Notes on Enveloping Algebras (following Dixmier)

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Abstract

The purpose of these notes is to provide a more-or-less self-contained proof of Theorem 5.1, which asserts that (borrowing some C^*-algebraic jargon) the universal enveloping algebra $U = U(\mathfrak{g})$ of a finite-dimensional Lie algebra $\mathfrak{g}$ is residually finite-dimensional, in the sense that given any nonzero element u of U we can produce a finite-dimensional representation π of U that does not vanish on u. In fact, we prove something stronger: we can arrange for the intersection of $\ker \pi$ and U^d to be trivial, where U^d is the subspace spanned by symmetric homogeneous tensors of degree d. These notes follow [2, Ch. 2] very closely – mostly they were written to digest and provide additional background information. A reader who knows Lie's theorem on solvable Lie algebras and Engel's theorem on nilpotent Lie algebras will have adequate background knowledge.

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1 The Poincaré-Birkhoff-Witt Theorem

Definition 1.1. Let A be an associative algebra. By A_- we denote the Lie algebra that has A as its underlying set and bracket given by $[xy] = xy - yx$. This is called the underlying Lie algebra of A.

Remark 1.1. There is a faithful functor **Alg** $\to$ **LieAlg** which is given by $A \mapsto A_-$ on objects and the identity on morphisms. It is kind of like a forgetful functor, so we seek its left adjoint. (One could probably try to invoke some heavy duty Freyd theorem to prove that it is a right adjoint, but who has the energy?)

The rough idea of a (universal) enveloping algebra is to reverse the construction in Definition 1.1: we take a Lie algebra $\mathfrak{g}$ and insert it in an associative algebra U in such a way that it generates the algebra (as an algebra! – not as a Lie algebra). The algebra representations of the algebra U ought to correspond precisely with the Lie algebra representations of $\mathfrak{g}$.

The idea of the construction is pretty simple: just take the elements of $\mathfrak{g}$ and treat them as elements in an associative algebra, while maintaining all relations coming from $\mathfrak{g}$. This corresponds with the *tensor algebra* of the vector space $\mathfrak{g}$.

Definition 1.2. Let $\mathfrak{g}$ be a Lie algebra. Let $T^0 = \mathbb{C}$ (or whatever ground field), let $T^1 = \mathfrak{g}$, and for $n > 1$ let

$$T^n = T^n(\mathfrak{g}) = \underbrace{\mathfrak{g} \otimes \mathfrak{g} \otimes \ldots \otimes \mathfrak{g}}_{n\text{-fold}}.$$

Set $T = T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} T^n$, which we refer to as the *tensor algebra* of $\mathfrak{g}$ (here we are only treating $\mathfrak{g}$ as a vector space).

Remark 1.2. The tensor algebra provides a good example of an inclusion $\mathfrak{g} \hookrightarrow T$ which is not a homomorphism:

$$x \otimes y - y \otimes x \neq [x, y].$$

Definition 1.3. Let $\mathfrak{g}$ and $T = T(\mathfrak{g})$ be as above. Define J to be the two-sided ideal of T generated by all elements of the form

$$x \otimes y - y \otimes x - [x, y] \in T^2 \oplus T^1 \subset T.$$

Denote T/J by $U = U(\mathfrak{g})$. We refer to this algebra as the universal enveloping algebra of $\mathfrak{g}$.

Remark 1.3. In view of remark 1.2, it is easy to see that U is the largest quotient of T so that the composite map $\mathfrak{g} \hookrightarrow T \twoheadrightarrow U(\mathfrak{g})$ is a Lie algebra homomorphism.

Definition 1.4. The map $\mathfrak{g} \to T \twoheadrightarrow U(\mathfrak{g})$ is denoted by σ and termed the canonical mapping of $\mathfrak{g}$ into $U(\mathfrak{g})$.

Remark 1.4. If $\mathfrak{g}$ is abelian (trivial Lie bracket), then J is equal to the ideal generated by all $x \otimes y - y \otimes x$, and U is equal to the symmetric algebra of $\mathfrak{g}$.

Remark 1.5. The ideal J in the definition of U is generated by elements belonging to the two-sided ideal $T_+ = T^1 \oplus T^2 \oplus \ldots$ be the subalgebra of T generated by $\mathfrak{g}$, which has trivial intersection with $T^0 = \mathbb{C}$. Thus the map $T^0 \twoheadrightarrow U$ is injective.

Definition 1.5. We denote the image of T_+ in $U(\mathfrak{g})$ by U_+. (This is the subalgebra of U generated by the image of σ.) We set $U(\mathfrak{g}) = U^0 \oplus U_+(\mathfrak{g})$. The summand U^0 is identified with the ground field. For $u \in U$ we refer to its projection onto the ground field as the constant term of u (imagining u as a polynomial). Note that U is generated as an algebra by $\{1\} \cup \sigma(\mathfrak{g})$.

We are finally ready for our first result pertaining to U, which justifies its lofty title.

Lemma 1.1 (cf. [2, Lemma 2.1.3])**.** *Let σ be the canonical mapping of $\mathfrak{g}$ into $U(\mathfrak{g})$ and let A be a unital algebra, and let $\tau : \mathfrak{g} \to A_-$ be a Lie homomorphism. There exists a unique unital homomorphism $\tau' : U \to A$ such that $\tau' \circ \sigma = \tau$.*

Proof. The uniqueness is evident from the fact that U is generated by $\{1\}\cup\sigma(\mathfrak{g})$. Let the inclusion $\mathfrak{g} \hookrightarrow T$ be denoted by ι. The universal property of the tensor algebra of $\mathfrak{g}$ provides an algebra homomorphism $\tilde{\tau} : T \to A$ satisfying $\tilde{\tau} \circ \iota = \tau$, which is unique if we require it to be unital. In particular, $\tilde{\tau}(x \otimes y) = \tau(x)\tau(y)$. From this and the fact that τ is a Lie homomorphism we see that $\tilde{\tau}$ vanishes on the two-sided ideal J discussed above. It therefore factors through the quotient $q : T \twoheadrightarrow U$ to provide an algebra homomorphism $\tau' : U \to A$ satisfying $\tilde{\tau} = \tau' \circ q$. Then we have

$$\tau' \circ \sigma = \tau' \circ q \circ \iota = \tilde{\tau} \circ \iota = \tau.$$

□

Remark 1.6. Lemma 1.1 shows that we have constructed a left adjoint to the "forgetful functor" $A \to A_-$. If A is any associative algebra, we have a natural bijection

$$\mathrm{Hom}_{\mathbf{LieAlg}}(g, A_-) \cong \mathrm{Hom}_{\mathbf{Alg}}(U(g), A).$$

Lemma 1.1 provides the left-to-right map and the right-to-left map is given by composition with σ (although we will shortly see that $\mathfrak{g}$ embeds injectively in U via σ).

Definition 1.6. Assume that $\mathfrak{g}$ is finite-dimensional (although as in [4] its possible to make this work for countable-dimension Lie algebras as well) with ordered basis $(x_1, x_2, \dots, x_n)$. Set $y_i := \sigma(x_i)$. If $I = (i_1, \dots, i_p) \in \{1, 2, \dots, n\}^p$ let $y_I = y_{i_1} y_{i_2} \dots y_{i_p}$. If $i \in \mathbb{Z}$, write $i \leq I$ if for all k we have $i \leq i_k$. If $q : T \twoheadrightarrow U$ is the quotient we set $U_d(\mathfrak{g})$ to be $q(T^0 + T^1 + \dots + T^d)$. We also allow an empty string I (of length 0) with $y_\emptyset = 1 \in U$.

We regard the U_d as the polynomials of degree $\leq d$. The next Lemma shows that we can rearrange any monomial at the cost of introducing lower-order terms.

Lemma 1.2 ([2, Lem. 2.1.5])**.** *let $a_1, \dots, a_p \in \mathfrak{g}$ and let $\sigma : \mathfrak{g} \to U$ be canonical. If π is a permutation of the set $\{1, 2, \dots, p\}$ then*

$$\sigma(a_1)\sigma(a_2)\dots\sigma(a_p) - \sigma(a_{\pi(1)})\sigma(a_{\pi(2)})\dots\sigma(a_{\pi(p)}) \in U_{p-1}(\mathfrak{g}).$$

Proof. Every π can be written as the product of transpositions of the form $(j\ j+1)$, so we assume that π is of that form. Then

$$\begin{aligned}
&\sigma(a_1)\dots\sigma(a_p) - \sigma(a_{\pi(1)})\dots\sigma(a_{\pi(p)}) \\
&= \sigma(a_1)\dots\sigma(a_j)\sigma(a_{j+1})\dots\sigma(a_p) - \sigma(a_1)\dots\sigma(a_{j+1})\sigma(a_j)\dots\sigma(a_p) \\
&= \sigma(a_1)\dots\sigma(a_{j-1})[\sigma(a_j),\sigma(a_{j+1})]\sigma(a_{j+2})\dots\sigma(a_p) \\
&= \sigma(a_1)\dots\sigma(a_{j-1})\sigma([a_j,a_{j+1}])\sigma(a_{j+2})\dots\sigma(a_p) \in U_{p-1}(\mathfrak{g}).
\end{aligned}$$

□

Lemma 1.2 immediately implies the following.

