). Here denotes σ_t the right shift on $L^2(\mathbb{R})$ and the family $(\chi_{[0,t]})_{t\geq 0}$ satisfies the cocycle equation $\chi_{[0,t+s]} = \chi_{[0,t]} + \sigma_t \chi_{[0,s]}$. In the discussion of 'whiteness', it is instructive to think of 'white noise' as the family of formal derivatives 'db_t/dt', thus as the family of Dirac distributions δ_t centred at t in the equivalent picture. The latter is better expressed in the spectral representation of the shift. By Fourier transformation one obtains the triple $(L^2(\mathbb{R}), (\mathrm{e}^{\mathrm{i}t\lambda})_{t\in\mathbb{R}}, ((\mathrm{i}\lambda)^{-1}(\mathrm{e}^{\mathrm{i}t\lambda} - 1))_{t\geq 0})$. The derivatives '$db_t/dt$' are now given by the family of functions $(\mathrm{e}^{\mathrm{i}t\lambda})_{t\in\mathbb{R}}$ (not contained in $L^2(\mathbb{R})$). 'White' because these functions ('frequency modes') evolve independently in time t *and* they are equally weighted by the Lebesgue measure. It is worthwhile recalling that $\mathbb{C}$-expected white noises, each of them generated by a single unitary cocycle, *all* have the same linear theory as sketched above. It is determined by the second order correlation functions of the additive cocycle. The differences between $\mathbb{C}$-expected non-commutative white noises (besides of multiplicities) appear by looking at the correlation functions of higher orders.

6.6. **Examples of the correspondence.** We illustrate the explicit form of additive cocycles which satisfy the structure equation of Theorem 6.4.4 and thus lead to unital cocycles. Note that all examples presented below are $\mathcal{A}_0$-expected non-commutative white noises in the sense of Definition 6.5.2.

Example 6.6.1 (Gaussian white noise)**.** We continue the discussion of Example 4.6.1. The $\mathbb{C}$-expected GNS Bernoulli shift is given explicitly by $(L^2(\mathscr{S}', \Sigma, \mu), \mathbb{1}, S, L^2(\mathscr{S}', \Sigma_I, \mu_I))$. Here μ_I denotes the restriction of μ to Σ_I. The Brownian motion $B_t \in L^2(\mathscr{S}', \Sigma, \mu)$ is the limit of $(X_{f_n})_{n\in\mathbb{N}}$ where $f_n \to \chi_{[0,t]}$ in the L^2-norm. All additives cocycle b_t are of the form $\lambda B_t + Kt$, where $\lambda, K \in \mathbb{C}$. It is elementary to check that b_t satisfies the structure equation stated in Theorem 6.4.4 (ii) if and only if $\operatorname{Re} K = -|\lambda|^2/2$ and $\operatorname{Im} K = h$ for some $h \in \mathbb{R}$. Thus we obtain $b_t = \lambda B_t - (|\lambda|^2/2 - \mathrm{i}h)t$. By straightforward calculations the corresponding unital cocycle is given by

$$\operatorname{Exp} b_t = \exp(\lambda B_t - (\lambda \operatorname{Re}(\lambda) - \mathrm{i}h)t).$$

Example 6.6.2 (Poisson white noise)**.** Consider the Poisson process N with intensity $\lambda > 0$ of Example 4.6.2. Then $c_t := \varepsilon(N_t - \lambda t)$ defines a centred additive cocycle. From $\psi_\mu(|c_t|^2) = |\varepsilon|^2 \lambda t$ we conclude that the additive cocycle $b_t := c_t + Kt$ satisfies the structure equation if and only if $K = -|\varepsilon|^2\lambda/2 + \mathrm{i}h$ for some $h \in \mathbb{R}$. The explicit form of the corresponding unital cocycle is found after some calculations to be of the form

$$\operatorname{Exp} b_t = (1+\varepsilon)^{N_t} \exp\left(-(\tfrac{1}{2}|\varepsilon|^2\lambda + \varepsilon\lambda - \mathrm{i}h)t\right).$$

Example 6.6.3 (CCR white noises)**.** We continue Examples 2.2.2 and 4.7.1. To find the form of additive cocycles, we pass to a concrete GNS representation of the C*-algebra $\mathrm{CCR}(L^2(\mathbb{R}), \mathrm{Im}\langle \cdot \,|\, \cdot \rangle)$ with the state $\psi_\lambda(W(f)) = \exp\left(-(2\lambda+1)/4 \,\|f\|^2\right)$ $(\lambda > 0)$. This representation of Araki-Woods type is given on the tensor product of two symmetric Fock spaces $\mathcal{F}_+(L^2(\mathbb{R})) \otimes \mathcal{F}_+(L^2(\mathbb{R}))$ with cyclic separating vector $\Omega \otimes \Omega$ such that

$$\psi_\lambda(W(f)) = \langle \Omega \otimes \Omega \,|\, W_{\mathcal{F}}(\sqrt{\lambda+1} f)\Omega \otimes W_{\mathcal{F}}(\sqrt{\lambda} Jf)\Omega \rangle$$

Here $W_{\mathcal{F}}(f)$ denotes the Weyl operator on the symmetric Fock space and J the complex conjugation on $L^2(\mathbb{R})$. One finds from the represented Weyl operators, via Stone's Theorem, the annihilation operator

$$a_\lambda(f) := \sqrt{\lambda+1}\, a(f) \otimes \mathbb{1} + \sqrt{\lambda}\, \mathbb{1} \otimes a^*(Jf)$$

and the annihilation operator $a_\lambda(f)^*$ as its adjoint, in terms of the usual annihilation operator $a(f)$ on $\mathcal{F}_+(L^2(\mathbb{R}))$. This gives immediately the general form of a centred additive cocycle as a linear combination of $a_\lambda(\chi_{[0,t]})$ and $a^*_\lambda(\chi_{[0,t]})$:

$$c_t = a_1 \sqrt{\lambda}\, \Omega \otimes \chi_{[0,t]} + a_2 \sqrt{\lambda+1}\, \chi_{[0,t]} \otimes \Omega \qquad (a_1, a_2 \in \mathbb{C}).$$

One verifies easily from this form that an additive cocycle $b_t = c_t + Kt$ satisfies the structure equation whenever $K = -1/2(\lambda|a_1|^2 + (\lambda+1)|a_2|^2) + \mathrm{i}h$ for some $h \in \mathbb{R}$. Obviously, the fixed constants $\sqrt{\lambda}$ and $\sqrt{\lambda+1}$ are superfluous. Their effect can be compensated by rescaling the coefficients, as long as one calculates only time-ordered higher moments factorizing into second moments.

The examples 6.6.1 to 6.6.3 generalize straightforward to the case of expected white noises with infinite multiplicity

$$(\mathcal{A}_0 \otimes (\bigotimes_{n\in\mathbb{N}} \mathcal{C}), \psi \otimes (\bigotimes_{n\in\mathbb{N}} \psi), \mathrm{id} \otimes (\bigotimes_{n\in\mathbb{N}} S), (\mathcal{A}_0 \otimes (\bigotimes_{n\in\mathbb{N}} \mathcal{C}_I))_{I\in\mathcal{I}}; \mathcal{A}_0 \otimes (\bigotimes_{n\in\mathbb{N}} \mathbb{1})).$$

Here $(\mathcal{C}, \psi, S, (\mathcal{C}_I)_{I\in\mathcal{I}})$ denotes a $\mathbb{C}$-expected white noise as considered in the last three examples. The general form of an additive cocycle is now

$$b_t = \sum_n a_n \otimes c_t^{(n)} + (K \otimes \mathrm{id})t,$$

where $K \in \mathcal{A}_0$ and $c_t^{(n)}$ denotes the canonical embedding of the $\mathbb{C}$-expected centred additive cocycle with variance $\langle c_1 \,|\, c_1 \rangle = 1$ into the infinite tensor product at the nth position. Moreover, it is required for the sequence $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}_0$

that $\sum_{i=1}^n |a_i|^2$ is SOT-convergent for $n \to \infty$. The structure equation is satisfied by b if and only if $K \in \mathcal{A}_0$ and $\operatorname{Re} K = -1/2 \sum_n |a_n|^2$. One obtains, as usual, the Markovian semigroup

$$\mathcal{A}_0 \ni x \mapsto R_t(x) := \langle \operatorname{Exp}(b_t) \,|\, x \operatorname{Exp}(b_t) \rangle_0,$$

where we identify $\mathcal{A}_0$ and $\mathcal{A}_0 \otimes \mathbb{1}$. As is well-known, this semigroup has a generator of Lindblad form:

$$\mathcal{L}(x) = \sum_n (a_n^* x a_n - \tfrac{1}{2}\{a_n^* a_n, x\}) + \mathrm{i}[h, x]$$

(here denote $\{a, b\}$ and $[a, b]$ the anti-commutator resp. the commutator).

Example 6.6.4 (CAR white noises)**.** We continue the discussion of Examples 2.4.1 and 4.7.3. Similarly to the CCR-algebra, we pass from the C*-algebra $\mathrm{CAR}(L^2(\mathbb{R}))$ with the quasi-free state $\psi_\lambda(a^*(f)a(g)) = \lambda\langle g \,|\, f\rangle$ $(0 < \lambda < 1)$ to an Araki-Woods representation on the tensor product of two antisymmetric Fock spaces $\mathcal{F}_-(L^2(\mathbb{R})) \otimes \mathcal{F}_-(L^2(\mathbb{R}))$ and find that an additive cocycle is always of the form

$$b_t = a_1 \Omega \otimes \chi_{[0,t]} + a_2 \chi_{[0,t]} \otimes \Omega \qquad (a_1, a_2 \in \mathbb{C}).$$

In the case of $\mathrm{CAR}(K_0 \oplus L^2(\mathbb{R}, \mathcal{K}_1))$, where one obtains a $\mathcal{B}_0$-expected white noise, the general form of an additive cocycle is now

$$b_t = \sum_n x_n \Omega \otimes (\chi_{[0,t]} \otimes e_n) + y_n (\chi_{[0,t]} \otimes e_n) \otimes \Omega \qquad (x_n, y_n \in \mathcal{B}_0).$$

Here, $\{e_n\}_n$ is an orthonormal basis of $\mathcal{K}_1$. Moreover, we identified $L^2(\mathbb{R}, \mathcal{K}_1)$ and $L^2(\mathbb{R}) \otimes \mathcal{K}_1$. If $\mathcal{K}_1$ is infinite dimensional, one needs also that $\sum_i^n x_i^* x_i + y_i^* y_i$ is SOT-convergent for $n \to \infty$. The computation of the Christensen-Evans generator is straightforward.

Example 6.6.5 (q-Gaussian white noises)**.** We continue the discussion of Examples 2.4.2, 2.4.3 and 4.7.5. Let us immediately consider the case

$$(\mathcal{F}_q(\mathcal{K}_0 \oplus L^2(\mathbb{R})), \tau, S, (\mathcal{F}_q(\mathcal{K}_0 \oplus L^2(I))_{I \in \mathcal{I}}; \mathcal{F}_q(\mathcal{K}_0 \oplus 0)).$$

The general form of an $\mathcal{F}_0(\mathcal{K}_0)$-expected additive cocycle is now

$$b_t = \sum_{i=1}^n a_i \Phi(0 \oplus \chi_{[0,t]}) \tilde{a}_i + Kt,$$

where $a_i, \tilde{a}_i, K \in \mathcal{F}_q(\mathcal{K}_0 \oplus 0)$ $(1 \le i \le n)$. The structure equation is satisfied if $K = \frac{1}{2} \sum_{i,j=1}^n \tilde{a}_i^* \Gamma_q(q)(a_i^* a_j) \tilde{a}_j + \mathrm{i}h$ for some selfadjoint operator $h \in \mathcal{B}_0$. Here $\Gamma_q(q)$ denotes the second quantization of the multiplication operator $M_q(g) := qg$ with $g \in \mathcal{K}_0 \oplus L^2_{\mathbb{R}}(\mathbb{R})$. Moreover we used the identity

$$E_0(\tilde{a}^* \Phi(0 \oplus \chi_{[0,t]}) a^* a \Phi(0 \oplus \chi_{[0,t]}) \tilde{a}) = t \tilde{a}^* \Gamma_q(q)(a^* a) \tilde{a},$$

where $a, \tilde{a} \in \mathcal{F}_q(\mathcal{K}_0 \oplus 0)$ (see [DM03]). Notice that $\Gamma_0(0)(x) = \tau(x)$ in free probability. The generator of the semigroups now has the Christensen-Evans form

$$\mathcal{L}(x) = \sum_{i,j=1}^{n} \left(\tilde{a}_i^* \Gamma_q(q)(a_i^* x a_j)\tilde{a}_j - \tfrac{1}{2}\left\{\tilde{a}_i^* \Gamma_q(q)(a_i^* a_j)\tilde{a}_j, x\right\}\right) + \mathrm{i}[h, x].$$

Remark 6.6.6. This list of examples can be enlarged by examples coming from generalized Brownian motions, as soon as white noise functors are available and the vacuum vector of the deformed Fock space is separating for the von Neumann algebra, generated by these generalized Brownian motions.

7. Non-commutative Itô integration

This section is devoted to the development of operator-valued non-commutative Itô integration, as it is needed for the correspondence stated in Theorem 6.4.4. In the case of a $\mathbb{C}$-expected Bernoulli shift with a commutative von Neumann algebra, our approach reduces to an L^2-theory of stochastic Itô integration, as understood for Lévy processes, in particular Brownian motion [App04].

Throughout this section, we work in the presence of a fixed expected continuous GNS Bernoulli shift $(\mathcal{E}, \mathbb{1}, S, (\mathcal{E}_I)_{I\in\mathcal{I}}; \mathcal{A}_0)$. Moreover, we assume that this shift has at least one non-zero centred additive cocycle.

7.1. **Non-commutative Itô integrals for simple adapted processes.** In the following, a centred additive cocycle c will serve as the non-commutative generalization of a stochastic process with stationary independent increments, such as Brownian motion or the Poisson process.

We start with the usual notion of adapted process in L^2-Itô integration. A *process* is a family $x = (x_t)_{t\geq 0} \subset \mathcal{E}$. It is called *(locally) adapted* if $x_t \in \mathcal{E}_{(-\infty,t]}$ (resp. $\mathcal{E}_{[0,t]}$) for any $t \geq 0$. A *simple adapted process* $x = (x_t)_{t\geq 0} \subset \mathcal{E}$ is given by

$$x := \sum_{i\geq 0} x_i \, \chi_{[s_i, s_{i+1})}\,, \qquad x_i \in \mathcal{E}_{(-\infty, s_i]}\,.$$

Here $Z := \{s_i \,|\, s_i < s_{i+1},\, i \in \mathbb{N}_0\}$ defines a partition of $\mathbb{R}^+$. For a simple adapted process x (putting $s_m := t_0$ and $s_n := t$ for suitable $0 \leq m < n$), we introduce the ((left) non-commutative) Itô integral

$$\int_{t_0}^{t} \mathrm{d}c_s x_s := \sum_{i=m}^{n-1} (c_{s_{i+1}} - c_{s_i})\, x_i\,. \tag{7.1.1}$$

It is sufficient to define the integrals of simple adapted processes just for intervals $[t_0, t]$ with boundaries $t_0, t \in Z$ (using a sub-partition of Z if necessary). Indeed, this expression is well-defined, since $c_{s_{i+1}} - c_{s_i} = S_{s_i} c_{s_{i+1}-s_i}$ and x_i are $\mathcal{A}_0$-independent, and thus, by Proposition 5.3.1, their product makes sense. The following operator identity in $\mathcal{A}_0$, the so-called *(non-commutative) Itô* *identity*, will be crucial for the extension of the integral to a larger class of adapted processes.

Lemma 7.1.1. *If x and y are two simple adapted processes, then*

$$\Big\langle \int_{t_0}^t \mathrm{d}c_s x_s \,\Big|\, \int_{t_0}^t \mathrm{d}c_s y_s \Big\rangle_0 = \int_{t_0}^t \langle x_s \,|\, \Lambda(\mathbb{1})\, y_s\rangle_0 \mathrm{d}s\,,$$

where Λ is the uniformly bounded covariance operator of b, namely $\langle c_1 \,|\, \cdot\, c_1\rangle_0$.

Proof. By refinement, we may assume that x and y are simple adapted processes with respect to the same partition. From equation (5.3.2) we conclude

$$\langle (c_{s_{i+1}} - c_{s_i})\, x_i \,|(c_{s_{j+1}} - c_{s_j})\, y_j\rangle_0 = 0 \qquad \text{for } i \neq j\,.$$

Thus,

$$\begin{aligned}
\Big\langle \int_{t_0}^t \mathrm{d}c_s x_s \,\Big|\, \int_{t_0}^t \mathrm{d}c_s y_s \Big\rangle_0 &= \sum_{i=m}^{n-1} \langle (c_{s_{i+1}} - c_{s_i})\, x_i \,|(c_{s_{i+1}} - c_{s_i})\, y_i\rangle_0 \\
&\overset{(5.3.2)}{=} \sum_{i=m}^{n-1} \langle x_i \,|\, |c_{s_{i+1}-s_i}|_0^2\, y_i\rangle_0 \overset{(6.3.2)}{=} \sum_{i=m}^{n-1} \langle x_i \,|\, \Lambda(\mathbb{1})\, y_i\rangle_0 (s_{i+1} - s_i) \\
&= \int_{t_0}^t \langle x_s \,|\, \Lambda(\mathbb{1})\, y_s\rangle_0 \mathrm{d}s\,.
\end{aligned}$$

□

Remark 7.1.2. In this paper we only make use of left Itô integrals. Using the same techniques, the right non-commutative Itô integral $\int_{t_0}^t x_s \mathrm{d}c_s$ may also be introduced. It enjoys an Itô identity of the form $\big\langle \int_{t_0}^t x_s \mathrm{d}c_s \,\big|\, \int_{t_0}^t y_s \mathrm{d}c_s \big\rangle_0 = \int_{t_0}^t \Lambda(\langle x_s \,|\, y_s\rangle_0) \mathrm{d}s$. We emphasize that here, in contrast to Itô integration or Hudson-Parthasarathy quantum stochastic integration, the left and right integrals differ in most cases, since the past/future structure may not commute, even in the case $\mathcal{A}_0 \simeq \mathbb{C}$. A typical example is Itô integration in the case of free Brownian motion (see also [BS98]).

7.2. **An extension of the non-commutative Itô integral.** For the purposes of this paper, it will be sufficient to extend the Itô integral, introduced in Subsection 7.1, to the vector space $\mathcal{V}$ of piecewise SOT-continuous adapted processes

$$\mathbb{R}_0^+ \ni t \mapsto x_t \in \mathcal{E}$$

(which are locally $\|\,\|_0$-bounded by the uniform boundedness principle). Notice that $\mathcal{V}$ includes locally bounded $\|\,\|_0$-continuous adapted processes, in particular centred additive cocycles.

The key to this extension is provided by the Itô identity. We introduce on $\mathcal{V}$ the family of seminorms

$$\mathrm{d}_{\xi_0,I}(x) := \Big[\int_I \big\| |x_s|_0 \xi_0 \big\|^2 \mathrm{d}s\Big]^{1/2},$$

where $\xi_0 \in \mathcal{H}_0$ and I ranges over all compact intervals in $\mathbb{R}^+$. We have

$$\mathrm{d}_{\xi_0,I}(x) \leq \|\xi_0\| \sup_{s\geq 0} \|x_s\|_0 \, |I|.$$

Notice that for a simple adapted process $x \in \mathcal{V}$, the estimate

$$\begin{aligned} \Big\| \Big| \int_I \mathrm{d}c_s x_s \Big|_0 \xi_0 \Big\|^2 &= \Big\langle \xi_0 \,\Big|\, \int_I \langle x_s \,|\, \Lambda(\mathbb{1})\, x_s \rangle_0 \mathrm{d}s\, \xi_0 \Big\rangle = \int_I \big\langle \xi_0 \,\big|\, \langle x_s \,|\, \Lambda(\mathbb{1})\, x_s \rangle_0 \xi_0 \big\rangle \mathrm{d}s \\ &\leq \|\Lambda(\mathbb{1})\| \, \mathrm{d}_{\xi_0,I}(x)^2. \end{aligned}$$

is valid. Here, we used the Itô identity and $\Lambda(\mathbb{1}) \leq \|\Lambda(\mathbb{1})\| \mathbb{1}$.

Next, we show that processes in $\mathcal{V}$ can be approximated by simple processes in $\mathcal{V}$. By a simple reduction argument, we may assume that the process $x \in \mathcal{V}$ is SOT-continuous on I. We define the simple adapted processes $x^Z := \sum_{i\in\mathbb{Z}^+} x_{s_i} \chi_{[s_i,s_{i+1})} \in \mathcal{V}$, where $Z := \{s_i \,|\, s_i < s_{i+1},\, i \in \mathbb{N}_0\}$ denotes a partition of $\mathbb{R}^+$. The continuity of x and

$$\mathrm{d}_{\xi_0,I}(x - x^Z)^2 \leq \sum_{s_i \in I} \int_{s_i}^{s_{i+1}} \big\| |x_r - x_{s_i}|_0 \xi_0 \big\|^2 \mathrm{d}r + \int_{s_{i-1}}^{s_i} \big\| |x_r - x_{s_{i-1}}|_0 \xi_0 \big\|^2 \mathrm{d}r\,,$$

imply that $\mathrm{d}_{\xi_0,I}(x - x^Z) \to 0$, whenever the mesh size of the partition Z tends to 0. From this and from

$$\begin{aligned} \Big\| \int_I \mathrm{d}c_s x_s^Z \Big\|_0 &= \sup_{\|\xi_0\|\leq 1} \Big\| \Big| \int_I \mathrm{d}c_s x_s^Z \Big|_0 \xi_0 \Big\| \leq \|\Lambda(\mathbb{1})\|^{1/2} \sup_{s\geq 0} \|x_s^Z\|_0 |I| \\ &\leq \|\Lambda(\mathbb{1})\|^{1/2} \sup_{s\geq 0} \|x_s\|_0 |I|, \end{aligned} \tag{7.2.1}$$

and furthermore

$$\Big\| \Big| \int_I \mathrm{d}c_s x_s^Z - \int_I \mathrm{d}c_s x_s^{Z'} \Big|_0 \xi_0 \Big\| \leq \|\Lambda(\mathbb{1})\|^{1/2} \big(\mathrm{d}_{\xi_0,I}(x^Z - x) + \mathrm{d}_{\xi_0,I}(x - x^{Z'}) \big)\,,$$

we conclude that $\big(\int_I \mathrm{d}c_s x_s^Z\big)_Z$ is a bounded SOT-Cauchy net in $\mathcal{E}$. Its limit in $\mathcal{E}$ is denoted by $\int_0^t \mathrm{d}c_s x_s$. Finally, one verifies by routine arguments that the definition of the integral is independent of the chosen net of partitions. Moreover, it is elementary to see from the definition that the integral carries all the usual properties, such as $\int_{t_0}^t \mathrm{d}c_s x_s = \int_{t_0}^{t_1} \mathrm{d}c_s x_s + \int_{t_1}^t \mathrm{d}c_s x_s$ $(t_0 \leq t_1 \leq t)$, and linearity with respect to vector space structure of $\mathcal{V}$. We summarize:

Proposition 7.2.1. *The non-commutative Itô integral extends from simple adapted processes in $\mathcal{E}$ to the vector space $\mathcal{V}$ of piecewise* SOT*-continuous adapted processes. Moreover, the Itô identity*

$$\Big\langle \int_{t_0}^t \mathrm{d}c_s x_s \,\Big|\, \int_{t_0}^t \mathrm{d}c_s x_s \Big\rangle_0 = \int_{t_0}^t \big\langle x_s \,\big|\, \Lambda(\mathbb{1})\, x_s \big\rangle_0 \mathrm{d}s \tag{7.2.2}$$

is valid for any $x \in \mathcal{V}$.

Note that the integral on the right-hand side is a weak* integral in $\mathcal{A}_0$.

Notice also that $X_t := \int_{t_0}^t \mathrm{d}c_r x_r$ is a $\| \|_0$-continuous non-commutative martingale with respect to the filtration $(\mathcal{E}_{(-\infty,s]})_{s\geq 0}$, i.e., $E_{(-\infty,s]}X_t = X_s$ for any $t_0 \leq s \leq t$. This is easily seen for a simple process x:

$$E_{(-\infty,s]} \int_s^t \mathrm{d}c_r x_r = \int_s^t E_{(-\infty,s]} \circ E_{(-\infty,r]} \mathrm{d}c_r x_r$$
$$= \int_s^t E_{(-\infty,s]} E_{(-\infty,r]}(\mathrm{d}c_r) x_r = 0\,.$$

This equation extends to processes $x \in \mathcal{V}$ by approximation, and thus proves that $(X_t)_{t_0\leq t}$ is a martingale. The $\| \|_0$-continuity follows directly from inequality (7.2.1), which, by (7.2.2), extends to the piecewise SOT-continuous adapted processes. Moreover, X_t is locally adapted if x is locally adapted.

Since a (non)-centred additive cocycle b decomposes uniquely into its centred part $c := b - E_0 b$ and its drift $E_0 b_t = tE_0 b_1$, we define

$$\int_{t_0}^t \mathrm{d}b_s\, x_s := \int_{t_0}^t \mathrm{d}c_s x_s + \int_{t_0}^t (E_0 b_1)\, x_s \,\mathrm{d}s\,,$$

for any $x \in \mathcal{V}$. By routine calculations it is shown that

$$\Big\| \int_{t_0}^t \mathrm{d}b_s\, x_s \Big\|_0^2 \leq 2(1 + t - t_0)\|b_1\|_0^2 \int_{t_0}^t \|x_s\|_0^2 \mathrm{d}s\,.$$

Remark 7.2.2. The following argument ensures, roughly speaking, that the vector space $\mathcal{V}$ contains many processes. Let $P := (E_{(-\infty,t]})_{t\geq 0}$ be the projection from the vector space of processes onto the vector space of adapted processes in $\mathcal{E}$. For a SOT-continuous process x, the map $s \mapsto E_{(-\infty,s]}x_s$ is also SOT-continuous, since $s \mapsto E_{(-\infty,s]}$ is pointwise SOT-continuous. Consequently, a SOT-continuous process is mapped by the projection P to a SOT-continuous adapted process. Notice that these arguments do not apply to locally adapted processes and the family $(E_{[0,s]})_{s\geq 0}$ if $s \mapsto E_{[0,s]}$ fails to be pointwise SOT-continuous.

A further extension of the non-commutative Itô integral to the class of L^2-integrable adapted processes is possible, but this requires considerably more technical effort. These processes should have $t \mapsto \|x_t \xi\|^2$ locally square-integrable, for each $\xi \in \mathcal{H}_0$. The relevant tools for the development of such a theory are well-known and can be found, for example, in [Tak03a], Chapter IV. We not elaborate further this direction, because the integration class of SOT-continuous processes will be sufficient for the purpose of this paper.

7.3. **Non-commutative Itô differential equations.** We close our digression on non-commutative Itô integration with an existence and uniqueness theorem for solutions of (non-commutative) Itô differential equations (IDEs).