Lemma 1.3 ([2, Lem. 2.1.6])**.** *The set of monomials $\{y_I : I$ increasing of length $p\}$ spans the space $U_p(\mathfrak{g})$.*

Definition 1.7. Let $P = \mathbb{C}[z_1,\dots,z_n]$ be the algebra of complex polynomials in n indeterminates. Set P_i to be the subspace consisting of elements of P of total degree $\le i$. For $I \in \{1,\dots,n\}^p$ set $z_I = z_{i_1}\dots z_{i_p}$.

Remark 1.7. The following rather technical result is used to prove that the increasing monomials are linearly independent.

Lemma 1.4. *For every integer $p \ge 0$ there exists a unique linear mapping $f_p : \mathfrak{g}\otimes P_p \to P$ satisfying the following conditions:*

(A_p) $f_p(x_i \otimes z_I) = z_i z_I$ if $i \le I$ and $z_I \in P_p$;

(B_p) $f_p(x_i \otimes z_I) - z_i z_I \in P_q$ for $z_I \in P_q$, $q \le p$;

(C_p) $f_p(x_i \otimes f_p(x_j \otimes z_J)) = f_p(x_j \otimes f_p(x_i \otimes z_J)) + f_p([x_i,x_j]\otimes z_J)$.

Moreover $f_p|_{\mathfrak{g}\otimes P_{p-1}} = f_{p-1}$.

Proof. The condition A_0 implies that $f_0(x_i \otimes 1) = z_i$. This implies B_0 and C_0 both hold. The main issue is to extend f_{p-1} inductively to f_p.

Define $f_p(x_i \otimes z_I) = z_i z_I$ as long as $i \le I$ and $z_i \in P_p$. If i is not bounded by I then $I = (j, J)$ where $j > i$ and $j \le J$. (If i is less than the least element in I, its bounded by everything in I.) Note that $z_I = f_{p-1}(x_j \otimes z_J)$.

Then

$$\begin{aligned}
f_p(x_i \otimes z_I) &= f_p(x_i \otimes f_{p-1}(x_j \otimes z_J)) \\
&= f_p(x_j \otimes f_{p-1}(x_i \otimes z_J)) + f_{p-1}([x_i,x_j]\otimes z_J).
\end{aligned}$$

Now $f_{p-1}(x_i \otimes z_J)) = z_i z_J + w$ for some $w \in P_{p-1}$ by induction. So we can define

$$\begin{aligned}
f_p(x_j \otimes f_{p-1}(x_i \otimes z_J)) &= z_j z_i z_J + f_{p-1}(x_j \otimes w) \\
&= z_i z_I + f_{p-1}(x_j \otimes w).
\end{aligned}$$

This is well defined as a linear map (we've basically given its value on the basis of $\mathfrak{g} \otimes P_p$). We need to verify that properties A_p, B_p, and C_p all hold. The first holds basically by definition, and $f_p(x_i \otimes z_I) - z_i z_I$ either equals 0 or $f_{p-1}(x_j \otimes w) + f_{p-1}([x_i, x_j] \otimes z_J) \in P_q$ as long as $z_I \in P_q$ and $q \leq p$.

The condition C_p is harder to check. We rewrite it as

$$x_i(x_j z_J) - x_j(x_i z_J) = [x_i, x_j] z_J$$

where $xz := f_p(x \otimes z)$ for $x \in \mathfrak{g}$ and $z \in P_p$. If $j \leq i$ and $j \leq J$, then C_p holds by the way that we have constructed f_p in the previous paragraph. Switching x_i and x_j we see that C_p still holds by negating both sides. This implies that either $i \leq J$ or $j \leq J$ implies C_p holds. It remains to check in the case when $J = (k, K)$ where $k \leq K$ and i, j (again remembering that J is an increasing sequence). since K is shorter than J, we can use induction to get

$$x_j z_J = x_j(x_k z_K) = x_k(x_j z_K) + [x_j, x_k] z_K. \qquad (1)$$

By B_{p-1} we know that $x_j z_K = z_j z_K + w$ where $w \in P_{p-2}$. As $k \leq K$ and $k < j$ we obtain

$$x_i(x_k(z_j z_K)) = x_k(x_i(z_j z_K)) + [x_i, x_k](z_j z_K);$$

by induction (on p) we have $x_i(x_k w) = x_k(x_i(w)) + [x_i, x_k](w)$. Linearity of the map and the equation $x_j z_K = z_j z_K + w$ implies that $x_i(x_k(x_j z_K)) = x_k(x_i(x_j z_K)) + [x_i, x_k](x_j z_K)$. Now we can calculate again, using (1) above as well as the general fact (*) that $A[B, C] = [B, C]A + [A, [B, C]]$, to get

$$\begin{aligned} x_i(x_j z_J) &= x_i(x_k(x_j z_K) + [x_j, x_k] z_K) \\ &= x_k(x_i(x_j z_K)) + [x_i, x_k](x_j z_K) + x_i [x_j, x_k] z_K \\ &= x_k(x_i(x_j z_K)) + [x_i, x_k](x_j z_K) + [x_j, x_k](x_i z_K) + [x_i, [x_j, x_k]] z_K. \end{aligned}$$

Since our only condition on i and j was that $i, j > k$, we can interchange them to obtain

$$x_j(x_i z_J) = x_k(x_j(x_i z_K)) + [x_j, x_k](x_i z_K) + [x_i, x_k](x_j z_K) + [x_j, [x_i, x_k]] z_K.$$

Subtracting the two equations, and again using (*), we obtain that $x_i(x_j z_J) - x_j(x_i z_J)$

$$\begin{aligned} & x_k(x_i(x_j z_K)) - x_k(x_j(x_i z_K)) + [x_i, [x_j, x_k]] z_K - [x_j, [x_i, x_k]] z_K \\ &= x_k([x_i, x_j] z_K) + [x_k, [x_i, x_j]] z_K + [x_i, [x_j, x_k]] z_K - [x_j, [x_i, x_k]] z_K \\ &= [x_i, x_j] x_k z_K \\ &= [x_i, x_j] z_J, \end{aligned}$$

where in the second-to last equation we apply the Jacobi identity and in the last we use property A_p and the fact that $k \leq K$. $\square$

Remark 1.8. We can view the sequence of f_p as comprising a single linear map $f : \mathfrak{g} \otimes P \to P$, which endows P with the structure of a Lie module. The main feature of this action is that $x_i.z_I = z_i z_I$ as long as $i \leq I$.

Lemma 1.5 ([2, Lem 2.1.8])**.** *The set* $\{y_I : I$ *increasing*$\}$ *forms a basis for* $U(\mathfrak{g})$.

Proof. This follows rather easily from Remark 1.8 and Lemma 1.3. Let $1 \in P$ be the ring unit; the map $\mathfrak{g} \ni y \mapsto y.1$ is linear in the y variable. If I is an increasing finite sequence of indices, then we have $y_I.1 = z_I$, and the set $\{z_I\}$ is linearly independent. The linear image of a linearly dependent set cannot be linearly independent. Hence $\{y_I : I$ increasing $\}$ must be independent. Lemma 1.3 implies that the same set is a basis. □

Proposition 1.1. *The canonical mapping of* $\mathfrak{g}$ *into* $U(\mathfrak{g})$ *is injective.*

Proof. The canonical map σ sends each basis vector $x_i \in \mathfrak{g}$ to the corresponding y_i (where we can view i as an increasing sequence of length 1), and the set $\{y_i\}$ is linearly independent by Lemma 1.5. □

Remark 1.9. The map σ that embeds $\mathfrak{g}$ into $U(\mathfrak{g})$ is henceforth suppressed.

The following is a restatement of Lemma 1.5.

Theorem 1.1 (Poincaré-Birkhoff-Witt)**.** *Let* $(x_1, \ldots, x_n)$ *be a basis for the vector space* $\mathfrak{g}$*. Then the elements* $x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$*, where* $k_j \in \mathbb{N} \cup \{0\}$*, form a basis for* $U(\mathfrak{g})$.

2 Functorial properties of U

Proposition 2.1. *Let* A *be a unital algebra,* $\tau : \mathfrak{g} \to A_-$ *a Lie homomorphism. Then there is a unique unital algebra homomorphism* $\tilde{\tau} : U(\mathfrak{g}) \to A$ *so that* $\tilde{\tau}|_{\mathfrak{g}} = \tau$.

Proof. This is a restatement of Lemma 1.1. □

Remark 2.1. If we fix a vector space V, then there is a one-to-one correspondence between representations of $\mathfrak{g}$ on V and representations of $U(\mathfrak{g})$ on V, described essentially by Lemma 1.1. This correspondence extends to the submodule structure. Dixmier uses the same symbol to refer to $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ and $U(\rho) : U(\mathfrak{g}) \to \mathrm{End}(V)$. The only noteworthy thing is that, unless otherwise indicated, the kernel of ρ refers to the larger set (i.e. the zero set in $U(\mathfrak{g})$).

Proposition 2.2. *Let* $\mathfrak{g}, \mathfrak{g}'$ *be Lie algebras and* $\phi : \mathfrak{g} \to \mathfrak{g}'$ *a Lie homomorphism. Then there exists a unique unital algebra homomorphism* $\tilde{\phi} : U(\mathfrak{g}) \to U(\mathfrak{g}')$ *so that* $\tilde{\phi}|_{\mathfrak{g}} = \phi$.

Proof. Simply regard ϕ as a map into $U(\mathfrak{g}') \supset \mathfrak{g}'$ and apply Proposition 1.1. □

Proposition 2.3 ([2, Prop. 2.2.4])**.** *Let* $\mathfrak{g}' \subset \mathfrak{g}$ *be a Lie aubalgebra. Then there is a Lie algebra embedding* $U(\mathfrak{g}') \subset U(\mathfrak{g})$.

Proof. Let $\iota : \mathfrak{g}' \to \mathfrak{g}$ be the inclusion map. We will show that $U(\iota)$ is injective. Take a basis $(x_1, \ldots, x_m)$ for $\mathfrak{g}'$ and extend it to a basis $(x_1, \ldots, x_n)$ for $\mathfrak{g}$. The image under $U(\iota)$ of any basis element $x_1^{k_1} \ldots x_m^{k_m} \in U(\mathfrak{g}')$ is a basis element in $U(\mathfrak{g})$ (with exponents $k_{m+1}, \ldots, k_n$ all equal to 0). Since $U(\iota)$ sends basis elements to basis elements it is injective. □

Remark 2.2. Henceforth $U(\mathfrak{g}')$ is considered as a subalgebra of $U(\mathfrak{g})$ and the embedding $U(\iota)$ is suppressed. Also we use the notation $\mathfrak{g}' \leq \mathfrak{g}$ to indicate a Lie algebra containment (i.e. that $\mathfrak{g}'$ is a Lie subalgebra of $\mathfrak{g}$).

Proposition 2.4 ([2, Prop. 2.2.7])**.** *Let $\mathfrak{g}' \leq \mathfrak{g}$ and let $(y_1, \ldots, y_q)$ be a basis for a (vector space) complement of $\mathfrak{g}'$ in $\mathfrak{g}$. Then the set $\{y_1^{k_1} \ldots y_q^{k_q} : k_j \in \mathbb{N} \cup \{0\}\}$ is a $U(\mathfrak{g}')$-module basis for $U(\mathfrak{g})$.*

Proof. Fairly straightforward, just regrouping basis elements from the PBW theorem. □

Proposition 2.5 ([2, Prop. 2.2.8])**.** *Let $\mathfrak{h}$ and $\mathfrak{f}$ be Lie subalgebras of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{f}$. Let $\mathfrak{l} = \mathfrak{h} \cap \mathfrak{f}$. Consider $U(\mathfrak{h})$ as a right $\mathfrak{l}$-module and $U(\mathfrak{f})$ as a left $U(\mathfrak{l})$-module. Then there is a unique linear map $f : U(\mathfrak{h}) \otimes U(\mathfrak{f}) \to U(\mathfrak{g})$ satisfying $f(v \otimes w) = vw$ for $v \in U(\mathfrak{h})$ and $w \in \mathfrak{f}$. The map f is a bijection.*

Proof. Consider the map $(v, w) \mapsto vw$ from $U(\mathfrak{h}) \times U(\mathfrak{f})$ to $U(\mathfrak{g})$. This is clearly $U(\mathfrak{l})$-bilinear, so it descends to a map $U(\mathfrak{h}) \otimes_{U(\mathfrak{l}} U(\mathfrak{f}) \to U(\mathfrak{g})$. Uniqueness is automatic because the elements $[h \otimes f]$ span the domain (brackets around the tensor denote the equivalence class in the $U(\mathfrak{l})$-balanced tensor product, as opposed to the ordinary tensor over $\mathbb{C}$).

It remains to prove that f is bijective. Let $(a_1, \ldots, a_m)$ be a basis for $\mathfrak{h}$ and $(b_1, \ldots, b_n)$ a basis for a supplement of $\mathfrak{l}$ in $\mathfrak{f}$. (Note that the dimension of $\mathfrak{h} + \mathfrak{f}$ is $m + n$.) Claim: the set of all tensors of the form

$$a_1^{k_1} \ldots a_m^{k_m} \otimes b_1^{j_1} \ldots b_n^{j_n}$$

forms a basis for the vector space $U(\mathfrak{h}) \otimes_{U(\mathfrak{l})} U(\mathfrak{f})$. (We will call these tensors *distinguished*. Dixmier rather cavalierly says that Prop. 2.4 implies this without spelling out the details.) First we show that these tensors span the vector space. Let $v = \sum z_i a_i \in \mathfrak{h}$ and $w = \ell + \sum z_j' b_j \in \mathfrak{f}$, where $\ell \in \mathfrak{l}$, say $\ell = \sum t_k a_k$. Now putting everything together (suppressing the balanced tensor product brackets) we get

$$v \otimes w = (\sum z_i a_i) \otimes (\sum t_k a_k + \sum z_j' b_j) = \sum_{i,k} z_i t_k a_i a_k \otimes 1 + \sum_{i,j'} z_i z_j' a_i \otimes b_j.$$

This shows that $\mathfrak{h} \otimes_{U(\mathfrak{l})} \mathfrak{f}$ is spanned by the distinguished tensors. From this and Poincaré-Birkhoff-Witt it is straightforward to replace v with an element of $U(\mathfrak{h})$ and w with an element of $U(\mathfrak{f})$, at cost of a lot of complicated multilinear algebra. Linear independence seems a bit more complicated – I think that the way to do it is to show that the image of the spanning set of distinguished vectors is linearly independent (being a bunch of spanning monomials). □

Proposition 2.6. *Let $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ be Lie subalgebras of $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_n$. There exists one and only one linear mapping $f : U(\mathfrak{g}_1) \otimes \ldots \otimes U(\mathfrak{g}_n) \to U(\mathfrak{g})$ satisfying $f(u_1 \otimes \ldots \otimes u_n) = u_1 \ldots u_n$ for $u_1, \ldots, u_n$ in respective summands. The map f is bijective.*

Proof. This is a corollary to Proposition 2.5, using induction and the fact that $U(0) = \mathbb{C}$. □

Remark 2.3. The isomorphism from Proposition 2.6 is called canonical, identifying the two vector spaces. If (and only if) $\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_n$ is a Lie algebra direct sum, then f is an algebra isomorphism.

Corollary 2.1 ([2, Cor. 2.2.12])**.** *Let $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ be Lie algebras and $\mathfrak{g}$ their product. The canonical vector space isomorphism $f : U(\mathfrak{g}_1) \otimes \ldots \otimes U(\mathfrak{g}_n)$ is an algebra isomorphism.*

Proof. In a tensor product $(a \otimes 1)(1 \otimes b) = (1 \otimes b)(a \otimes 1)$. Thus in order for f to be an algebra map the elements in distinct summands must commute – this is exactly the condition that we have a Lie algebra product. □

Proposition 2.7 ([2, Prop. 2.2.14])**.** *Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$.*

(i) The left ideal R of $U(\mathfrak{g})$ generated by $\mathfrak{h}$ is equal to the right ideal of $U(\mathfrak{g})$ generated by $\mathfrak{h}$.

(ii) Let $j : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ be the Lie algebra quotient. The homomorphism $U(j) : U(\mathfrak{g}) \to U(\mathfrak{g}/\mathfrak{h})$ is surjective with kernel R.

Proof. By $U_+(\mathfrak{h})$ we denote the subalgebra of $U(\mathfrak{h})$ generated by $\mathfrak{h}$. Let $(x_1, \ldots, x_m)$ be a basis for a vector space complement of $\mathfrak{h}$ in $\mathfrak{g}$. For $i = 1, \ldots, m$ set $y_i = j(x_i) \in \mathfrak{g}/\mathfrak{h}$. Proposition 2.4 implies that

$$U(\mathfrak{g}) = \sum_{k_1, \ldots, k_m \in \mathbb{N}} x_1^{k_1} \ldots x_m^{k_m} U(\mathfrak{h}).$$

Let $\epsilon : U(\mathfrak{h}) \to \mathbb{C}$ send each v to its constant term (in $\mathbb{C} = U^0$). □

Definition 2.1 (Principal anti-automorphism of $U(\mathfrak{g})$)**.** The map $(u_1 \ldots u_n) \mapsto (-1)^n u_n u_{n-1} \ldots u_1$ is termed the principal anti-automorphism of $U(\mathfrak{g})$. On $\mathfrak{g} \subset U(\mathfrak{g})$ it is given by $x \mapsto -x$; it is the unique unital algebra anti-automorphism of $U(\mathfrak{g})$ which is given by $x \mapsto -x$ on $\mathfrak{g}$. This is denoted by $u \mapsto u^T$ for $u \in U(\mathfrak{g})$.

Definition 2.2 ($\mathfrak{g}$-module structure on $U(\mathfrak{g})$)**.** For all $u \in U(\mathfrak{g})$ let $L(u)$ and $R(u)$ denote the mappings $v \mapsto uv$ and $v \mapsto vu$ of $U(\mathfrak{g})$ into itself. The associativity of $U(\mathfrak{g})$ implies that L and R are both algebra representations, termed the *left* and *right regular representations* of $U(\mathfrak{g})$. The mapping $\mathfrak{g} \ni x \mapsto L(x) \in \mathrm{End}(U(\mathfrak{g}))$ is called the left regular representation of $\mathfrak{g}$. The mapping $u \mapsto R(u^T)$ is an algebra representation (check) and the restriction $\mathfrak{g} \ni x \mapsto -R(x)$ is termed the right regular representation of $\mathfrak{g}$ in $U(\mathfrak{g})$.

For all $u \in U(\mathfrak{g})$ it is immediate from associativity that $R(u)L(u) = L(u)R(u)$ for any u. The mapping $\mathfrak{g} \ni x \mapsto L(x) - R(x)$ is a Lie representation of $\mathfrak{g}$, which we'll denote $\mathfrak{a}$. This is nothing but the adjoint representation of the Lie algebra $U(\mathfrak{g})_-$ restricted to the Lie subalgebra $\mathfrak{g}$.