Let c be a centred additive cocycle in $\mathscr{C}_0(\mathcal{E}, +)$. We say that the process $x \in \mathcal{V}$ has the differential $\mathrm{d}x_t = \alpha_t \mathrm{d}t + \mathrm{d}c_t \beta_t$, if it has the form $x_t = x_0 +$

$\int_0^t \alpha_s \, \mathrm{d}s + \int_0^t \mathrm{d}c_s \beta_s$ for all $t \geq t_0$, where α and β are processes in $\mathcal{V}$. We call a function $\alpha : \mathbb{R}^+ \times \mathcal{E} \to \mathcal{E}$ *adapted*, if $\alpha(t, y) \in \mathcal{E}_{(-\infty,t]}$ for any $t \geq 0$ and $y \in \mathcal{E}_{(-\infty,t]}$. Moreover, we say that the function α is *locally adapted*, if the previous statement is satisfied with respect to $\mathcal{E}_{[0,t]}$, instead of $\mathcal{E}_{(-\infty,t]}$. Finally, we say that such a function is SOT-continuous, if $t \mapsto \alpha(t, y)$ is SOT-continuous for any $y \in \mathcal{E}$ and $y \mapsto \alpha(t, y)$ is SOT-SOT-continuous for any $t \geq 0$.

If α and β are adapted, a process $x \in \mathcal{V}$ is called a solution of the non-commutative Itô differential equation (IDE)

$$\mathrm{d}x_t = \alpha(t, x_t) \, \mathrm{d}t + \mathrm{d}c_t \beta(t, x_t) \, , \qquad x_{t_0} \in \mathcal{E}_{(-\infty,t_0]} \, , \tag{7.3.1}$$

if x satisfies the integral equation

$$x_t = x_0 + \int_{t_0}^t \alpha(t, x_s) \, \mathrm{d}s + \int_{t_0}^t \mathrm{d}c_s \beta(s, x_s) \, . \tag{7.3.2}$$

In particular, the following result ensures that the IDE, as stated in Theorem 6.4.4, has a unique solution.

Theorem 7.3.1. *Let the expected continuous GNS Bernoulli shift* $(\mathcal{E}, \mathbb{1}, S, (\mathcal{E}_I)_{I \in \mathcal{I}}; \mathcal{A}_0)$ *be given. Let* $c \in \mathscr{C}_0(\mathcal{E}, +)$ *be a centred additive cocycle and let* α, β *be adapted* SOT*-continuous functions from* $\mathbb{R}^+ \times \mathcal{E}$ *to* $\mathcal{E}$*. Furthermore, assume that, for any compact interval* $[t_0, T]$ *with* $t_0 \geq 0$ *and* $\xi \in \mathcal{H}_0$*, there exists a constant* $C_\xi \geq 0$ *such that for all* $x, y \in \mathcal{E}$ *and* $s, t \in [t_0, T]$

$$\mathrm{d}_\xi(\alpha(t, x) - \alpha(s, x)) \leq C_\xi \left(\|\xi\|^2 + \mathrm{d}_\xi(x) \right) |f_\xi(t) - f_\xi(s)| \, , \tag{7.3.3}$$

$$\mathrm{d}_\xi(\beta(t, x) - \beta(s, x)) \leq C_\xi \left(\|\xi\|^2 + \mathrm{d}_\xi(x) \right) |g_\xi(t) - g_\xi(s)| \, , \tag{7.3.4}$$

$$\mathrm{d}_\xi(\alpha(t, x) - \alpha(t, y)) \leq C_\xi \, \mathrm{d}_\xi(x - y) \, , \tag{7.3.5}$$

$$\mathrm{d}_\xi(\beta(t, x) - \beta(t, y)) \leq C_\xi \, \mathrm{d}_\xi(x - y) \, . \tag{7.3.6}$$

Here, f_ξ *and* g_ξ *are continuous, real-valued functions on* $\mathbb{R}^+$.

If all these assumptions are satisfied, then there is a unique process x *in* $\mathcal{V}$ *that satisfies the IDE*

$$\mathrm{d}x_t = \alpha(t, x_t) \, \mathrm{d}t + \mathrm{d}c_t \, \beta(t, x_t) \, , \qquad x_{t_0} \in \mathcal{E}_{(-\infty,t_0]} \, . \tag{7.3.7}$$

Moreover, this solution x *is* $\| \, \|_0$*-continuous. In addition, if* $x_{t_0} \in \mathcal{E}_{[0,t_0]}$ *and if* α, β *are locally adapted functions, then* x *is locally adapted too.*

Remark 7.3.2. A similar theorem is valid for IDEs which contain both left and right non-commutative Itô integrals.

Proof. By the Picard iteration, we construct on the interval $[t_0, t_1]$ a unique SOT-continuous solution x of the IDE (7.3.7). In particular, we produce from the initial data (t_0, x_{t_0}) the new data (t_1, x_{t_1}). The latter ones will serve as initial data for the Picard iteration to produce a solution on the interval $[t_1, t_2]$ (with $t_2 \leq t_1 + 1$). By this iterative method, we cover the interval $[t_0, T]$ by finitely many intervals $[t_i, t_{i+1}]$, since the iteration procedure will show that we can choose all intervals $[t_i, t_{i+1}]$ to be of the same length $0 < \Delta t \leq 1$.

Thus, we produce successively a unique solution for each interval $[t_i, t_{i+1}]$, and consequently a solution for any time $t \geq 0$.

We already know from (7.3.3) and (7.3.5) that, for a SOT-continuous process x, the function $t \mapsto \alpha(t, x_t)$ is SOT-continuous and consequently $\| \|_0$-bounded on any compact interval. The same is true for the function $t \mapsto \beta(t, x_t)$.

We choose some fixed Δt with $0 < \Delta t \leq 1$ and start the iteration on the interval $[t_0, t_1]$ with $t_1 := t_0 + \Delta t$. Let $x_t^0 := x_{t_0}$ for all $t \in [t_0, t_1]$. The functions $t \mapsto \alpha(t, x_t^0)$ and $t \mapsto \beta(t, x_t^0)$ are SOT-continuous and (locally) adapted if α and β are (locally) adapted and if x_{t_0} is an element in $L^2(\mathcal{A}_{(-\infty, t_0]}, E_0)$ (resp. $L^2(\mathcal{A}_{[0,t_0]}, E_0)$). Thus the nth iteration step

$$x_t^n := x_{t_0} + \int_{t_0}^t \mathrm{d}s\, \alpha(s, x_s^{n-1}) + \int_{t_0}^t \mathrm{d}c_s \beta(s, x_s^{n-1})$$

is well-defined by induction. Notice that x^n is SOT-continuous and (locally) adapted. We let $M := \sup_{s \in [t_0, t_1]} \|x_s^1 - x_s^0\|_0$. Moreover, we define $q_\xi := C_\xi((\Delta t)^{1/2} + \|\Lambda(\mathbb{1})\|^{1/2})$. From (7.3.5) and (7.3.6) we find for any $t \in [t_0, t_1]$ and normalized $\xi \in \mathcal{H}_0$ the estimate

$$\begin{aligned}
\mathrm{d}_\xi(x_t^{n+1} - x_t^n) &\leq \int_{t_0}^t \mathrm{d}_\xi(\alpha(s, x_s^n) - \alpha(s, x_s^{n-1}))\,\mathrm{d}s \\
&\quad + \|\Lambda(\mathbb{1})\|^{1/2} \left[\int_{t_0}^t \mathrm{d}_\xi(\beta(s, x_s^n) - \beta(s, x_s^{n-1}))^2 \mathrm{d}s \right]^{1/2} \\
&\leq q_\xi \Big[\int_{t_0}^t \mathrm{d}_\xi(x_s^n - x_s^{n-1})^2 \mathrm{d}s \Big]^{1/2} \leq M\, q_\xi^n \frac{(\Delta t)^{n/2}}{\sqrt{n!}}.
\end{aligned}$$

Thus, for any $n > q_\xi^2$,

$$\begin{aligned}
\max_{t \in [t_0, t_1]} \mathrm{d}_\xi(x_t^{n+m} - x_t^n) &\leq \sum_{k=0}^{m-1} \max_{t \in [t_0, t_1]} \mathrm{d}_\xi(x_t^{n+k+1} - x_t^{n+k}) \\
&\leq M\, q_\xi^n \frac{(\Delta t)^{n/2}}{\sqrt{n!}} \sum_{k=0}^{m-1} \sqrt{\frac{n!}{(n+k)!}} (\Delta t)^{k/2} q_\xi^k \\
&\leq M\, q_\xi^n \frac{(\Delta t)^{n/2}}{\sqrt{n!}} \sum_{k=0}^{\infty} \Big(\frac{q_\xi}{\sqrt{n}} \Big)^k (\Delta t)^{k/2} \xrightarrow{n \to \infty} 0\,.
\end{aligned}$$

We conclude that the limit $x_t := \text{SOT-}\lim_{n\to\infty} x_t^n$ exists uniformly on $[t_0, t_1]$ and defines a SOT-continuous, (strongly) adapted process on this interval. Next, we obtain from (7.3.5) and (7.3.6) that $\int_{t_0}^t \alpha(s, x_s)\mathrm{d}s = \lim_{n\to\infty} \int_{t_0}^t \alpha(s, x_s^n)\mathrm{d}s$ and $\int_{t_0}^t \mathrm{d}c_s \beta(s, x_s) = \lim_{n\to\infty} \int_{t_0}^t \mathrm{d}c_s \beta(s, x_s^n)$ in the strong operator topology. Now one concludes with routine arguments that x solves the IDE on $[t_0, t_1]$. Since q_ξ depends only on Δt, this procedure applies iteratively to all right next neighbor intervals with the same length Δt. The $\| \|_0$-continuity of the solution x follows from the fact that the integrals, as they appear in the integral equation for x, contain SOT-continuous integrands and thus define $\|\,\|_0$-continuous functions. Finally, uniqueness of the solution follows with the help of the estimates (7.3.3) - (7.3.6) in the usual way. □

Remark 7.3.3. A more complete treatment of non-commutative Itô integration, compatible with the setting of continuous GNS Bernoulli shifts is contained in [Hel01]. If the shift is scalar-expected, then our approach reduces to non-commutative Itô integration in the GNS Hilbert space of the shift, as is already described in [Pri89]. If the state of the $\mathcal{A}_0$-expected shift is tracial, then the continuous GNS Bernoulli shifts are realized as subspaces of non-commutative L^2-spaces in [Kös00]. This approach produces similar results on non-commutative Itô integration. Notice also that one-sided integrands in [BS98] fall into our setting.

8. Non-commutative exponentials and logarithms

This section is devoted to the development of non-commutative exponentials (Subsection 8.1) and non-commutative logarithms (Subsection 8.2). The constructions presented generalize beyond the frame of non-commutative exponentials Exp of additive cocycles and non-commutative logarithms Ln of unital cocycles, but here we shall just focus on the development of sufficient tools to complete the proof of Theorem 6.4.1 and Theorem 6.4.4. This will be done in Subsection 8.3, where we shall show that the mappings Exp and Ln are injective and thus mutually inverse.

We assume throughout this section that a fixed expected continuous Bernoulli shift $\big(\mathcal{A}, \psi, S, (\mathcal{A}_I)_{I\in\mathcal{I}}; \mathcal{A}_0\big)$ is given, together with its GNS representation $(\mathcal{E}, \mathbb{1}, S, (\mathcal{E}_I)_{I\in\mathcal{I}}; \mathcal{A}_0)$ are given.

8.1. Non-commutative exponentials of additive cocycles.

Non-commutative exponentials will be obtained as solutions of non-commutative Itô differential equations (IDEs).

Theorem 8.1.1. *Let the centred additive cocycle $c \in \mathscr{C}_0(\mathcal{E}, +)$ and the operator $K \in \mathcal{A}_0$ be given. Then the unique solution u of the IDE*

$$u_t = \mathbb{1} + \int_0^t \mathrm{d}c_s u_s + \int_0^t \mathrm{d}t\, K u_s \tag{8.1.1}$$

is a locally adapted $\|\,\|_0$-continuous process in $\mathcal{E}$ that satisfies the cocycle identity $u_{s+t} = (S_t u_s)u_t$ $(s, t \geq 0)$. Moreover, it enjoys the following additional properties:

(i) *The compression $A := (E_0 u_t)_{t\geq 0}$ defines a $\|\,\|$-continuous semigroup with generator K.*

(ii) *The compression $R := (\langle u_t \,|\, \cdot\, u_t\rangle_0)_{t\geq 0}$ defines a $\|\,\|$-continuous semigroup of completely positive mappings on $\mathcal{A}_0$. It has the Christensen-Evans generator*

$$\mathcal{L}(a) = \Lambda(a) + K^* a + aK,$$

where $\Lambda = \langle c_1 \,|\, \cdot\, c_1 \rangle_0$ *is the covariance of* c.

In particular, the following are equivalent:

(a) u *is a unital cocycle;*

(b) $R_t(\mathbb{1}) = \mathbb{1}$ *for any* $t \geq 0$, *or equivalently* $\mathcal{L}(\mathbb{1}) = 0$;

(c) $|c_t|_0^2 + (K + K^*)t = 0$.

If the conditions (a) to (c) are satisfied, then the semigroup R *is contractive.*

Let b denote the additive cocycle defined by $b_t = c_t + Kt$, $t \geq 0$.

Definition 8.1.2. *The solution* u *of the IDE* (8.1.1) *is called the* exponential of the additive cocycle b *and is denoted* $\operatorname{Exp}(b)$.

In stochastic analysis the additive cocycle b falls into the class of semi-martingales and a solution of (8.1.1) is called an exponential semi-martingale.

Proof of Theorem 8.1.1. Putting $\alpha(t, u_t) = Ku_t$ and $\beta(t, u_t) = u_t$, the functions α and β are locally adapted and evidently satisfy the Lipschitz conditions of Theorem 7.3.1. Moreover, the initial condition is locally adapted, i.e., $u_0 = \mathbb{1} \in \mathcal{A}_0$. Thus the IDE (8.1.1) has a unique $\|\,\|_0$-continuous locally adapted solution $u \subset \mathcal{E}$.