Definition 2.3. Let $n \geq 0$ be an integer. The vector subspace of $U(\mathfrak{g})$ spanned by the products $x_1 x_2 \ldots x_n$, where $x_1, \ldots, x_n \in \mathfrak{g} \cup \{1\}$, is denoted by $U_n(\mathfrak{g})$. (In the case $n = 0$, we only take the empty product, which equals $1 \in U(\mathfrak{g})$, and generates $\mathbb{C}$.)

The following claims are stated without proof in [2, 2.3.1].

Lemma 2.1. *Let $U_n(\mathfrak{g})$ be as in Definition 2.3.*

(i) For all $n \geq 0$ we have $U_n(\mathfrak{g}) \subset U_{n+1}(\mathfrak{g})$.

(ii) The union $\bigcup_{n=0}^{\infty} U_n(\mathfrak{g}) = U(\mathfrak{g})$.

(iii) The first part $U_1(\mathfrak{g})$ equals $\mathbb{C} \oplus \mathfrak{g}$.

(iv) There is a grading-like structure $U_n(\mathfrak{g})U_p(\mathfrak{g}) \subset U_{n+p}(\mathfrak{g})$.

Definition 2.4. The sequence $(U_n(\mathfrak{g}))_{n \geq 0}$ is called the *canonical filtration* of $U(\mathfrak{g})$. If $u \in U(\mathfrak{g})$ then the smallest integer n such that $u \in U_n(\mathfrak{g})$ is called the *filtration* of u.

Remark 2.4. If $(e_1, \ldots, e_r)$ is a basis for $\mathfrak{g}$ then the elements of the form $e_1^{k_1} \ldots e_r^{k_r}$ with $\sum k_j \leq n$ together form a basis for $U_n(\mathfrak{g})$. This follows by expanding the $x_1, \ldots, x_n$ relative to the basis and expanding out linearly.

Definition 2.5 (Graded algebra G)**.** Let $G^n = U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$ the quotient vector space and define $G = G^0 \oplus G^1 \oplus \ldots$. Convention: $U_{-1}(\mathfrak{g}) = 0$, so that $G^0 = \mathbb{C}$. The multiplication in $U(\mathfrak{g})$ defines a bilinear mapping $G^m \times G^n \to G^{m+n}$. This endows G with the structure of a unital associative algebra.

Lemma 2.2. *The algebra G is commutative.*

Proof. It is clear that $G^1 = \mathfrak{g}$ generates G as an algebra, and that for $g_1, g_2 \in \mathfrak{g}$ we have $g_1 g_2 \in U^2$ and $g_1 g_2 - g_2 g_1 = [g_1, g_2] \in U^1$. Thus, in the quotient G^2 we have $g_1 g_2 = g_2 g_1$. Similar reasoning shows that this holds for all summands, using Lemma 1.2. ☐

Remark 2.5. The symmetric algebra of $\mathfrak{g}$ is universal for linear maps into commutative algebras. Hence we obtain a (unique) homomorphism $\phi : S(\mathfrak{g}) \to G$ satisfying $\phi(1) = 1$. This is referred to as the *canonical homomorphism.* If $S^n(\mathfrak{g})$ denotes the subspace of degree n homogeneous elements, then $\phi(S^n(\mathfrak{g})) \subset G^n$.

Proposition 2.8 ([2, Prop. 2.3.6])**.** *The canonical homomorphism $\phi : S(\mathfrak{g}) \to G$ is an isomorphism.*

Proof. This is really straightforward from Definition 2.3. The basis elements in $S^n(\mathfrak{g})$ have the form

$$e_{k_1} \otimes \ldots \otimes e_{k_n}$$

for a sequence $k_1 \leq k_2 \leq \ldots \leq k_n$. The map ϕ sends such an element to the class of $e_{k_1} \ldots e_{k_n}$, one of the basis elements for U_n that does not belong to U_{n-1}. $\square$

Definition 2.6. A ring is called *Noetherian* if it satisfies the maximal condition for left ideals and right ideals. That is, every nonempty set of left ideals contains a maximal left ideal, and similarly for right ideals.

The following result is not too hard to prove, but we omit the proof here.

Theorem 2.1 ([5, Thm. 6.9])**.** *If S is a filtered ring and the associated graded ring $\operatorname{gr} S$ is right Noetherian, then S is right Noetherian.*

Remark 2.6. By considering opposite rings we obtain the corresponding result for left Noetherian rings.

Definition 2.7. A (possibly noncommutative) algebra is said to be Noetherian if it satsifies the asceding chain condition for left ideals and for right ideals.

Corollary 2.2. *The universal enveloping algebra $U(\mathfrak{g})$ of a finite-dimensional Lie algebra $\mathfrak{g}$ is Noetherian.*

Proof. The graded algebra G is commutative and generated by a finite set of elements (namely those corresponding to a basis for $\mathfrak{g}$). Thus the Hilbert basis theorem implies that G is Noetherian and then Theorem 2.1 and the remark which follows it together imply that $U(\mathfrak{g})$ is Noetherian. $\square$

3 The symmetrization map

Let $n \geq 0$ be an integer, $T^n(\mathfrak{g}) = \underbrace{\mathfrak{g} \otimes \ldots \otimes \mathfrak{g}}_{n\text{-fold}}$, let $S^n(\mathfrak{g})$ be the set of homoegeneous elements of degree n in the symmetric algebra $S(\mathfrak{g})$, and let $G^n(\mathfrak{g}) = U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$. Consider the diagram below:

$$\begin{array}{ccc} T^n(\mathfrak{g}) & \xrightarrow{\psi_n} & U_n(\mathfrak{g}) \\ \downarrow{\scriptstyle \tau_n} & & \downarrow{\scriptstyle \theta_n} \\ S^n(\mathfrak{g}) & \xrightarrow{\phi_n} & G^n(\mathfrak{g}) \end{array}$$

Lemma 3.1 ([2, Lem. 2.4.2])**.** *The diagram 3.2 commutes.*

Proof. Let $x_1, \ldots, x_n \in \mathfrak{g}$. Then $\psi_n(x_1 \otimes \ldots \otimes x_n) = x_1 \ldots x_n$ in $U(\mathfrak{g})$, and hence $\theta_n(\psi_n(x_1 \otimes \ldots \otimes x_n)) = x_1 \ldots x_n$ in $G = G^0(\mathfrak{g}) \oplus G^1(\mathfrak{g}) \oplus \ldots$. Similarly, $\tau_n(x_1 \otimes \ldots \otimes x_n) = x_1 \ldots x_n \in S(\mathfrak{g})$, hence $\phi_n(\tau_n(x_1 \otimes \ldots \otimes x_n)) = x_1 \ldots x_n \in G$. $\square$

Remark 3.1. This proof doesn't say much, does it?

Definition 3.1. An element of $U(\mathfrak{g})$ is said to be *symmetric homogeneous of degree* n if it is the canonical image in $U(\mathfrak{g})$ of a homogeneous symmetric tensor of degree n over $\mathfrak{g}$. [Here the map is ψ_n.] The set of elements of $U(\mathfrak{g})$ which are symmetric homoegeneous of degree n is denoted by $U^n(\mathfrak{g})$.

Proposition 3.1 ([2, Prop. 2.4.4])**.** *There is a direct summand decomposition* $U_n(\mathfrak{g}) = U_{n-1}(\mathfrak{g}) \oplus U^n(\mathfrak{g})$.

Proof. Let us use the notation of Diagram 3.2. Let $T' = T'^n(\mathfrak{g}) \subset T^n(\mathfrak{g})$ be the set of symmetric elements of $T^n(\mathfrak{g})$. Then $\tau_n|_{T'}$ is a bijection onto $S^n(\mathfrak{g})$. By Proposition 2.8 we know that ϕ_n is a bijection. This implies that $\theta_n \circ \psi_n|_{T'}$ is a bijection of T' onto $G^n(\mathfrak{g})$. This implies by the quotient vector space construction that $\psi_n|_{T'}$ is a bijection onto a complement of $U_{n-1}(\mathfrak{g})$ in $U_n(\mathfrak{g})$.

This gives the direct sum decomposition: $U^n = \psi_n(T')$, etc. □

Definition 3.2. The diagram 3.2 combined with the above discussion gives the commutative diagram of bijections

$$\begin{array}{ccc} T'^n(\mathfrak{g}) & \xrightarrow{\psi_n} & U^n(\mathfrak{g}) \\ \Big\downarrow{\scriptstyle \tau_n} & & \Big\downarrow{\scriptstyle \theta_n} \\ S^n(\mathfrak{g}) & \xrightarrow{\phi_n} & G^n(\mathfrak{g}) \end{array}$$

This gives us a bijection ω_n, termed *canonical*, of $S^n(\mathfrak{g})$ onto $U^n(\mathfrak{g})$. This is defined on the basis via

$$\omega_n(x_1 x_2 \dots x_n) = \frac{1}{n!} \sum_{\pi \in S_n} x_{\pi(1)} x_{\pi(2)} \dots x_{\pi(n)}.$$

Remark 3.2. I don't really see where the canonical homomorphism comes from.

Definition 3.3 (Following [3])**.** Let V be a vector space and A an associative algebra. Say that a linear map $f : S(V) \to A$ is a *symmetrization map* if $f(x^n) = f(x)^n$ for all $x \in V$. A (really, the) *universal symmetrization map* with values in an associative algebra Q is a symmetrization map $g : S(V) \to Q$ such that if $f : S(V) \to A$ is another symmetrization map, then there is an associative algebra map $h : Q \to A$ so that $h \circ g = f$.