In the following we verify the cocycle property of the solution u. Let $(Z_n[a,b])_{n\in\mathbb{N}}$ be a sequence of partitions of the interval $[a,b]$ with mesh size tending to zero. In a first step we determine for an arbitrary $w \in \mathcal{E}_{[0,t]}$

$$
\begin{aligned}
\Big(S_t \int_0^s \mathrm{d}c_r u_r\Big) w &= \lim_{n\to\infty} \sum_{r_i \in Z_n[0,s]} (S_t(c_{r_{i+1}} - c_{r_i}))(S_t u_{r_i}) w \\
&= \lim_{n\to\infty} \sum_{r_i \in Z_n[0,s]} (c_{t+r_{i+1}} - c_{t+r_i})(S_t u_{t+r_i-t}) w \\
&= \lim_{n\to\infty} \sum_{t_i \in Z_n[t,s+t]} (c_{t_{i+1}} - c_{t_i})\,(S_t u_{t_i-t}) w \\
&= \int_t^{t+s} \mathrm{d}c_r (S_t u_{r-t}) w\,,
\end{aligned}
$$

where the limits are meant in the SOT-sense on $\mathcal{E}$. Throughout these calculations we have used the continuity properties of the product of $\mathcal{A}_0$-independent elements in $\mathcal{E}$ (Proposition 5.3.1). Moreover, we used triple products (cf. 5.4) of the three $\mathcal{A}_0$-independent, increasingly ordered factors w, $S_t u_{r_i}$ and $c_{t+r_{i+1}} - c_{t+r_i}$. (Here, 'increasingly ordered' means $[0,t] \leq [t, t+r_i] \leq [t+r_i, t+r_{i+1}]$.) We define for a fixed $t > 0$

$$
v_r := \begin{cases} u_r\,, & 0 \leq r \leq t, \\ S_t(u_{r-t}) u_t\,, & t < r. \end{cases}
$$

The continuity of the product of $\mathcal{A}_0$-independent elements implies the SOT-continuity of $r \mapsto v_r$ and hence the integrability. Clearly, v_r satisfies the IDE

(8.1.1), whenever $0 \le r \le t$. Hence we conclude

$$\begin{aligned}
v_{t+s} &= S_t\Big(\mathbb{1} + \int_0^s \mathrm{d}c_r u_r + \int_0^s \mathrm{d}r K u_r\Big) u_t \\
&= u_t + \int_t^{t+s} \mathrm{d}c_r (S_t u_{r-t}) u_t + \int_0^s \mathrm{d}r K (S_t u_r) u_t \\
&= \mathbb{1} + \int_0^t \mathrm{d}c_r u_r + \int_t^{t+s} \mathrm{d}c_r (S_t u_{r-t}) u_t + \int_0^t \mathrm{d}r K u_r + \int_t^{t+s} \mathrm{d}r K (S_t u_{r-t}) u_t \\
&= \mathbb{1} + \int_0^{t+s} \mathrm{d}c_r v_r + \int_0^{t+s} \mathrm{d}r K v_r \,.
\end{aligned}$$

This calculation shows that v satisfies (8.1.1). Now, uniqueness of the solution implies that $u_r = v_r$ for all $r \ge 0$ and consequently the cocycle property $u_{t+s} = v_{t+s} = (S_t u_s) u_t$ for the solution u.

(i) The compression $E_0 u$ of the solution u defines a semigroup. Indeed, we apply E_0 on both sides of (8.1.1) and obtain

$$E_0 u_t = \mathbb{1} + \int_0^t \mathrm{d}s\, K E_0 u_s \,.$$

This integral equation has the unique solution $t \mapsto E_0 u_t = \mathrm{e}^{Kt}$.

(ii) R_t is completely positive by construction. By the cocycle property, it is immediately seen that $R_{s+t}(a) = R_s(R_t(a))$ for any $s, t \ge 0$ and $a \in \mathcal{A}_0$. Since u is $\|\,\|_0$-continuous and $u_0 = \mathbb{1}$, the uniform continuity of R follows from (6.2.2). Thus R has a bounded generator $\mathcal{L}$. Next, we identify the form of $\mathcal{L}$. For this purpose we rewrite R_t with the help of the IDE (8.1.1). Recall that $\Lambda = \langle c_1 \,|\, \cdot\, c_1 \rangle_0$. An elementary calculation shows that, for any $a \in \mathcal{A}_0$,

$$\begin{aligned}
R_t(a) &= \langle u_t \,|\, a u_t \rangle_0 \\
&= a + \int_0^t \langle \mathbb{1} \,|\, a K u_s \rangle_0 \mathrm{d}s + \int_0^t \langle K u_s \,|\, a \rangle_0 \mathrm{d}s + \int_0^t \langle u_s \,|\, \Lambda(a) u_s \rangle_0 \mathrm{d}s \\
&\quad + \int_0^t \Big\langle K u_s \,\Big|\, a \int_0^s \mathrm{d}c_r u_r \Big\rangle_0 \mathrm{d}s + \int_0^t \Big\langle \int_0^s \mathrm{d}c_r u_r \,\Big|\, a K u_s \Big\rangle_0 \mathrm{d}s \\
&\quad + \int_0^t \int_0^s \langle K u_s \,|\, a K u_r \rangle_0 \mathrm{d}r\, \mathrm{d}s + \int_0^t \int_0^r \langle K u_s \,|\, a K u_r \rangle_0 \mathrm{d}s\, \mathrm{d}r \,.
\end{aligned}$$

Notice that, due to the $\|\,\|_0$-continuity of u, all integrals of the form $\int \langle \cdot \,|\, \cdot \rangle_0 \mathrm{d}s$ are Bochner integrals on $\mathcal{A}_0$. Consequently, we are allowed to differentiate separately each term and obtain, for any $a \in \mathcal{A}_0$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} R_t(a) &= \langle \mathbb{1} \,|\, a K u_t \rangle_0 + \langle K u_t \,|\, a \rangle_0 + \langle u_t \,|\, \Lambda(a) u_t \rangle_0 \\
&\quad + \Big\langle K u_t \,\Big|\, a \int_0^t \mathrm{d}c_r u_r \Big\rangle_0 + \Big\langle \int_0^t \mathrm{d}c_r u_r \,\Big|\, a K u_t \Big\rangle_0
\end{aligned}$$

$$+\left\langle Ku_t \,\middle|\, a\int_0^t \mathrm{d}r\, Ku_r\right\rangle_0 + \left\langle \int_0^t \mathrm{d}r\, Ku_r \,\middle|\, aKu_t\right\rangle_0$$

– we again use the IDE (8.1.1) –

$$\begin{aligned} &= \langle \mathbb{1} \,|\, aKu_t\rangle_0 + \langle Ku_t \,|\, a\rangle_0 + R_t(\Lambda(a)) \\ &\quad + \langle Ku_t \,|\, a(u_t - \mathbb{1})\rangle_0 + \langle u_t - \mathbb{1} \,|\, aKu_t\rangle_0 \\ &= R_t(\Lambda(a)) + R_t(K^*a + aK)\,. \end{aligned}$$

Thus, the generator $\mathcal{L}$ is identified as $\mathcal{L}(a) = \Lambda(a) + (K^*a + aK)$, where $a \in \mathcal{A}_0$.

Finally, the equivalence of (a) to (c) and the contractivity of R is evident from the form of the generator $\mathcal{L}$ and the definition of a unital cocycle. □

8.2. **Non-commutative logarithms of unital cocycles.** We construct an additive cocycle in $\mathcal{E}$ from a $\|\,\|_0$-continuous unital cocycle. Motivated by Corollary 6.4.5 and probability theory, we shall call the constructed additive cocycle the non-commutative logarithm of the unital cocycle. Our main result of this subsection is stated in Theorem 8.2.2.

Notation 8.2.1. $\mathcal{Z}(t)$ will denote a net of partitions $Z := \{t_i \geq 0 \,|\, 0 = t_0 < t_1 < \ldots < t_{n_Z} = t\}$ of the interval $[0,t]$. The set of partitions in $\mathcal{Z}(t)$ is partially ordered by inclusion such that their mesh $|Z| := \max\{|t_{i+1} - t_i| \,|\, i = 0,\ldots,n_Z - 1\}$ tends to zero.

Theorem 8.2.2. *A $\|\,\|_0$-continuous unital cocycle $u \subset \mathcal{E}$ with associated contractive semigroup $A_t := E_0 u_t = \mathrm{e}^{Kt}$ defines via*

$$b_t := \|\,\|_0\text{-}\lim_{Z\in\mathcal{Z}(t)} \sum_{i=0}^{n_Z-1} S_{t_i}(u_{t_{i+1}-t_i} - \mathbb{1}), \tag{8.2.1}$$

$$c_t := \|\,\|_0\text{-}\lim_{Z\in\mathcal{Z}(t)} \sum_{i=0}^{n_Z-1} S_{t_i}u_{t_{i+1}-t_i} - A_{t_{i+1}-t_i} \tag{8.2.2}$$

two additive cocycles b, c in $\mathscr{C}_0(\mathcal{E},+)$ satisfying $b_t = c_t + Kt$ and $K = E_0 b_1$. Moreover, the additive cocycle b is in $\mathscr{C}_0^0(\mathcal{E},+)$, in other words, it satisfies the structure equation

$$|b_t - E_0 b_t|_0^2 + t(E_0 b_1)^* + tE_0 b_1 = 0$$

and the pair (c, K), containing the centred additive cocycle c and the drift K, satisfies the structure equation

$$|c_t|_0^2 + t(K^* + K) = 0\,.$$

Definition 8.2.3. *The additive cocycle b (resp. c), constructed in Theorem 8.2.2, is called the* (centred) non-commutative logarithm of the unital cocycle *u and is denoted* $\mathrm{Ln}(u)$ *(resp.* $\mathrm{Ln}_0(u)$*).*

For brevity, we also say that b is the logarithm and that c is the centred logarithm of u. Notice that for a trivial unital cocycle u, i.e. $u_t = \exp(itH)$ with $H = H^* \in \mathcal{A}_0$, one finds $\mathrm{Ln}(u_t) = itH$ and $\mathrm{Ln}_0(u_t) = 0$. Let us further motivate this definition:

Corollary 8.2.4. *Let $u, v \subset \mathcal{A}$ be two $\|\,\|_0$-continuous unital cocycles. If u and v are $\mathcal{A}_0$-independent and satisfy the commutation relation $u_t(S_t v_s) = (S_t v_s)u_t$ for any $s, t \geq 0$, then $uv := (u_t v_t)_{t\geq 0}$ is again a unital $\|\,\|_0$-continuous cocycle and*

$$\begin{aligned}\mathrm{Ln}(uv) &= \mathrm{Ln}(u) + \mathrm{Ln}(v) = \mathrm{Ln}(vu)\,,\\ \mathrm{Ln}_0(uv) &= \mathrm{Ln}_0(u) + \mathrm{Ln}_0(v) = \mathrm{Ln}_0(vu)\,.\end{aligned}$$

Proof. The $\mathcal{A}_0$-independence of u and v ensures that the product $u_t v_t$ is well-defined in $\mathcal{E}$ for any $t \geq 0$. The commutation property for u and v guarantees that uv is a cocycle. The $\|\,\|_0$-continuity of $t \mapsto u_t v_t$ is also easily checked. In view of the identity

$$\begin{aligned}S_{t_i}(u_{t_{i+1}-t_i} v_{t_{i+1}-t_i} - u_0 v_0) = {}& S_{t_i}(u_{t_{i+1}-t_i} - u_0)v_0 + u_0 S_{t_i}(v_{t_{i+1}-t_i} - v_0)\\ &+ S_{t_i}((u_{t_{i+1}-t_i} - u_0)(v_{t_{i+1}-t_i} - v_0)),\end{aligned}$$

it is sufficient to establish that

$$\|\,\|_0\text{-}\lim_{Z\in\mathcal{Z}(t)} \sum_{i=0}^{n_Z-1} S_{t_i}((u_{t_{i+1}-t_i} - u_0)(v_{t_{i+1}-t_i} - v_0)) = 0.$$

Indeed this follows from estimates similar to those that appear in the proof of Theorem 8.2.2. □

We prepare the proof of Theorem 8.2.2 with a technical result.

Lemma 8.2.5. *For $r, s, t \geq 0$ and $a \in \mathcal{A}_0$ the following identities hold:*

(i) $\langle S_r u_s \,|\, a u_t\rangle_0 = R_s(aA_{t-r-s})\,A_r\,, \quad 0 \leq s \leq t-r,$

(ii) $\langle S_r u_s \,|\, a u_t\rangle_0 = R_{t-r}(A^*_{s-t+r}a)\,A_r\,, \quad 0 \leq t-r \leq s.$

Proof. For $t - r \geq 0$ one calculates

$$\begin{aligned}\langle S_r u_s \,|\, a u_t\rangle_0 &= \langle S_r u_s \,|(S_r a u_{t-r})\,u_r\rangle_0 = \langle \mathbb{1} \,|\langle u_s \,|\, a u_{t-r}\rangle_0\, u_r\rangle_0\\ &= \langle u_s \,|\, a u_{t-r}\rangle_0\, A_r\,.\end{aligned}$$

In case (i) with $0 \leq s \leq t - r$ we conclude further

$$\begin{aligned}\langle S_r u_s \,|\, a u_t\rangle_0 &= \langle u_s \,|\, a(S_s u_{t-r-s})\,u_s\rangle_0\, A_r\\ &= \langle u_s \,|\, a\langle \mathbb{1} \,|\, u_{t-r-s}\rangle_0\, u_s\rangle_0\, A_r = R_s(aA_{t-r-s})\,A_r\,.\end{aligned}$$

In the case (ii) with $0 \leq t - r \leq s$ one finds

$$\begin{aligned}\langle S_r u_s \,|\, a u_t\rangle_0 &= \langle (S_{t-r} u_{s-(t-r)})\,u_{t-r} \,|\, a u_{t-r}\rangle_0\, A_r\\ &= \langle u_{t-r} \,|\, A^*_{s-t+r} a u_{t-r}\rangle_0\, A_r = R_{t-r}(A^*_{s-t+r}a)\,A_r\,.\end{aligned}$$

□

Proof of Theorem 8.2.2. We let

$$c_Z(t) := \sum_{i=0}^{n_Z-1} S_{t_i} u_{t_{i+1}-t_i} - A_{t_{i+1}-t_i} \in \mathcal{E}_{[0,t]} \tag{8.2.3}$$

and will prove that $(c_Z(t))_{Z\in\mathcal{Z}(t)}$ is a $\| \|_0$-Cauchy net in $\mathcal{E}$. This convergence implies immediately that

$$b_Z(t) := \sum_{i=0}^{n_Z-1} S_{t_i} u_{t_{i+1}-t_i} - \mathbb{1} \in \mathcal{E}_{[0,t]} \tag{8.2.4}$$

converges to b_t, since the difference $b_Z(t) - c_Z(t)$ converges evidently to Kt. Moreover the equivalence of the structure equations of b and c is ensured, since the decomposition of an additive cocycle into its centred part and its drift is unique. Thus, we concentrate in the following on the convergence of $c_Z(t)$ to c_t, and the cocycle property of c and its structure equation.