Remark 3.3. The uniqueness (up to isomorphism) of a universal symmetrization map is standard stuff.

Some simple calculations:

$$f((x+y)^2) = f(x+y)^2 = (f(x)+f(y))^2 = f(x)^2 + f(x)f(y) + f(y)f(x) + f(y)^2$$

but also

$$f((x+y)^2) = f(x^2) + f(xy) + f(yx) + f(y^2).$$

But $xy = yx$ in the symmetric algebra $S(V)$, hence after some cancellation we obtain

$$2f(xy) = f(x)f(y) + f(y)f(x),$$

whence $f(xy) = \frac{1}{2}(f(x)f(y) + f(y)f(x))$.

Remark 3.4. The associative algebra that is the codomain of the symmetrization map is almost never commutative, so we don't expect f to be an algebra homomorphism.

Now let $x_1, \ldots, x_n$ be a basis for V and let $t_1, \ldots, t_n$ be scalar indeterminates. If $j : S(V) \to A$ is a symmetrization map then

$$j((t_1x_1 + \ldots + t_nx_n)^n) = j(t_1x_1 + \ldots + t_nx_n)^n.$$

Both of these are polynomials in the indeterminates $t_1, \ldots, t_n$ with coefficients in A. The coefficient in front of $t_1 \cdot \ldots \cdot t_n$ on the left-hand side is $n! \cdot j(x_1 \ldots x_n)$, whereas on the right it is

$$\sum_{\pi \in S_n} j(x_{\pi(1)}x_{\pi(2)} \ldots x_{\pi(n)}).$$

This finally gives the formula

$$j(x_1 \ldots x_n) = \frac{1}{n!} \sum_{\pi \in S_n} j(x_{\pi(1)}) \ldots j(x_{\pi(n)}).$$

4 Existence of finite-dimensional representations

Lemma 4.1 ([2, Lem. 2.5.1])**.** *Let $I_1, \ldots, I_m$ be right (or left) ideals of finite codimension in $U(\mathfrak{g})$. Then the product ideal $I_1 I_2 \ldots I_m$ has finite codimension.*

Proof. Induction allows us to focus on the case with $m = 2$ right ideals. The right $U(\mathfrak{g})$-module I_1 is generated by a finite set of elements $U_1, \ldots, u_p$ (because we proved somewhat vaguely that as long as $\mathfrak{g}$ is finite dimensional, the universal enveloping algebra is Noetherian). Let $\nu_1, \ldots, \nu_q$ be elements of $U(\mathfrak{g})$ be elements of $U(\mathfrak{g})$ which span a vector space complement of I_2 in $U(\mathfrak{g})$.

Notice that $I_1 = \{u_1, \ldots, u_p\}U(\mathfrak{g})$ and $U(\mathfrak{g}) = I_2 + span\{\nu_1, \ldots, \nu_q\}$. This implies that

$$\begin{aligned} I_1 &= \{u_1, \ldots, u_p\} \cdot U(\mathfrak{g}) \\ &= \{u_1, \ldots, u_p\} \cdot (I_2 + span\{\nu_1, \ldots, \nu_q\} \\ &\subseteq I_1 I_2 + span\{u_i \nu_j\}. \end{aligned}$$

Thus each element of I_1 is congruent modulo $I_1 I_2 I_1 I_2$ to a linear combination of the $U_i \nu_j$. We hav ea chain of vector space inclusions: $I_1 I_2 \subset I_1 \subset U(\mathfrak{g})$. Consequently

$$\dim(U(\mathfrak{g})/I_1 I_2) = \dim(U(\mathfrak{g})/I_1) + \dim(I_1/I_1 I_2) < \infty.$$

Here the first codimension is finite because I_1 is assumed to have finite codimension, and the second is finite because we have given a finite spanning set for the vector space $I_1/I_1 I_2$, viz. $\{u_i \nu_j\}$. □

Remark 4.1. The fact that we are working with the universal enveloping algebra of a Lie algebra is not really being used here. All that is needed is that $U(\mathfrak{g})$ is a Noetherian algebra.

Lemma 4.2 ([2, Lem. 1.4.5]). *Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$, V a finite-dimensional vector space, $\rho : \mathfrak{g} \to \mathrm{End}(V)$ a simple representation such that each element of $\rho(\mathfrak{a})$ is nilpotent. Then $\rho(\mathfrak{a}) = 0$.*

Proof. Set $W = \{v \in V : \rho(\mathfrak{a})v = 0\}$. Engel's theorem implies that W is non-trivial. Claim: W is $\rho(\mathfrak{g})$-invariant. Let $g \in \mathfrak{g}$, $a \in \mathfrak{a}$, and $v \in W$. Then

$$\begin{aligned}\rho(a)(\rho(g)v) &= \rho(g)\rho(a)v + [\rho(a), \rho(g)]v \\ &= 0 + \rho([a, g])v \\ &= 0,\end{aligned}$$

where the first 0 occurs from the fact that $v \in W$ and the second comes from the fact that $[a, g] \in \mathfrak{a}$. □

Definition 4.1 ([2, Defn. 1.2.6]). Let $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be a representation. A sequence $(V_0, V_1, \ldots, V_n)$ of sub-$\mathfrak{g}$-modules of V so that $V = V_0 \supset V_1 \supset \ldots \supset V_n = 0$ is termed a *composition series* of ρ (or of the $\mathfrak{g}$-module V). Such a composition series is termed a *Jordan-Holder series* if each quotient V_i/V_{i+1} is simple as a $\mathfrak{g}$-module.

Following isn't stated in [2]. The first statement is standard (the length of a composition series for V is bounded by the dimension of V, so take a composition series of maximum length), the second part isn't too tricky but I'm too lazy to go through it.

Lemma 4.3. *If V is a finite dimensional $\mathfrak{g}$-module, then we can always obtain a Jordan-Holder series. Moreover any two Jordan-Holder series are isomorphic up to permutation of quotients (in the same sense as the Jordan-Holder theorem from theory of finite groups).*

The following lemma isn't stated in [2], but it's useful to state.

Lemma 4.4. *Let $\sigma : \mathfrak{g} \to \mathrm{End}(V)$ be a finite-dimensional representation and*

$$V = V_0 \supset V_1 \supset \ldots \supset V_n \supset 0$$

be a Jordan-Holder series for the $\mathfrak{g}$-module V. Suppose that the action of $\mathfrak{g}$ on each quotient V_i/V_{i+1} is via nilpotent operators. Then the action of $\mathfrak{g}$ on V is via nilpotent operators.

Proof. Let $v \in V$ and let $g \in \mathfrak{g}$. Then the assumption gives us for each j an integer k_j so that $(\sigma(g))^{k_j} V_j \subset V_{j+1}$. Set $k = \sum_{j=0}^{n} k_j$ and notice that $(\sigma(g))^k v = 0$. Thus each operator $\sigma(g)$ is □

Lemma 4.5 ([2, Lem. 2.5.2]). *Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$, $\mathfrak{b}$ a vector subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$, and $\sigma : \mathfrak{g} \to \mathrm{End}(V)$ a finite-dimensional representation of $\mathfrak{g}$. Assume that $\sigma(x)$ is nilpotent for all $x \in \mathfrak{a} \cup \mathfrak{b}$. Then $\sigma(x)$ is nilpotent for all $x \in \mathfrak{g}$.*

Proof. Using Lemma 4.3 we write

$$V = V_0 \supset V_1 \supset \ldots \supset V_n = 0$$

where each V_i is stable under the action of $\mathfrak{g}$ and each quotient V_i/V_{i+1} is simple. This means that the action of $\mathfrak{a}$ is trivial on V_i/V_{i+1}; thus, letting σ_i denote the representation of $\mathfrak{g}$ on V_i/V_{i+1}, we obtain $\sigma_i(\mathfrak{g}) = \sigma_i(\mathfrak{a}) + \sigma_i(\mathfrak{b}) = \sigma_i(\mathfrak{b})$. As the action of $\mathfrak{b}$ on V is via nilpotent operators, the action of $\mathfrak{b}$ on V_i/V_{i+1} is also by nilpotent operators (this is the easy direction). Thus the action of $\mathfrak{g}$ on V_i/V_{i+1} is via nilpotent operators. Then Lemma 4.4 implies that the action of $\mathfrak{g}$ on V is via nilpotent operators. □

Lemma 4.6 ([2, Lem. 2.5.3]). *Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$, $\mathfrak{b}$ a Lie subalgebra of $\mathfrak{b}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ (vector space direct sum), π the left regular representation of $\mathfrak{a}$ in $U(\mathfrak{a})$ (which is a Lie algebra homomorphism!), and ϕ the adjoint representation of $\mathfrak{g}$ in $U(\mathfrak{a})$. The linear mapping ψ of $\mathfrak{g}$ into $\mathrm{End}(U(\mathfrak{a}))$ such that $\psi|_{\mathfrak{a}} = \pi$ and $\psi|_{\mathfrak{b}} = \phi|_{\mathfrak{b}}$ is a representation of $\mathfrak{g}$.*

Proof. The map ψ is well-defined and linear because of the vector space decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$. It's necessary to prove that $\psi([x, y]) = [\psi(x), \psi(y)]$ for any $x, y \in \mathfrak{g}$. Since ψ is a representation when restricted to either of the summands in the direct sum decomposition, it's sufficient to show that $\psi([x, y]) = [\psi(x), \psi(y)]$ if $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$.

Let $u \in U = U(\mathfrak{a})$. Then

$$\begin{aligned}
[\psi(x), \psi(y)]u &= \psi(x)\psi(y)u - \psi(y)\psi(x)u \\
&= x(yu - uy) - (yxu - xuy) \\
&= xyu - yxu \\
&= [x, y]u \\
&= \psi([x, y])u,
\end{aligned}$$

where in the last line we've used the fact that $\mathfrak{a}$ is an ideal. □

4.1 Nilpotency ideals

It's pretty standard to learn about the largest solvable ideal of a Lie algebra (the radical); the corresponding construction for nilpotent ideals is a little less standard.

Lemma 4.7 ([2, Lem. 1.4.6]). *Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$, $\sigma : \mathfrak{g} \to \mathrm{End}(V)$ a finite-dimensional representation, and $V_0 \supset V_1 \supset \ldots \supset V_n$ a J-H series for V. The following are equivalent:*

(i) for every $x \in \mathfrak{a}$ the operator $\sigma(x)$ is nilpotent;

(ii) for every $x \in \mathfrak{a}$ and $k = 0, \dots, n-1$ we have $\sigma(x)V_k \subset V_{k+1}$.

Proof. That (ii) implies (i) is a result of Lemma 4.4. That (i) implies (ii) is a result of Lemma 4.2. □

Proposition 4.1 ([2, Prop. 1.4.7])**.** *Let $\sigma : \mathfrak{g} \to Endo(V)$ be a finite-dimensional representation, and $b(x, y) = \mathrm{Tr}(\sigma(x) \circ \sigma(y))$ be the bilinear form associated with σ.*

(i) Among the ideals $\mathfrak{a}$ of $\mathfrak{g}$ such that $\sigma(x)$ is nilpotent for every $x \in \mathfrak{a}$ there is a largest such ideal, denoted $\mathfrak{n}$.

(ii) Let $(V_0, V_1, \dots, V_n)$ be a J-H series of V and σ_i be the representation induced on V_i/V_{i+1}. Then

$$\mathfrak{n} = \bigcap_{i=0}^{n-1} \ker \sigma_i.$$

(iii) $\mathfrak{n}$ is orthogonal to $\mathfrak{g}$ with respect to b (i.e. $b(x, y) = 0$ if $x \in \mathfrak{n}$ and $y \in \mathfrak{g}$)

Proof. Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$. From Lemma 4.7 it is clear that $\sigma(x)$ is nilpotent for each $x \in \mathfrak{a}$ if and only if $x \in \bigcap_{i=0}^{n-1} \ker \sigma_i$.

This immediately gives that $\mathfrak{n} = \bigcap_{i=0}^{n-1} \ker \sigma_i$. Claim: if $x \in \mathfrak{n}$ and $y \in \mathfrak{g}$ then $\sigma(x)\sigma(y)$ is nilpotent. To verify this, we only need to check that $\sigma(x)\sigma(y)V_i \subset V_{i+1}$ for $i = 0, \dots, n-1$. But $\sigma(x)\sigma(y)V_i \subset \sigma(x)V_i \subset V_{i+1}$, using the fact that $\sigma(x)$ is nilpotent and V_i is stable under the action of $\mathfrak{g}$. This establishes that $\sigma(x)\sigma(y)$ is nilpotent and so has trace equal to 0. □

Definition 4.2. The ideal $\mathfrak{n}$ in Proposition 4.1 is termed the *largest nilpotency ideal* of σ.

Proposition 4.2 ([2, Prop. 1.4.9])**.** *Let $\mathfrak{n}$ be the largest nilpotency ideal of the adjoint representation of $\mathfrak{g}$ (on itself). Then $\mathfrak{n}$ is the largest nilpotent ideal of $\mathfrak{g}$. It is orthogonal to $\mathfrak{g}$ with respect to the Killing form.*

Proof. Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$; first we establish that $\mathfrak{a}$ is nilpotent (as a Lie algebra) if and only if for all $x \in \mathfrak{a}$ the operator $\mathrm{ad}_{\mathfrak{g}}(x) \in \mathrm{Endo}(\mathfrak{g})$ is nilpotent. If $\mathfrak{a}$ is nilpotent, then each $\mathrm{ad}_{\mathfrak{a}}(x)$ is nilpotent for $x \in \mathfrak{a}$, so that for each x we can find k satisfying $\mathrm{ad}_{\mathfrak{a}}(x)^k = 0$. Because $\mathfrak{a}$ is an ideal we have that $\mathrm{ad}_{\mathfrak{g}}(x)$ maps $\mathfrak{g}$ into $\mathfrak{a}$; hence $\mathrm{ad}_{\mathfrak{a}}(x)^{k+1} = 0$. The other direction is just Engel's theorem (restricting the adjoint operators to $\mathfrak{a}$).

Thus for an ideal $\mathfrak{a}$ nilpotency and $\mathrm{ad}_{\mathfrak{g}}$-nilpotency coincide. The largest nilpotency ideal of the adjoint representation of $\mathfrak{g}$ is therefore the largest nilpotent ideal of $\mathfrak{g}$. The last bit follows from the last bit of Prop 4.1. □

Lemma 4.8 ([2, Lem. 2.5.4])**.** *Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$, $\mathfrak{b}$ a Lie subalgebra of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$, V a finite-dimensional vector space and $\rho : \mathfrak{a} \to \mathrm{End}(V)$ a representation whose largest nilpotency ideal $\mathfrak{n}$ contains $[\mathfrak{b}, \mathfrak{a}]$.*

(i) *There exists a finite-dimensional representation $\sigma : \mathfrak{g} \to Endo(W)$ of $\mathfrak{g}$, whose largest nilpotency ideal contains $\mathfrak{n}$, such that ρ is a quotient representation of $\sigma|_{\mathfrak{a}}$.*

(ii) *If, for all $y \in \mathfrak{b}$, $\mathrm{ad}_{\mathfrak{g}}\, y$ acts as a nilpotent operator on $\mathfrak{a}$, then we can choose σ so that, in addition, the largest nilpotency ideal of σ contains $\mathfrak{b}$.*

Remark 4.2. A quotient representation means a representation of the form W/W' for some $\mathfrak{a}$-submodule $W' \subset W$. Alternatively the range of a surjective $\mathfrak{a}$-module map $W \to Q$.

Proof of Lemma 4.8. As earlier set $U = U(\mathfrak{a})$. Let $V^1, \ldots, V^r$ be sub-$\mathfrak{a}$-modules of V, with sum V, such that each V^i is generated as a U-module by a single element. The map $V^1 \oplus V^2 \oplus \ldots \oplus V^r \to V$ given by $(v_1, \ldots, v_r) \mapsto v_1 + v_2 \ldots, v_r$ is a surjective $\mathfrak{a}$-module map.

Claim: it therefore suffices to prove the Lemma in the case where V is cyclic (Dixmier uses the term *monogeneous*). The direct sum of modules $W^1 \oplus \ldots \oplus W^r$ has nilpotency ideal equal to the intersection $\mathfrak{n}_1 \cap \ldots \cap \mathfrak{n}_r$, where $\mathfrak{n}_k$ is the nilpotency ideal of W^k. If each of these contained $\mathfrak{n}$, then so to would the intersection. Furthermore, if V^k were a quotient representation of W^k for each k, then $V^1 \oplus \ldots \oplus V^r$ would be a quotient of $W^1 \oplus \ldots \oplus W^r$. By the forgoing this would imply that V is a quotient of $W^1 \oplus \ldots \oplus W^r$. The condition (ii) is similarly satisfied when the adjoint action of $\mathfrak{b}$ restricted to $\mathfrak{a}$ is nilpotent. This establishes the claim.

Henceforth require that V is cyclic as a U-module, with $v \in V$ a cyclic vector. Let I be the kernel in U of the representation ρ; as $U/I = \rho(U) \subset End(V)$, and V is finite dimensional, we see that I has finite codimension. Equip U with the left regular representation $\mathfrak{a}$-module structure. Consider the map $U \to V$ given by $u \mapsto uv$. This map vanishes on I, so we obtain a well-defined map $U/I \to V$; this map is surjective because v is cyclic for the U-module V. Thus V is a quotient of the $\mathfrak{a}$-module U/I.

Let $(V_0, V_1, \ldots, V_n)$ be a J-H series for the $\mathfrak{a}$-module V. Let ρ_i be the representation of $\mathfrak{a}$ via ρ induced on V_i/V_{i+1}. Let I' be the intersection of the kernels in U of all the ρ_i.

Claim: $I'^n \subset I \subset I'$ and $I' \cap \mathfrak{a} = \mathfrak{n}$. Proof of claim: let $x = j_1 j_2 \ldots j_n$ be a spanning element in I'^n, so that each $j_k \in I'$. Then $j_1 j_2 \ldots j_n V_0 \subset j_1 j_2 \ldots j_{n-1} V_1$, and $j_1 j_2 \ldots j_{n-1} V_1 \subset V_2$, so that eventually we get $xV_0 = V_n = 0$. Thus $x \in I$. The fact that $I \subset I'$ is immediate: if $\rho(x) = 0$, then $\rho_i(x) = 0$ on each quotient representation V_i/V_{i+1}. If $x \in \mathfrak{n}$, then x belongs to $\mathfrak{a}$ by definition and it belongs to I' by Proposition 4.1, so that $\mathfrak{n} \subset \mathfrak{a} \cap I'$. The converse is basically the same, so the claim is established.