Let us denote the common refinement of $Z, W \in \mathcal{Z}(t)$ by ZW. Since

$$\|c_Z(t) - c_W(t)\|_0 \le \|c_Z(t) - c_{ZW}(t)\|_0 + \|c_W(t) - c_{ZW}(t)\|_0, \tag{8.2.5}$$

it is sufficient to investigate $\|c_Z(t) - c_{ZW}(t)\|_0^2$. This expression is controlled by $\langle c_Z(t) \,|\, c_{ZW}(t)\rangle_0$. For the refinement ZW of Z we let $t_i + s_{i,j}$, $j = 1, \ldots, n^i - 1$, be the additional points in the interval $[t_i, t_{i+1}]$. Moreover, we put $s_{i,0} := 0$ and $s_{i,n^i} := t_{i+1} - t_i$. Setting $v_s := u_s - A_s$ for $s \in [0,t]$, one calculates

$$\begin{aligned}
\langle c_Z(t) \,|\, c_{ZW}(t)\rangle_0 &= \sum_{i=0}^{n_Z-1} \sum_{k=0}^{n_Z-1} \sum_{j=0}^{n^k-1} \langle S_{t_i} v_{t_{i+1}-t_i} \,|\, S_{t_k+s_{k,j}} v_{s_{k,j+1}-s_{k,j}}\rangle_0 \\
&= \sum_{i=0}^{n_Z-1} \sum_{j=0}^{n^i-1} \langle u_{t_{i+1}-t_i} - A_{t_{i+1}-t_i} \,|\, S_{s_{i,j}} u_{s_{i,j+1}-s_{i,j}} - A_{s_{i,j+1}-s_{i,j}}\rangle_0 \\
&= \sum_{i=0}^{n_Z-1} \sum_{j=0}^{n^i-1} \langle u_{t_{i+1}-t_i} \,|\, S_{s_{i,j}} u_{s_{i,j+1}-s_{i,j}}\rangle_0 - A^*_{t_{i+1}-t_i} A_{s_{i,j+1}-s_{i,j}} \\
&= \sum_{i=0}^{n_Z-1} \sum_{j=0}^{n^i-1} A^*_{s_{i,j}} R_{s_{i,j+1}-s_{i,j}}(A^*_{t_{i+1}-t_i-s_{i,j+1}}) - A^*_{t_{i+1}-t_i} A_{s_{i,j+1}-s_{i,j}}
\end{aligned}$$

– since $s_{i,j+1} - s_{i,j} \le t_{i+1} - t_i - s_{i,j}$ the application of Lemma 8.2.5 (i) is permitted –

$$\begin{aligned}
&= \sum_{i=0}^{n_Z-1} \sum_{j=0}^{n^i-1} A^*_{s_{i,j}} (R_{s_{i,j+1}-s_{i,j}} - \mathrm{id})(A^*_{t_{i+1}-t_i-s_{i,j+1}} - \mathbb{1}) \\
&+ \sum_{i=0}^{n_Z-1} \sum_{j=0}^{n^i-1} A^*_{(t_{i+1}-t_i)-(s_{i,j+1}-s_{i,j})} (\mathbb{1} - A^*_{s_{i,j+1}-s_{i,j}} A_{s_{i,j+1}-s_{i,j}}) \,.
\end{aligned}$$

We investigate the two double sums separately.

From the norm continuity of the semigroups R and A we conclude that there exists an upper bound $M > 0$ such that $\|R_t - \mathrm{id}\| \leq Mt$, $\|A_t - \mathbb{1}\| \leq Mt$ and $\|A_t^* A_t - \mathbb{1}\| \leq Mt$. With this bound we obtain the following estimate for the first summand:

$$\begin{aligned}
&\Big\| \sum_{i=0}^{n_Z-1} \sum_{j=0}^{n^i-1} A^*_{s_{i,j}} (R_{s_{i,j+1}-s_{i,j}} - \mathrm{id})(A^*_{t_{i+1}-t_i-s_{i,j+1}} - \mathbb{1}) \Big\| \\
&\qquad \leq M^2 \sum_{i=0}^{n_Z-1} \sum_{j=0}^{n^i-1} (s_{i,j+1} - s_{i,j})(t_{i+1} - t_i - s_{i,j+1}) \\
&\qquad \leq M^2 \sum_{i=0}^{n_Z-1} \sum_{j=0}^{n^i-1} (s_{i,j+1} - s_{i,j})(t_{i+1} - t_i) \\
&\qquad = M^2 \sum_{i=0}^{n_Z-1} (t_{i+1} - t_i)^2 \leq M^2 \max\{|Z|, |W|\} t\,.
\end{aligned}$$

The second summand converges to λt with $\lambda := -(K + K^*)$:

$$\begin{aligned}
&\Big\| \sum_{i=0}^{n_Z-1} \sum_{j=0}^{n^i-1} A^*_{(t_{i+1}-t_i)-(s_{i,j+1}-s_{i,j})} (\mathbb{1} - A^*_{s_{i,j+1}-s_{i,j}} A_{s_{i,j+1}-s_{i,j}}) - \lambda t \Big\| \\
&\quad \leq \sum_{i=0}^{n_Z-1} \sum_{j=0}^{n^i-1} \|A^*_{(t_{i+1}-t_i)-(s_{i,j+1}-s_{i,j})} - \mathbb{1}\| \|\mathbb{1} - A^*_{s_{i,j+1}-s_{i,j}} A_{s_{i,j+1}-s_{i,j}}\| \\
&\qquad + \sum_{i=0}^{n_Z-1} \sum_{j=0}^{n^i-1} \Big\| \frac{\mathbb{1} - A^*_{s_{i,j+1}-s_{i,j}} A_{s_{i,j+1}-s_{i,j}}}{s_{i,j+1} - s_{i,j}} - \lambda \Big\| (s_{i,j+1} - s_{i,j}) \\
&\quad \leq \sum_{i=0}^{n_Z-1} \sum_{j=0}^{n^i-1} M^2 [(t_{i+1} - t_i) - (s_{i,j+1} - s_{i,j})](s_{i,j+1} - s_{i,j}) \\
&\qquad + \sum_{i=0}^{n_Z-1} \sum_{j=0}^{n^i-1} \varepsilon (s_{i,j+1} - s_{i,j}) \\
&\quad \leq \sum_{i=0}^{n_Z-1} \sum_{j=0}^{n^i-1} M^2 (t_{i+1} - t_i)(s_{i,j+1} - s_{i,j}) + \varepsilon t \\
&\quad = \sum_{i=0}^{n_Z-1} M^2 (t_{i+1} - t_i)^2 + \varepsilon t \\
&\quad \leq (M^2 |Z| + \varepsilon) t \leq (M^2 \max\{|Z|, |W|\} + \varepsilon) t\,.
\end{aligned}$$

During this calculation we used the fact that $\left\|\frac{\mathbb{1}-A_\delta^* A_\delta}{\delta} - \lambda\right\| \le \varepsilon$ when $0 < \delta \le \delta_\varepsilon$, for an appropriate $\delta_\varepsilon > 0$ such that $\max\{|Z|, |W|\} < \delta_\varepsilon$. Consequently we obtain, for any partitions $Z, W \in \mathcal{Z}(t)$ with $\max\{|Z|, |W|\} < \min\{\delta_\varepsilon, \varepsilon\}$, the estimates

$$\left\|\langle c_Z(t) \,|\, c_{ZW}(t)\rangle_0 - \lambda t\right\| \le \varepsilon(M^2+1)t$$

and

$$\|c_Z(t) - c_{ZW}(t)\|_0^2 \le 4\varepsilon(M^2+1)t\,.$$

The same line of arguments gives a similar estimate for $\|c_W(t) - c_{ZW}(t)\|_0^2$. According to the inequality (8.2.5), we conclude that $(c_Z(t))_{Z\in\mathcal{Z}(t)}$ is a Cauchy net with limit $c_t := \|\,\|_0\text{-}\lim_{Z\in\mathcal{Z}(t)} c_Z(t) \in \mathcal{E}_{[0,t]}$. The independence of the limit c_t from the net used also follows using this inequality.

It remains only to prove the cocycle property of $c := (c_t)_{t\ge 0}$. Given the nets $\mathcal{Z}(s)$ and $\mathcal{Z}(t)$ of partitions of the intervals $[0,s]$ resp. $[0,t]$, we define the net $\mathcal{Z}(s+t)$ of partitions associated to the interval $[0, s+t]$ in the following manner. For $Z_s := \{s_i \ge 0 \,|\, 0 = s_0 < s_1 < \ldots < s_{n_s} = s\} \in \mathcal{Z}(s)$ let $t + Z_s$ denote the partition $\{t + s_i \,|\, 0 = s_0 < s_1 < \ldots < s_{n_s} = s\}$ of the interval $[t, t+s]$. Now we define for any pair (Z_t, Z_s) of partitions $Z_t \in \mathcal{Z}(t)$ and $Z_s \in \mathcal{Z}(s)$ an element Z of the net $\mathcal{Z}(t+s)$ by $Z := Z_t \cup (t + Z_s)$. Obviously, the mesh size of the partitions in this net tends to zero. Now note that

$$\begin{aligned} c_Z(t+s) &= \sum_{i=0}^{n_t-1} S_{t_i}(v_{t_{i+1}-t_i}) + \sum_{i=0}^{n_s-1} S_{t+s_i}(v_{s_{i+1}-s_i}) \\ &= c_{Z_t}(t) + S_t c_{Z_s}(s)\,. \end{aligned}$$

From the net convergence of the left-hand side of this equality and the convergence of each summand of the right-hand side to c_{t+s}, c_t resp. $S_t c_s$, we obtain the cocycle identity $c_{t+s} = c_t + S_t c_s$.

In Corollary 6.4.6 we have already identified the Christensen-Evans generator $\mathcal{L}$ of R as

$$\mathcal{L}(a) = \Lambda(a) + K^* a + aK\,.$$

Since $\mathcal{L}(\mathbb{1}) = 0$ and $|c_t|_0^2 = t|c_1|_0^2$ by (6.3.2), the structure equation follows immediately from

$$\Lambda(\mathbb{1}) = |c_1|_0^2 = -(K + K^*)\,. \qquad \square$$

We close this section with a technical result which will be needed in Section 8.3 for finishing the proof of Theorem 6.4.4.

Lemma 8.2.6. *Let u be a $\|\,\|_0$-continuous unital cocycle and let $c := \mathrm{Ln}_0(u)$ the centred logarithm of u. Setting $A_t := E_0 u_t$ and $\Lambda := \langle c_1 \,|\, \cdot\, c_1\rangle_0$, it follows that*

$$\langle c_t \,|\, a u_t\rangle_0 = \int_0^t \Lambda(a A_{t-s}) A_s \,\mathrm{d}s\,, \qquad a \in \mathcal{A}_0\,. \tag{8.2.6}$$

Proof. Recalling Lemma 8.2.5 (i) we see that $\langle S_r u_s \,|\, a u_t \rangle_0 = R_s(aA_{t-r-s})A_r$, whenever $0 \le s \le t-r$ and $a \in \mathcal{A}_0$. Next we choose for $[0,t]$ the equidistant partition $Z_n := \{i\delta_n \,|\, i = 0, \ldots, 2^n\} \in \mathcal{Z}(t)$, where $\delta_n := 2^{-n}t$. Since the approximants c_{Z_n} in (8.2.2) converge to c_t in the $\|\,\|_0$-topology, we obtain (in the norm topology on $\mathcal{A}_0$) that

$$\begin{aligned}
\langle c_t \,|\, a u_t \rangle_0 &= \lim_{n\to\infty} \sum_{i=0}^{2^n-1} \langle S_{i\delta_n} u_{\delta_n} - A_{\delta_n} \,|\, a u_t \rangle_0 \\
&= \lim_{n\to\infty} \sum_{i=0}^{2^n-1} \left[R_{\delta_n}(aA_{t-(i+1)\delta_n})A_{i\delta_n} - A^*_{\delta_n} a A_{\delta_n} \right] \\
&= \lim_{n\to\infty} \sum_{i=0}^{2^n-1} \left[\frac{R_{\delta_n} - id}{\delta_n} - \mathcal{L} \right] (aA_{t-(i+1)\delta_n})A_{i\delta_n}\delta_n \\
&\quad + \lim_{n\to\infty} \sum_{i=0}^{2^n-1} \mathcal{L}(aA_{t-(i+1)\delta_n})A_{i\delta_n}\delta_n \\
&\quad + \lim_{n\to\infty} t \left[a\frac{A_{t-\delta_n} - A_t}{\delta_n} + \frac{\mathbb{1} - A^*_{\delta_n}}{\delta_n} a A_t \right] \\
&= \int_0^t \mathcal{L}(aA_{t-s})A_s \,\mathrm{d}s - t(aK + K^*a)A_t
\end{aligned}$$

– here we use the form of the generator $\mathcal{L}$ from Corollary 6.4.6 –

$$\begin{aligned}
&= \int_0^t \Lambda(aA_{t-s})A_s \,\mathrm{d}s + tK^*aA_t \\
&\quad + \int_0^t aA_{t-s}KA_s \,\mathrm{d}s - t(aK + K^*a)A_t \\
&= \int_0^t \Lambda(aA_{t-s})A_s \,\mathrm{d}s \,.
\end{aligned}$$

□

8.3. **Proof of the correspondence.** In this section we complete the proof of Theorem 6.4.1 and Theorem 6.4.4. We shall show that the mappings Ln and Exp, as introduced in Section 8.2 resp. Section 8.1, are injective. This proves that Ln and Exp are mutually inverse. Thus we have completed the proof of Theorem 6.4.1. Notice that the injectivity of both mappings will also establish the bijectivity of the correspondence in Theorem 6.4.4. We divide the proof into several intermediate results.

Let the $\mathcal{A}_0$-expected continuous GNS Bernoulli shift $(\mathcal{E}, \mathbb{1}, S, (\mathcal{E}_I)_{I\in\mathcal{I}}; \mathcal{A}_0)$ be given.

Proposition 8.3.1. *For any additive cocycle b in $\mathscr{C}_0^0(\mathcal{E}, +)$,*

$$\mathrm{Ln}(\mathrm{Exp}(b)) = b\,.$$

Consequently, the mapping $\mathrm{Exp}\colon \mathscr{C}^0_0(\mathcal{E},+) \to \mathscr{C}^0_0(\mathcal{E},\cdot\,)$ *is injective.*

Lemma 8.3.2. *Let b be an additive cocycle with centred part c and drift K, satisfying the structure equation $\langle c_t \,|\, c_t\rangle_0 + t(K^* + K) = 0$. Let $u_t := \mathrm{Exp}(c_t + Kt)$ and $A_t := \mathrm{e}^{Kt}$. For any $a \in \mathcal{A}_0$,*

$$\Big\langle c_t \,\Big|\, a \int_0^t \mathrm{d}c_s u_s \Big\rangle_0 = \int_0^t \Lambda(a) A_s \,\mathrm{d}s\,.$$

Proof. Since u satisfies the IDE (8.1.1), the map $t \mapsto u_t$ is $\|\,\|_0$-continuous and thus the Itô integral $\int_0^t \mathrm{d}c_s u_s$ is well-defined. Moreover, $(a^* c_t)_{t\geq 0}$ is a centred additive cocycle. We calculate with the Itô identity (7.2.2)

$$\begin{aligned}\Big\langle c_t \,\Big|\, a \int_0^t \mathrm{d}c_s u_s \Big\rangle_0 &= \Big\langle \int_0^t \mathrm{d}(a^* c_s) \,\Big|\, \int_0^t \mathrm{d}c_s u_s \Big\rangle_0 = \int_0^t \langle \mathbb{1} \,|\, \Lambda(a) u_s\rangle_0 \,\mathrm{d}s \\ &= \int_0^t \Lambda(a) A_s \,\mathrm{d}s\,. \qquad \square\end{aligned}$$

Proof of Proposition 8.3.1. By Theorem 8.1.1 $u := \mathrm{Exp}(c)$ is a unital cocycle and by Theorem 8.2.2 we know that $\mathrm{Ln}(\mathrm{Exp}(c))$ is again an additive cocycle satisfying the structure equation. We are left with the task of identifying this cocycle as c. For this we consider, as stated in the previous Lemma, the unique decomposition $b_t = c_t + Kt$ of the additive cocycle b and $A_t = E_0 \,\mathrm{Exp}(b_t)$. We prove that

$$\mathrm{Ln}(\mathrm{Exp}(b_t)) - b_t = \lim_{n\to\infty} \sum_{i=0}^{2^n-1} S_{i\delta_n}(\mathrm{Exp}(b_{\delta_n}) - \mathbb{1}) - b_t = 0\,,$$

where $\delta_n := t2^{-n}$, by showing that

$$\lim_{n\to\infty} \sum_{i=0}^{2^n-1} S_{i\delta_n}(A_{\delta_n} - \mathbb{1}) - Kt = 0 \tag{8.3.1}$$

and

$$\lim_{n\to\infty} \sum_{i=0}^{2^n-1} S_{i\delta_n}(\mathrm{Exp}(b_{\delta_n}) - A_{\delta_n}) - c_t = 0 \tag{8.3.2}$$

in the $\|\,\|_0$-topology. The equation (8.3.1) is obvious, since $\mathcal{A}_0$ is the fixed point algebra of S and A_t is a norm continuous semigroup with generator K. Next we focus on the limit stated in equation (8.3.2), and let $u_t := \mathrm{Exp}(b_t)$:

$$\begin{aligned}\sum_{i=0}^{2^n-1} S_{i\delta_n}(u_{\delta_n} - A_{\delta_n}) - c_t &= \sum_{i=0}^{2^n-1} S_{i\delta_n}(u_{\delta_n} - \mathbb{1} - c_{\delta_n}) - \sum_{i=0}^{2^n-1}(A_{\delta_n} - \mathbb{1}) \\ &= \sum_{i=0}^{2^n-1} S_{i\delta_n} \int_0^{\delta_n} \mathrm{d}c_s(u_s - \mathbb{1}) - \sum_{i=0}^{2^n-1} S_{i\delta_n} \int_0^{\delta_n} \mathrm{d}s\, K u_s - \sum_{i=0}^{2^n-1} \frac{A_{\delta_n} - \mathbb{1}}{\delta_n}\delta_n\,.\end{aligned}$$

Obviously, the last summand tends to $-Kt$. The second summand tends to Kt, since

$$\Big\| \int_0^{\delta_n} K(u_{\delta_n} - \mathbb{1})\,\mathrm{d}s \Big\|_0 \le \sqrt{2}\|K\| \int_0^{\delta_n} \|A_s - \mathbb{1}\|^{1/2}\mathrm{d}s$$

and thus $\int_0^{\delta_n} K(u_{\delta_n} - \mathbb{1})\,\mathrm{d}s$ tends to zero with order $\delta_n^{3/2}$. Finally, the first summand converges to zero:

$$\begin{aligned}\Big|\sum_{i=0}^{2^n-1} S_{i\delta_n} \int_0^{\delta_n} \mathrm{d}c_s\,(u_s - \mathbb{1})\Big|_0^2 &= \sum_{i=0}^{2^n-1} \Big|\int_0^{\delta_n} \mathrm{d}c_s\,u_s - c_{\delta_n})\Big|_0^2 \\ &= \sum_{i=0}^{2^n-1} \Big(\int_0^{\delta_n} R_s(\Lambda(\mathbb{1}))\,\mathrm{d}s + \delta_n\Lambda(\mathbb{1})\Big) - \sum_{i=0}^{2^n-1} 2\,\mathrm{Re}\Big\langle c_{\delta_n}\,\Big|\int_0^{\delta_n} \mathrm{d}c_s\,u_s\Big\rangle_0\end{aligned}$$

– here we use Lemma 8.3.2 –

$$= \sum_{i=0}^{2^n-1} \int_0^{\delta_n} R_s(\Lambda(\mathbb{1}))\,\mathrm{d}s + \Lambda(\mathbb{1})t - \sum_{i=0}^{2^n-1} 2\,\mathrm{Re}\int_0^{\delta_n} \Lambda(\mathbb{1})A_s\,\mathrm{d}s$$

which tends to zero as $n \to \infty$, since both sums converge to $2\Lambda(\mathbb{1})t$ in the $\|\,\|$-topology on $\mathcal{A}_0$. □

We next prove the injectivity of the mapping Ln.

Proposition 8.3.3. *For any unital cocycle u in $\mathscr{C}_0^0(\mathcal{E},\cdot\,)$,*

$$\mathrm{Exp}(\mathrm{Ln}(u)) = u\,.$$

Consequently, the mapping $\mathrm{Ln}\colon \mathscr{C}_0^0(\mathcal{E},\cdot\,) \to \mathscr{C}_0^0(\mathcal{E},+)$ *is injective.*

We show that each $\|\,\|_0$-continuous unital cocycle u satisfies the IDE (6.4.1), where $c = \mathrm{Ln}_0(u)$, the centred non-commutative logarithm of u.

Proof. We prepare the proof with some results which we need in the sequel. By Theorem 8.2.2 we already know that the semigroup $R := (\langle u_t\,|\,\cdot\, u_t\rangle_0)_{t\ge 0}$ has generator $\mathcal{A}_0 \ni a \mapsto \mathcal{L}(a) := \Lambda(a) + K^*a + aK$, where $\Lambda := \langle c_1\,|\,\cdot\, c_1\rangle_0$, $A_t := E_0u_t = \mathrm{e}^{Kt}$, and $\Lambda(\mathbb{1}) = -(K^* + K)$. The non-commutative logarithm of u we obtain as $c_t = \|\,\|_0\text{-}\lim_{n\to\infty} \sum_{i=0}^{2^n-1} S_{i\delta_n}(u_{\delta_n} - A_{\delta_n})$, with $\delta_n := t2^{-n}$. We will show that

$$\Big|u_t - \mathbb{1} - \int_0^t \mathrm{d}c_s u_s - \int_0^t \mathrm{d}s\,Ku_s\Big|_0^2 = 0\,. \tag{8.3.3}$$

Expanding the inner product leads to the following ten expressions:

(i) $|u_t|_0^2 = \mathbb{1}$ (ii) $|\mathbb{1}|_0^2 = \mathbb{1}$

(iii) $\Big|\int_0^t \mathrm{d}c_s u_s\Big|_0^2 = \int_0^t R_s(\Lambda(\mathbb{1}))\,\mathrm{d}s$

$$\begin{aligned}
(iv)\quad & \Big|\int_0^t \mathrm{d}s\, Ku_s\Big|_0^2 = \int_0^t R_s(\Lambda(\mathbb{1}))\,\mathrm{d}s \\
& \qquad\qquad -2\,\mathrm{Re}\Big[\int_0^t R_s(\Lambda(A_{t-s}))\,\mathrm{d}s - (\mathbb{1}-A_t)\Big] \\
(v)\quad & -2\,\mathrm{Re}\langle u_t \,|\, \mathbb{1}\rangle_0 = -2\,\mathrm{Re}\, A_t \\
(vi)\quad & 2\,\mathrm{Re}\Big\langle \mathbb{1}\Big| \int_0^t \mathrm{d}c_s u_s\Big\rangle_0 = 0 \\
(vii)\quad & -2\,\mathrm{Re}\Big\langle u_t\Big| \int_0^t \mathrm{d}c_s u_s\Big\rangle_0 = -2\,\mathrm{Re}\int_0^t R_s(\Lambda(A_{t-s}))\,\mathrm{d}s \\
(viii)\quad & -2\,\mathrm{Re}\Big\langle u_t\Big| \int_0^t \mathrm{d}s\, Ku_s\Big\rangle_0 = 2\,\mathrm{Re}\Big[\int_0^t R_s(\Lambda(A_{t-s}))\,\mathrm{d}s - (\mathbb{1}-A_t)\Big] \\
(ix)\quad & 2\,\mathrm{Re}\Big\langle \mathbb{1}\Big| \int_0^t \mathrm{d}s\, Ku_s\Big\rangle_0 = -2\,\mathrm{Re}(\mathbb{1}-A_t) \\
(x)\quad & 2\,\mathrm{Re}\Big\langle \int_0^t \mathrm{d}s\, Ku_s\Big| \int_0^t \mathrm{d}c_s u_s\Big\rangle_0 = 2\,\mathrm{Re}\Big[\int_0^t R_s(\Lambda(A_{t-s}))\,\mathrm{d}s \\
& \qquad\qquad - \int_0^t R_s(\Lambda(\mathbb{1}))\,\mathrm{d}s\Big].
\end{aligned}$$

Equation (8.3.3) is obtained by adding (i) to (x). The equations (i), (ii), (iii), (v) and (vi), as well as (ix), are evident; we verify the rest.
(iv): We use Lemma 8.2.5 (i) and (ii) to obtain

$$\begin{aligned}
\Big|\int_0^t \mathrm{d}s\, Ku_s\Big|_0^2 &= \int_0^t\int_0^r \langle u_s \,|\, K^*Ku_r\rangle_0\,\mathrm{d}s\,\mathrm{d}r + \int_0^t\int_r^t \langle u_s \,|\, K^*Ku_r\rangle_0\,\mathrm{d}s\,\mathrm{d}r \\
&= \int_0^t\int_0^r R_s(K^*KA_{r-s})\,\mathrm{d}s\,\mathrm{d}r + \int_0^t R_r\Big(\int_r^t A^*_{s-r}K^*K\,\mathrm{d}s\Big)\,\mathrm{d}r\,.
\end{aligned}$$

The second integrals equals $\int_0^t R_r((A^*_{t-r}-\mathbb{1})K)\,\mathrm{d}r$. In the first integral we exchange the order of integration and obtain $\int_0^t R_s\big(\int_s^t K^*KA_{r-s}\,\mathrm{d}r\big)\,\mathrm{d}s = \int_0^t R_s(K^*(A_{t-s}-\mathbb{1}))\,\mathrm{d}s$, thus

$$\Big|\int_0^t \mathrm{d}s\, Ku_s\Big|_0^2 = \int_0^t R_s(\Lambda(\mathbb{1}))\,\mathrm{d}s + \int_0^t R_s(A^*_{t-s}K + K^*A_{t-s})\,\mathrm{d}s\,.$$