Equip U with the $\mathfrak{g}$-module structure as in Lemma 4.6 so that for $x \in \mathfrak{a}$, $y \in \mathfrak{b}$, and $u \in U$, we have $x.u = xu$, $y.u = yu - uy$. If $x \in \mathfrak{b}$, then $\phi(x)$ (adjoint action of $\mathfrak{g}$ on U) is a derivation of U carrying $\mathfrak{a}$ into $[\mathfrak{b}, \mathfrak{a}]$. Moreover, by hypothesis, $[\mathfrak{b}, \mathfrak{a}] \subset \mathfrak{n}$ and hence $[\mathfrak{b}, \mathfrak{a}] \subset I'$ by the previous claim. We also have that I' and I'^n are invariant under the action of $\phi(x)$: if $y \in I'$ and

$i = 0, \dots, n-1$ then

$$(xy - yx)V_i \subset xV_{i+1} + yV_i \subset V_{i+1}.$$

Hence $xy - yx$ belongs to $\ker \rho_i$ for all i, implying that I' is invariant under the action of $\phi(x)$. The fact that $\phi(x)$ is a derivation then implies that I'^n is invariant under $\phi(x)$ as well.

Since the kernel of each ρ_i in U is an $\mathfrak{a}$-module, we get that I' is stable under both $\mathfrak{a}$ and $\mathfrak{b}$, hence under all of $\mathfrak{g}$. Similarly I'^n is a sub-$\mathfrak{g}$-module of U. Then the quotient module U/I'^n hosts a representation σ of $\mathfrak{g}$, which has *finite dimension*: as $I \subset I'$, and U/I is finite dimension, the Lemma 4.1 implies that I'^n has finite codimension.

If $x \in I' \cap \mathfrak{a} = \mathfrak{n}$, then $x^n U \in I'^n$. This means that $\sigma(x^n) = \sigma(x)^n$ acts trivially on the module U/I'^n under σ. So $\sigma(x)$ is a nilpotent operator, and $\mathfrak{n}$ is contained in the largest nilpotency ideal of σ. The inclusion $I'^n \subset I$ implies that we have surjective $\mathfrak{a}$-module maps $U/I'^n \to U/I \to V$. Thus V is a quotient representation of $\sigma|_{\mathfrak{a}}$. This establishes part (i) of the Lemma.

Now suppose that $\operatorname{ad}_{\mathfrak{g}} y|_{\mathfrak{a}}$ is nilpotent for every $y \in \mathfrak{b}$. Claim: for all $y \in \mathfrak{b}$ the operator $\phi(y)$ acts locally nilpotently on U. This is due to the fact, that if $D : \mathfrak{a} \to \mathfrak{a}$ is a nilpotent derivation, then the canonical extension to a derivation $D' : U \to U$ is locally nilpotent. This fact is a little tedious to prove, but should be plausible from [2, Prop. 2.4.9] and an induction argument. Claim: $\sigma(y)$ is nilpotent. This follows because $\sigma(y)$ acts via $\phi(y)$ on the finite dimensional module U/I'^n, and a locally nilpotent transformation of a finite dimensional module must be nilpotent. By the equation $\mathfrak{n} = I' \cap \mathfrak{a}$ and the hypothesis we obtain $[\mathfrak{b}, \mathfrak{n}] \subset [\mathfrak{b}, \mathfrak{a}] \subset \mathfrak{n}$.

By definition of σ, we have that for $y \in \mathfrak{b}$ and $n \in \mathfrak{n}$ the composition $\sigma(y)\sigma(n) = \sigma([y, n])$. This and nilpotency of $\mathfrak{n}$ and $\sigma(y)$ implies that $\sigma(y + n)$ is nilpotent as well, similar to the proof that the set of nilpotent elements in a ring is closed under addition. This shows that $\mathfrak{b} + \mathfrak{n}$ acts via nilpotent operators under σ; to show that it is contained in the largest nilpotency ideal for σ, we must show that it is an ideal. The equation $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ implies that

$$[\mathfrak{g}, \mathfrak{b} + \mathfrak{n}] \subset [\mathfrak{b}, \mathfrak{b}] + [\mathfrak{b}, \mathfrak{a}] + [\mathfrak{a}, \mathfrak{n}] \subset \mathfrak{b} + \mathfrak{n},$$

where we've used the fact that $[\mathfrak{b}, \mathfrak{n}] \subset [\mathfrak{b}, \mathfrak{a}] \subset \mathfrak{n}$ as well as the ideal property of $\mathfrak{n}$. This verifies that $\mathfrak{b} + \mathfrak{n}$ is an ideal of $\mathfrak{g}$, and since we have already verified that it acts via nilpotent operators under σ, it must be contained in the the largest nilpotency ideal of σ. □

The following theorem seems like a technical refinement of Ado's theorem (every finite-dimensional Lie algebra has a faithful finite-dimensional representation), but Dixmier cites it to Bourbaki, Harish-Chandra, and Jacobson (shrug emoji).

Theorem 4.1 ([2, Thm. 2.5.5])**.** *Let* $\mathfrak{n}$ *be the largest nilpotent ideal of* $\mathfrak{g}$*. There is a finite-dimensional injective representation* ρ *of* $\mathfrak{g}$ *whose largest nilpotency ideal contains* $\mathfrak{n}$*.*

Remark 4.3. The proof proceeds in stages: first you produce the representation for the center $\mathfrak{c}$ of $\mathfrak{g}$ (i.e. the largest commutative ideal), then you extend to $\mathfrak{n}$ (i.e. the largest nilpotent ideal), then you extend to $\mathfrak{r}$ (the largest solvable ideal), then you invoke Levi's theorem to extend it to all of $\mathfrak{g}$. Thennn you show that this is faithful on the center, and then you direct sum with the adjoint representation to get something that's faithful on the entire algebra.

Theorem 4.2 (Levi decomposition, [2, Thm. 1.6.9])**.** *Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$. Then there exists a Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ as a vector space direct sum.*

We also need a little extra tidbit, which we've reprhased a bit.

Lemma 4.9 ([2, Prop. 1.7.1])**.** *If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{r}$ is its radical, then $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{r} = [\mathfrak{g}, \mathfrak{r}]$, and the ideal $[\mathfrak{g}, \mathfrak{r}]$ is nilpotent.*

Proof of Theorem 4.1. The 1-dimensional Lie algebra k has the representation

$$\lambda \mapsto \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}.$$

This is injective and every operator acts nilpotently. Thus if $\mathfrak{c} \subset \mathfrak{g}$ is the center, we can produce a faithful finite-dimensional representation of $\mathfrak{c}$ via nilpotent operators by direct summing suitably many copies of this representation together. Refer to this representation as ϕ.

The inclusion of the ideal $\mathfrak{c} \subset \mathfrak{n}$ is immediate as abelian Lie algebras are nilpotent; if we take a J-H series for $\mathfrak{n}/\mathfrak{c}$ then all the subquotients V_i/V_{i+1} must be 1-dimensional as they are simple modules under the action of a nilpotent Lie algebra (see Engel's theorem). Lift this under the correspondence between ideals of $\mathfrak{n}/\mathfrak{c}$ and ideals of $\mathfrak{n}$ that contain $\mathfrak{c}$ to obtain a chain

$$\mathfrak{c} = \mathfrak{n}_0 \subset \mathfrak{n}_1 \subset \ldots \subset \mathfrak{n}_p = \mathfrak{n}$$

and moreover each codimension equals 1. Each $\mathfrak{n}_i$ is the vector space sum of $\mathfrak{n}_{i-1}$ and a 1-dimensional subspace (any 1-dimensional subspace is automatically a Lie subalgebra). We have a faithful representation of $\mathfrak{c}$ via nilpotent operators, say $\sigma_{\mathfrak{c}} : \mathfrak{c} \to End(V_{\mathfrak{c}})$. Since $\mathfrak{c}$ is an ideal of $\mathfrak{n}_1$, and all the other conditions of Lemma 4.8 are satisfied, we can extend σ to an action of $\mathfrak{n}_1$ on some $V_{\mathfrak{n}_1}$ which surjects onto $V_{\mathfrak{c}}$ via a $\mathfrak{c}$ map. Repeat this over and over to obtain a finite-dimensional representation ψ of $\mathfrak{n}$ via nilpotent operators so that ϕ is a quotient of $\psi|_{\mathfrak{c}}$. (The possibly non-obvious bit to verify among the hypotheses of Lemma 4.8 is to show that the action of $\mathfrak{n}_i$ on $\mathfrak{n}_{i-1}$ is nilpotent, but this follows from the fact that $\mathfrak{n}$ is nilpotent.)

Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$. It's a general fact that $[\mathfrak{r}, \mathfrak{r}]$ is a nilpotent ideal of $\mathfrak{g}$ (true if you replace $\mathfrak{r}$ by any solvable ideal), and hence $[\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{n} \subset \mathfrak{r}$. Lifting a J-H series for $\mathfrak{r}/\mathfrak{n}$, and arguing similarly to the previous paragraph but with solvable in place of nilpotent, we get a sequence of ideals

$$\mathfrak{n} = \mathfrak{r}_0 \subset \mathfrak{r}_1 \subset \ldots \subset \mathfrak{r}_q = \mathfrak{r}$$

so that each $\mathfrak{r}_i/\mathfrak{r}_{i-1}$ has dimension 1 (by Lie's theorem). Just as in the previous paragraph, each algebra $\mathfrak{r}_i$ is the (vector space) direct sum of $\mathfrak{r}_{i-1}$ and a 1-dim Lie subalgebra. The equation $[\mathfrak{r},\mathfrak{r}] \subset \mathfrak{n}$ ensures that the hypotheses of part (i) of Lemma 4.8 are again satisfied for each $i = 0, \ldots, q-1$. So we can extend a representation τ of $\mathfrak{r}$ so that $\tau(\mathfrak{n})$ consists of nilpotent operators and such that ψ is a quotient of $\tau|_{\mathfrak{n}}$.