The next calculation makes use of the identity

$$-\int_0^t R_s(K^*A^*_{t-s})\,\mathrm{d}s = \int_0^t R_s\big(\frac{\mathrm{d}}{\mathrm{d}s}A^*_{t-s}\big)\,\mathrm{d}s = \mathbb{1} - A^*_t - \int_0^t R_s(\mathcal{L}(A^*_{t-s}))\,\mathrm{d}s\,,$$

and a similar one for the adjoint expression $-\int_0^t R_s(A_{t-s}K)\,\mathrm{d}s$. Taking these into account, and the form of the generator $\mathcal{L}$,

$$\Big|\int_0^t \mathrm{d}s\, Ku_s\Big|_0^2 = \quad \int_0^t R_s(\Lambda(\mathbb{1}))\,\mathrm{d}s$$

$$
\begin{aligned}
&+\int_0^t R_s(A^*_{t-s}K+K^*A^*_{t-s})\,\mathrm{d}s+\int_0^t R_s(A_{t-s}K+K^*A_{t-s})\,\mathrm{d}s\\
&-\int_0^t R_s(K^*A^*_{t-s})\,\mathrm{d}s-\int_0^t R_s(A_{t-s}K)\,\mathrm{d}s\\
=\quad&\int_0^t R_s(\Lambda(\mathbb{1}))\,\mathrm{d}s\\
&+\int_0^t R_s((A_{t-s}+A^*_{t-s})K+K^*(A_{t-s}+A^*_{t-s}))\,\mathrm{d}s\\
&-\int_0^t R_s(\mathcal{L}(A_{t-s}+A^*_{t-s}))\,\mathrm{d}s+\mathbb{1}-A_t+\mathbb{1}-A^*_t\\
=\quad&\int_0^t R_s(\Lambda(\mathbb{1}))\,\mathrm{d}s-2\operatorname{Re}\int_0^t R_s(\Lambda(A_{t-s}))\,\mathrm{d}s+2\operatorname{Re}(\mathbb{1}-A_t)\,.
\end{aligned}
$$

(vii): Next, $0\le s\le t-\delta$,

$$
\begin{aligned}
\langle u_t\,|(S_sc_\delta)\,u_s\rangle_0&=\langle (S_su_{t-s})u_s\,|(S_sc_\delta)u_s\rangle_0=R_s(\langle u_{t-s}\,|\,c_\delta\rangle_0)\\
&=R_s(\langle A_{t-s-\delta}u_\delta\,|\,c_\delta\rangle_0)\overset{(8.2.6)}{=}R_s\Big(\int_0^\delta A^*_r\Lambda(A^*_{\delta-r}A^*_{t-s-\delta})\,\mathrm{d}r\Big)\\
&=R_s\Big(\int_0^\delta A^*_r\Lambda(A^*_{t-s-r})\,\mathrm{d}r\Big)\,.
\end{aligned}
$$

Thus we obtain:

$$
\begin{aligned}
\Big\langle u_t\,\Big|\int_0^t \mathrm{d}c_s\,u_s\Big\rangle_0&=\lim_{n\to\infty}\sum_{i=0}^{2^n-1}\langle u_t\,|(S_{i\delta_n}c_{\delta_n})u_{i\delta_n}\rangle_0\\
&=\lim_{n\to\infty}\sum_{i=0}^{2^n-1}\underbrace{R_{i\delta_n}\Big(\frac{1}{\delta_n}\int_0^{\delta_n}A^*_r\Lambda(A^*_{t-i\delta_n-r})\,\mathrm{d}r-\Lambda(A^*_{t-i\delta_n})\Big)\,\delta_n}_{o(\delta_n)}\\
&\quad+\lim_{n\to\infty}\sum_{i=0}^{2^n-1}R_{i\delta_n}(\Lambda(A^*_{t-i\delta_n}))\delta_n=\int_0^t R_s(\Lambda(A^*_{t-s}))\,\mathrm{d}s\,.
\end{aligned}
$$

(viii):

$$
\begin{aligned}
\Big\langle u_t\,\Big|\int_0^t \mathrm{d}s\,Ku_s\Big\rangle_0&=\int_0^t\langle (S_su_{t-s})u_s\,|\,Ku_s\rangle_0\,\mathrm{d}s=\int_0^t R_s(A^*_{t-s}K)\,\mathrm{d}s\\
&=\int_0^t R_s(A^*_{t-s}K+K^*A^*_{t-s})\,\mathrm{d}s-\int_0^t R_s(K^*A^*_{t-s})\,\mathrm{d}s\\
&=\int_0^t R_s\mathcal{L}(A^*_{t-s})\,\mathrm{d}s-\int_0^t R_s\Lambda(A^*_{t-s})\,\mathrm{d}s-\int_0^t R_s(K^*A^*_{t-s})\,\mathrm{d}s\\
&=\mathbb{1}-A^*_t-\int_0^t R_s(\Lambda(A^*_{t-s}))\,\mathrm{d}s\,.
\end{aligned}
$$

The last equation is found, similar to (iv), by partial integration.
(x): The same calculation as for (vii) shows that $\langle Ku_s \,|\, \int_0^s \mathrm{d}c_r\, u_r\rangle_0 = \int_0^s R_r(\Lambda(K^*A^*_{s-r}))\,\mathrm{d}r$. Consequently, we have

$$\begin{aligned}\Big\langle \int_0^t \mathrm{d}s\, Ku_s \,\Big|\, \int_0^t \mathrm{d}c_s u_s \Big\rangle_0 &= \int_0^t \int_0^s R_r(\Lambda(K^*A^*_{s-r}))\,\mathrm{d}r\,\mathrm{d}s \\ &= \int_0^t \int_r^t R_r(\Lambda(K^*A^*_{s-r}))\,\mathrm{d}s\,\mathrm{d}r \\ &= \int_0^t R_r(\Lambda(A^*_{t-r}))\,\mathrm{d}r - \int_0^t R_r(\Lambda(\mathbb{1}))\,\mathrm{d}r\,. \qquad \square\end{aligned}$$

Proof of Theorems 6.4.1 and 6.4.4. According to Theorem 8.1.1 we can associate to each additive cocycle b in $\mathscr{C}^0_0(\mathcal{E},+)$ a unital cocycle $u = \mathrm{Exp}(b)$ in $\mathscr{C}^0_0(\mathcal{E},\cdot)$. By Proposition 8.3.1 the mapping Exp is injective and by Proposition 8.3.3 it is also surjective. This completes the proof of both theorems. $\square$

Appendix A. Hilbert W*-modules

For the convenience of the reader we provide some background results on Hilbert W*-modules, relevant to this paper.

We start with a (pre-) Hilbert C*-module $\mathcal{E}$ over a von Neumann algebra $\mathcal{A}_0 \subseteq \mathcal{B}(\mathcal{H}_0)$ and present the construction of a Hilbert W*-module which we used for the framework of continuous Bernoulli shifts. For a detailed approach to Hilbert modules we refer to [Lan95] and for the notion of Hilbert W*-modules to [Fra90, Pas73, Sch96]. The $\mathcal{A}_0$-valued inner product $\langle\cdot\,|\,\cdot\rangle_0$ induces by $|x|_0 := \langle x\,|\,x\rangle_0^{1/2}$ an $\mathcal{A}_0$-valued 'norm' which gives rise to the norm $\|x\|_0 := \||x|_0\|$ on $\mathcal{E}$. The completion of $\mathcal{E}$ in this norm is a Hilbert C*-module which we also denote by $\mathcal{E}$. The Cauchy-Schwarz inequality for the inner product is valid in the following form:

$$|\langle x\,|\,y\rangle_0|^2 \le \|x\|_0^2\,|y|_0^2, \qquad x,y\in\mathcal{E}\,. \tag{A.1}$$

$\mathcal{B}(\mathcal{E})$ is the Banach algebra of bounded module maps, i.e., the continuous $\mathcal{A}_0$-linear maps $T:\ \mathcal{E}\to\mathcal{E}$ such that $T(xa) = T(x)a$ for any $x\in\mathcal{E}$ and $a\in\mathcal{A}_0$. A bounded linear map $T:\mathcal{E}\to\mathcal{E}$ is $\mathcal{A}_0$-linear if and only if the inequality

$$\langle Tx\,|\,Tx\rangle_0 \le M\,\langle x\,|\,x\rangle_0, \qquad x\in\mathcal{E}\,. \tag{A.2}$$

is satisfied [Pas73]. $\mathcal{L}(\mathcal{E})$ denotes the C*-algebra of adjointable $\mathcal{A}_0$-linear maps, i.e., maps $T\in\mathcal{B}(\mathcal{E})$ for which a linear map T^* exists such that $\langle x\,|\,Ty\rangle_0 = \langle T^*x\,|\,y\rangle_0$ for any $x,y\in\mathcal{E}$. In general a bounded $\mathcal{A}_0$-linear map is not adjointable (but this property will always be the case for W*-modules). The 'dual' $\mathcal{E}'$ of $\mathcal{E}$ is defined to be the space of continuous $\mathcal{A}_0$-linear maps from $\mathcal{E}$ into $\mathcal{A}_0$. The map $\hat{x}:\mathcal{E}\ni y\mapsto\langle x\,|\,y\rangle$ defines an isometric embedding of $\mathcal{E}$ into $\mathcal{E}'$. Denote by $\widehat{\mathcal{E}}$ the image of $\mathcal{E}$ under this embedding. A Hilbert module $\mathcal{E}$ with the property $\widehat{\mathcal{E}} = \mathcal{E}'$ is called selfdual.

$\mathcal{K}(\mathcal{E}) := \mathrm{lh}\{\Theta_{x,y} \,|\, x, y \in \mathcal{E}\}^{-\|\,\|} \subseteq \mathcal{L}(\mathcal{E})$, with $\Theta_{x,y}z := x\langle y \,|\, z\rangle_0$ is the two-sided *-ideal of 'compact operators' on $\mathcal{E}$. The ideal $\mathcal{K}(\mathcal{E})$ has an approximate unit $(e_\alpha)_{\alpha\in I}$ of the form $e_\alpha := \sum_{i=1}^{n_\alpha} \Theta_{y_i^\alpha, y_i^\alpha}$ which can be constructed by an obvious adaption of the proof in [BR79, Prop. 2.2.18].
In the following, we present a concrete realization of the Hilbert W*-module via the minimal Kolmogorov decomposition of the kernel $\mathcal{E}\times\mathcal{E} \to \mathcal{A}_0$; $(x, y) \mapsto \langle x \,|\, y\rangle_0$ [EL77, Mur97]. By this decomposition the Hilbert module $\mathcal{E}$ is realized as a subspace of $\mathcal{B}(\mathcal{H}_0, \mathcal{H}')$ for some Hilbert space $\mathcal{H}'$ such that $\overline{\mathcal{E}\mathcal{H}_0} = \mathcal{H}'$ and thus, by an embedding, as a subspace of $\mathcal{B}(\mathcal{H}_0 \oplus \mathcal{H}')$. Hereby, the inner product then takes the form $\langle x \,|\, y\rangle_0 = x^*y$.

Definition A.1. *The* SOT*-closure of $\mathcal{E}$ of $\mathcal{B}(\mathcal{H}_0 \oplus \mathcal{H}')$ is called a Hilbert W*-module.*

The strong operator topology (SOT) resp. the σ-strong operator (topology (σ-SOT) on $\mathcal{E}$ is induced by the seminorms $x \mapsto \||x|_0\xi_0\|$, $\xi_0 \in \mathcal{H}_0$, resp. $x \mapsto |\varphi(\langle x \,|\, x\rangle_0)|^{1/2}$, $\varphi \in \mathcal{A}_{0*}$. The weak* topology on $\mathcal{E}_1$ is induced by the seminorms $x \mapsto |\varphi(\langle y \,|\, x\rangle_0)|$, $\varphi \in \mathcal{A}_{0*}$, $y \in \mathcal{E}$.

Theorem A.2. *A Hilbert W*-module $\mathcal{E}$ over the von Neumann algebra $\mathcal{A}_0$ has the predual $\mathcal{E}_* = \mathrm{lh}\{\varphi(\langle y \,|\, \cdot\rangle_0) \,|\, y \in \mathcal{E}, \varphi \in \mathcal{A}_{0*}\}^{-\|\,\|}$ and is selfdual in the sense of Hilbert modules.*

Proof. Since $\mathcal{E}$ is a weak*-closed subspace of $\mathcal{B}(\mathcal{H}_0 \oplus \mathcal{H}')$, it is the dual of the Banach space $\mathcal{T}(\mathcal{H}_0 \oplus \mathcal{H}')/\mathcal{E}^\circ$. Here, $\mathcal{T}(\mathcal{H}_0 \oplus \mathcal{H}')$ denotes the trace class operators on $\mathcal{H}_0 \oplus \mathcal{H}'$ and $\mathcal{E}^\circ$ the polar of $\mathcal{E}$ in $\mathcal{T}(\mathcal{H}_0 \oplus \mathcal{H}')$. Obviously, the functionals $x \mapsto \langle \xi \,|\, x\xi_0\rangle$, $\xi \in \mathcal{H}'$, $\xi_0 \in \mathcal{H}_0$ form a total set in $\mathcal{E}_*$. Due to the minimality of the Kolmogorov decomposition, they are approximated by functionals $x \mapsto \langle y\eta_0 \,|\, x\xi_0\rangle = \langle \eta_0 \,|\langle y \,|\, x\rangle_0\xi_0\rangle$, $\eta_0, \xi_0 \in \mathcal{H}_0$, $y \in \mathcal{E}$. The latter can be used to approximate $x \mapsto \varphi(\langle y \,|\, x\rangle_0)$, $\varphi \in \mathcal{A}_{0*}$. It follows that $\mathcal{E}_* = \mathrm{lh}\{\varphi(\langle y \,|\, \cdot\rangle_0) \,|\, y \in \mathcal{E}, \varphi \in \mathcal{A}_*\}^{-\|\,\|}$.

The self duality of $\mathcal{E}$ is proved with the help of the approximate unit $(e_\alpha)_{\alpha\in I}$ of $\mathcal{K}(\mathcal{E})$. For $\Psi \in \mathcal{E}'$ we check

$$\Psi(e_\alpha x) = \sum_{i=1}^{n_\alpha} \Psi(y_i^\alpha)\, \langle y_i^\alpha \,|\, x\rangle_0 = \Big\langle \sum_{i=1}^{n_\alpha} y_i^\alpha\, \Psi(y_i^\alpha)^* \,\Big|\, x\Big\rangle_0 =: \langle z_\alpha \,|\, x\rangle_0 \,.$$

This expression converges in norm to $\Psi(x)$. Thus $\varphi(\Psi(e_\alpha x))$ converges to $\varphi(\Psi(x))$ for any $\varphi \in \mathcal{A}_{0*}$. From this we conclude that $(z_\alpha)_{\alpha\in I}$ converges to some $z \in \mathcal{E}$ in the weak* topology on $\mathcal{E}$, since the net is bounded: $\|z_\alpha\|_0 = \|\langle z_\alpha \,|\, \cdot\rangle_0\| = \|\Psi \circ e_\alpha\| \le \|\Psi\|\,\|e_\alpha\|_0 \le \|\Psi\|$. Therefore we get $\varphi(\Psi(x)) = \varphi(\langle z \,|\, x\rangle_0)$ for any $\varphi \in \mathcal{A}_{0*}$ and any $x \in \mathcal{E}$, i.e., $\Psi = \langle z \,|\, \cdot\rangle_0$. $\square$

Corollary A.3. *$\mathcal{B}(\mathcal{E}) = \mathcal{L}(\mathcal{E})$, for a Hilbert W*-module$\mathcal{E}$.*

Proof. For $T \in \mathcal{B}(\mathcal{E})$ and $y \in \mathcal{E}$, $\mathcal{E} \ni x \mapsto \langle y \,|\, Tx\rangle_0$ defines an element in $\mathcal{E}'$, therefore there is a unique element $z_y \in \mathcal{E}$ such that $\langle z_y \,|\, x\rangle_0 = \langle y \,|\, Tx\rangle_0$ for any $x \in \mathcal{E}$. From [Lan95] it is known that $T^* : y \mapsto z_y$ defines an adjoint for T. This proves $\mathcal{B}(\mathcal{E}) \subseteq \mathcal{L}(\mathcal{E})$. The opposite inclusion is obvious. $\square$

Corollary A.4. *Let $\mathcal{E} \subseteq \mathcal{B}(\mathcal{H}_0 \oplus \mathcal{H}')$ be a Hilbert W*-module over the von Neumann algebra $\mathcal{A}_0 \subseteq \mathcal{B}(\mathcal{H}_0)$. If $y \in \mathcal{B}(\mathcal{H}_0 \oplus \mathcal{H}')$ and $y^*x \in \mathcal{A}_0$ for any $x \in \mathcal{E}$, then $y \in \mathcal{E}$.*

Proof. The map $\mathcal{E} \ni x \mapsto y^*x$ defines an element of $\mathcal{E}'$, therefore there is $z \in \mathcal{E}$ such that $y^*x\xi_0 = \langle z \,|\, x\rangle_0 \xi_0 = z^*x\xi_0$, for all $x \in \mathcal{E}$, $\xi_0 \in \mathcal{H}_0$. Since $\mathcal{H}'$ is generated by $\mathcal{E}\mathcal{H}_0$ we conclude $y^* = z^*$, and thus $y \in \mathcal{E}$. $\square$

Theorem A.5 (Kaplansky)**.** *Let $\mathcal{E}$ be the Hilbert W*-module generated by the pre-Hilbert module $\mathcal{E}_0$. Then the unit ball of $\mathcal{E}_0$ is σ-SOT-dense in the unit ball of $\mathcal{E}$.*

Proof. The so-called linking algebra $\mathcal{L}_\mathcal{E} := \begin{bmatrix} \mathcal{A}_0 & \mathcal{E}^* \\ \mathcal{E} & \mathcal{L}(\mathcal{E}) \end{bmatrix}$ is a W*-algebra in $\mathcal{B}(\mathcal{H}_0 \oplus \mathcal{H}')$, in which $\mathcal{E}^* := \{x^* \,|\, x \in \mathcal{E}\}$. The density follows from the contractivity of the embedding of $\mathcal{E}$ in $\mathcal{L}_\mathcal{E}$ and Kaplansky's density theorem for $\mathcal{L}_\mathcal{E}$. $\square$

Appendix B. The ψ-adjoint of morphisms

We provide results which are needed in the proof of Theorem 5.2.2.

Lemma B.1. *Let $(\mathcal{A}, \psi)$ be a probability space and $T\colon \mathcal{A} \to \mathcal{A}$ a completely positive map such that there is an $a \in \mathcal{A}$ with the property*

$$\psi(T(x)) = \psi(xa) \tag{B.1}$$

for all $x \in \mathcal{A}$. Then T is normal (i.e. weak-weak*-continuous), σ-SOT-σ-SOT- and σ-SOT *-σ-SOT *-continuous.*

Note that the operator a is necessary unique, if it exists. For the proof of the lemma we introduce the so called ψ-norm on $\mathcal{A}$ by $\|x\|_\psi := \psi(x^*x)^{1/2}$.

Proof. Using [Sak71, Prop. 1.24.1], we obtain the inequality

$$\psi(T(x)) \le \|a\|\psi(x)\,, \quad x \in \mathcal{A}^+ \tag{B.2}$$

from the positivity of $\psi \circ T$. The Kadison-Schwarz inequality for T leads to $\|T(x)\|_\psi \le \|a\|^{1/2}\|x\|_\psi$. Now the Cauchy-Schwarz inequality shows, for each $x \in \mathcal{A}$, the σ SOT-continuity of $\mathcal{A} \ni y \mapsto \psi_x(T(y))$, where $\psi_x(y) := \psi(xy)$. Since $\{\psi_x \,|\, x \in \mathcal{A}\}$ is norm dense in $\mathcal{A}_*$ by the bipolar theorem, the usual $\varepsilon/2$-argument shows the SOT-continuity of the maps $y \mapsto \varphi(T(y))$, $\varphi \in \mathcal{A}_*$, on bounded subsets of $\mathcal{A}$. From [Tak03a, Thm. II.2.6] therefore $\varphi \circ T \in \mathcal{A}_*$ for any $\varphi \in \mathcal{A}_*$, i.e. T is normal. Now let $(x_\alpha)_{\alpha\in I}$ be a net converging σ-strongly to zero. Then for any positive $\varphi \in \mathcal{A}_*$ we have by Kadison-Schwarz's inequality $\varphi(T(x_\alpha^*)T(x_\alpha)) \le \varphi \circ T(x_\alpha^* x_\alpha) \xrightarrow{\alpha} 0$, i.e. $T(x_\alpha) \xrightarrow{\alpha} 0$ σ-strongly. The σ SOT *-σ SOT *-continuity of T is shown analogously. $\square$

For the reader's convenience we include a direct proof of a result from [Küm84], which is needed in Theorem 5.2.2 (cf. [Hel01]). In the following Δ and J denotes, as usual, the modular operator and the modular conjugation with respect to the state $\psi = \langle \Omega \,|\, \cdot\, \Omega \rangle$.

Theorem B.2. *For a completely positive map T on the probability space $(\mathcal{A}, \psi)$ with property* (B.1) *the following are equivalent:*

(i) *There exists a completely positive map T^* on $\mathcal{A}$ with the property $\psi(xT(y)) = \psi(T^*(y)x)$ for all $x, y \in \mathcal{A}$.*

(ii) *T commutes with the modular automorphism group σ^ψ of ψ.*

Consequently, the operator a in equation (B.1) *belongs to the centralizer $\mathcal{A}^\psi$ of ψ and is given by $T^*(\mathbb{1})$.*

Definition B.0.4. *T^* is called the ψ-*adjoint *of T.*

Proof. By (B.2), T has an extension $\overline{T}$ to a bounded operator on the GNS Hilbert space $\mathcal{H}_\psi$, which is defined by $\overline{T}x\Omega := T(x)\Omega$, $x \in \mathcal{A}$.

(ii) $\Rightarrow$ (i): Since T and σ^ψ commute, T maps the SOT-dense subspace of entire analytic elements $\mathcal{A}_\mathrm{a}$ for σ^ψ into itself. Hence, for any $y \in \mathcal{A}_\mathrm{a}$, the analytic continuation of $\mathbb{R} \ni t \mapsto \overline{T}\Delta^{\mathrm{i}t}y^*\Omega = \Delta^{\mathrm{i}t}T(y^*)\Omega$, evaluated at $t = -\mathrm{i}/2$, yields

$$\overline{T}Jy\Omega = \overline{T}\Delta^{1/2}y^*\Omega = \Delta^{1/2}\overline{T}y^*\Omega = \Delta^{1/2}T(y^*)\Omega = JT(y)\Omega = J\overline{T}y\Omega\,.$$

It follows that $[\overline{T}, J] = 0$, since $\mathcal{A}_\mathrm{a}\Omega$ is dense in $\mathcal{H}_\psi$. Next we show $\overline{T}^*\mathcal{A}^+\Omega \subseteq \mathcal{A}^+\Omega$. To begin with, by the duality of the cones $\overline{\mathcal{A}^+\Omega}$ and $\overline{\mathcal{A}'^+\Omega}$ and the estimation

$$\langle Jx\Omega \,|\, \overline{T}^*y\Omega \rangle = \langle JT(x)\Omega \,|\, y\Omega \rangle = \langle T(x)^{1/2}\Omega \,|\, JyJT(x)^{1/2}\Omega \rangle \geq 0\,,$$

for all $x, y \in \mathcal{A}^+$, we obtain $\overline{T}^*y\Omega \in \overline{\mathcal{A}^+\Omega}$. Next we observe that the positive linear functional $x \mapsto \langle Jx\Omega \,|\, \overline{T}^*y\Omega \rangle$ is dominated by ψ: For $x \in \mathcal{A}^+$ we have

$$\langle T(x)^{1/2}\Omega \,|\, JyJT(x)^{1/2}\Omega \rangle \leq \|y\|\, \langle \Omega \,|\, T(x)\Omega \rangle \overset{(\mathrm{B}.2)}{\leq} \|y\|\,\|a\|\,\psi(x)\,.$$

Hence, by the Radon-Nikodym theorem, there is a unique $z \in \mathcal{A}^+$ with the property $\langle Jx\Omega \,|\, \overline{T}^*y\Omega \rangle = \langle Jz\Omega \,|\, x\Omega \rangle = \langle Jx\Omega \,|\, z\Omega \rangle$ for any $x \in \mathcal{A}$. That is, $\overline{T}^*y\Omega = z\Omega \in \mathcal{A}^+\Omega$.

Therefore, $T^*(y)\Omega := \overline{T}^*y\Omega$, $y \in \mathcal{A}$, defines a positive linear map T^* on $\mathcal{A}$, which satisfies $\psi(xT(y)) = \psi(T^*(x)y)$. Thus the uniqueness of T^* and $a = T^*(\mathbb{1}) \geq 0$ are evident. $T^*(\mathbb{1}) \in \mathcal{A}^\psi$ follows from

$$\psi(T^*(\mathbb{1})x) = \overline{\psi(T(x^*))} = \overline{\psi(T^*(\mathbb{1})x^*)} = \psi(xT^*(\mathbb{1}))\,.$$

Up to now we have only used the positivity of T. We are left to show that T^* is completely positive. To this end consider the map $T_{(n)} := \mathrm{id}_n \otimes T$ on the probability space $(M_n \otimes \mathcal{A}, \psi_{(n)})$, where $\psi_{(n)} := \tau_n \otimes \psi$, with τ_n the normed trace on M_n. The modular operator $\Delta_{(n)}$ and the modular conjugation $J_{(n)}$ of $\psi_{(n)}$ are, respectively, given by $\Delta_{(n)} := \mathbb{1}_n \otimes \Delta$ and $J_{(n)} := J_n \otimes J$, with J_n being the modular conjugation of τ_n. Obviously, $T_{(n)}$ commutes with the modular automorphism group of $\psi_{(n)}$, given by $\mathrm{id}_n \otimes \sigma^\psi$. An elementary calculation shows that $\psi_{(n)}(xT_{(n)}(y)) = \psi_{(n)}(T^*_{(n)}(x)y)$ for all $x, y \in M_n \otimes \mathcal{A}$, with $T^*_{(n)} := \mathrm{id}_n \otimes T^*$. Choosing $x = \mathbb{1}_n \otimes \mathbb{1}$, we arrive at (B.1) for $\psi_{(n)}$ and $T_{(n)}$, with a replaced by $\mathbb{1}_n \otimes a$. Since T is completely positive, $T_{(n)}$ is positive. Hence, our considerations above show the existence of the adjoint map $(T_{(n)})^*$, which by uniqueness is given by $T^*_{(n)}$. Since $(T_{(n)})^*$ is positive, T^* is completely positive.

(i) $\Rightarrow$ (ii): Let $y \in \mathrm{Dom}(\Delta)$ and $x \in \mathcal{A}$. Then we have, using $\overline{T}^* = \overline{T^*}$:

$$\begin{aligned}
\langle x\Omega \,|\, \overline{T}\Delta y\Omega\rangle &= \langle \Delta^{1/2}T^*(x)\Omega \,|\, \Delta^{1/2}y\Omega\rangle = \langle JT^*(x^*)\Omega \,|\, Jy^*\Omega\rangle \\
&= \langle y^*\Omega \,|\, T^*(x^*)\Omega\rangle = \langle T(y^*)\Omega \,|\, x^*\Omega\rangle \\
&= \langle J\Delta^{1/2}T(y)\Omega \,|\, J\Delta^{1/2}x\Omega\rangle = \langle \Delta^{1/2}x\Omega \,|\, \Delta^{1/2}\overline{T}y\Omega\rangle\,.
\end{aligned}$$

Since $\mathcal{A}\Omega$ is a core for $\Delta^{1/2}$, $\Delta^{1/2}\overline{T}y\Omega$ is in the domain of $\Delta^{1/2}$ and we have $\Delta\overline{T}y\Omega = \overline{T}\Delta y\Omega$. Hence $\overline{T}$ and Δ commute strongly. From this we obtain $[\overline{T}, \Delta^{\mathrm{i}t}] = 0$ and finally $T \circ \sigma^\psi_t = \sigma^\psi_t \circ T$. □

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