Now we can take a Levi decomposition (Theorem 4.2) $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$. The tidbit implies that $[\mathfrak{g},\mathfrak{r}]$ is a nilpotent ideal and hence contained in $\mathfrak{n}$; hence the subalgebra $[\mathfrak{s},\mathfrak{r}]$ is contained in $\mathfrak{n}$ as well. Invoking Lemma 4.8 gives a representation σ of $\mathfrak{g}$ such that every element of $\sigma(\mathfrak{n})$ is nilpotent and such that τ is a quotient of $\sigma|_{\mathfrak{r}}$. Let V_1 be the space for ϕ, V_2 the space for ψ, V_3 the space for τ, and V_4 the space for σ. Then we obtain a chain of surjective $\mathfrak{c}$-morphisms $V_4 \to V_3 \to V_2 \to V_1$. If $x \in \mathfrak{c}$ is nonzero, then $xV_4 = 0$ would imply that $xV_1 = 0$, contrary to the assumption that ϕ acts faithfully upon V_1. Thus $\sigma|_{\mathfrak{c}}$ is faithful.

Now let $\rho = \sigma \oplus \mathrm{ad}_{\mathfrak{g}}$, which is a direct sum of finite-dimensional representations and hence finite-dimensional. The kernrel of ρ is equal to $\ker\sigma \cap \ker \mathrm{ad}_{\mathfrak{g}}$, but $\ker \mathrm{ad}_{\mathfrak{g}} = \mathfrak{c}$. Hence ρ is injective. □

Corollary 4.1. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then there is a faithful finite-dimensional representation via traceless operators $\rho : \mathfrak{g} \to \mathfrak{sl}(V)$.*

Proof. Let $\sigma : \mathfrak{g} \to End(W)$ be a faithful finite-dimensional representation from Theorem 4.1. The map σ' defined via $\mathfrak{g} \ni x \mapsto -\mathrm{Tr}_W(\sigma(x))$ is a Lie algebra representation (this boils down to the observation that the trace of a commutator is zero). Now set $\rho = \sigma \oplus \sigma'$, and the trace of $\rho(x)$ is equal to 0 for each $x \in \mathfrak{g}$. □

5 Main result: residual finiteness of $U(\mathfrak{g})$

Now we finally arrive at the theorem that we were originally trying to figure out. I've paraphrased it from Dixmier's phrasing to obtain the version that Luminet uses.

Theorem 5.1 ([2, Thm. 2.5.7])**.** *If $\mathfrak{g}$ is a finite-dimensional Lie algebra, and $U^d(\mathfrak{g})$ is the set of elements of $U = U(\mathfrak{g})$ that are symmetric homogeneous of degree d, then there is a finite-dimensional representation π of $\mathfrak{g}$ such that π is injective when restricted to $U_d(\mathfrak{g})$.*

The following is not stated as a separate result in Dixmier, but I find it's helpful to isolate it.

Lemma 5.1. *Let V be a finite dimensional vector space and let $(x_0, x_1, \ldots, x_p)$ a basis for $End(V)$ such that $x_0 = 1$ (identity transformation). For an integer $d \geq 1$, set $V_d = \underbrace{V \otimes \ldots \otimes V}_{d\text{-fold}}$.*

(i) The tensors $x_{i_1} \otimes x_{i_2} \otimes \ldots \otimes x_{i_d}$, where $0 \le i_1, i_2, \ldots, i_d \le p$, form a basis for $End(V_d)$.

(ii) If we take F to be the vector subspace of $End(V_d)$ spanned by all tensors of the form $u_1 \otimes \ldots u_d$, where $u_1, \ldots, u_d \in End(V)$ and at least one $u_k = 1$, then the set of tensors of the form $x_{i_1} \otimes x_{i_2} \otimes \ldots \otimes x_{i_d}$, where $0 \le i_1, i_2, \ldots, i_d \le p$ and at least one $i_k = 0$

(iii) There is a vector space direct sum decomposition $End(V_d) = F \oplus F'$, where F' has basis consisting of all tensors $x_{i_1} \otimes x_{i_2} \otimes \ldots \otimes x_{i_d}$ such that no $i_k = 0$.

Proof. Part (i) is a fairly standard by dimension counting. Parts (ii) and (iii) are implied immediately by part (i) □

Now we can give a fairly easy proof of Theorem 5.1. Recall that if V and W are $\mathfrak{g}$-modules, then $V \otimes W$ becomes a $\mathfrak{g}$-module via $g.(v \otimes w) = (gv) \otimes w + v \otimes (gw)$.

Proof of Theorem 5.1. Corollary 4.1 and the fact that an inclusion $\mathfrak{g} \subset \mathfrak{h}$ induces $U^d(\mathfrak{g}) \subset U^d(\mathfrak{h})$, we can assume that $\mathfrak{g} = \mathfrak{sl}(V)$ for a finite dimensional vector space V. Let $(x_0, x_1, \ldots, x_p)$ be a basis for $\mathfrak{gl}(V)$ (just $End(V)$) such that $x_0 = 1$ (the identity matrix) and $(x_1, \ldots, x_p)$ is a basis for $\mathfrak{sl}(V)$.

The proof is by induction on d. When $d = 0$, we can take the trivial representation of $\mathfrak{g}$ on k. In this case $U_d = U_0 = k$, and the representation is the identity representation on U_0. So assume that for each $k < d$ we can produce a finite-dimensional representation π such that $\pi(u') \neq 0$ for all $u' \in U_k(\mathfrak{g})$.

Take $u \in U_d(\mathfrak{g}) \setminus U_{d-1}(\mathfrak{g})$; then Lemma 3.1 implies that $u = u_1 + u_2$ where $u_1 \in U^d(\mathfrak{g})$ and $u_2 \in U_{d-1}(\mathfrak{g})$ (recall that the former subspace is spanned by canonical images of symmetric homogeneous tensors of degree d, the latter by images of tensors of degree $\le d-1$). Moreover, $u_1 \neq 0$, and (allowing I_d to be the set of all non-decreasing elements of $\{1, \ldots, p\}^d$) we can write

$$u_1 = \sum_{\bar{i} \in I_d} \alpha_{i_1, \ldots, i_d} \left(\sum_{\tau \in S_d} x_{i_{\tau(1)}} x_{i_{\tau(2)}} \ldots x_{i_{\tau(d)}} \right).$$

This last equation is just the statement that u_1 is the image of a symmetric homogeneous tensor of degree d.

Let σ be the identity representation of $\mathfrak{g} = \mathfrak{sl}(V)$ on V. Set $\sigma_d = \underbrace{\sigma \otimes \ldots \otimes \sigma}_{d\text{-fold}}$.

If $x \in \mathfrak{g}$, then

$$\sigma_d(x) = (x \otimes 1 \ldots \otimes 1) + (1 \otimes x \otimes \ldots \otimes 1) + \ldots + (1 \otimes \ldots \otimes 1 \otimes x).$$

Moreover, if $x_1 \ldots x_k$ is a generating monomial in U_{d-1} then, using the fact that σ_d induces an algebra representation of $U(\mathfrak{g})$, we have

$$\sigma_d(x_1 \ldots x_k) = \sigma_d(x_1)\sigma_d(x_2) \ldots \sigma_d(x_k).$$

If $u_2 \in U_{d-1}$, we get by expanding using the distributive property many times that $\sigma_d(u_2) \in F$. To see this, note that in order to "cover up" all the copies of 1 in each summand of $\sigma_d(x)$ one has to multiply each indeterminate by at least $d-1$ other terms. There aren't enough indeterminates in an element of $U_{d-1}(\mathfrak{g})$ for this to happen.

Claim: $\sigma_n(u_1) \notin F$. To see this, note that (again using the distributive property many times) we have that

$$\sigma_d(x_{i_1} \dots x_{i_d}) = \sum_{\tau \in S_d} x_{i_{\tau(1)}} \otimes x_{i_{\tau(2)}} \otimes \dots x_{i_{\tau(d)}} + f$$

where $f \in F$. To see this, note that "covering" all the 1's in the various $\sigma_d(x_k)$ requires exactly one term from each $\sigma_d(x_k)$ – hence the permutations – and everything else that is multiplied out is an element of F because not all the 1's have been covered.

If $\pi \in S_d$, then we see by re-indexing that

$$\sigma_d(x_{i_{\pi(1)}} x_{i_{\pi(2)}} \dots x_{i_{\pi(d)}}) = \sum_{\tau \in S_d} x_{i_{\tau(1)}} \otimes x_{i_{\tau(2)}} \otimes \dots x_{i_{\tau(d)}} + f_\pi$$

where $f_\pi \in F$. Thus if we sum over all such π we obtain

$$\sigma_d(\sum_{\pi \in S_d} x_{i_{\pi(1)}} x_{i_{\pi(2)}} \dots x_{i_{\pi(d)}}) = \left(n! \sum_{\tau \in S_d} x_{i_{\tau(1)}} \otimes x_{i_{\tau(2)}} \otimes \dots x_{i_{\tau(d)}} \right) + f'$$

where $f' \in F$. The tensors of the form $x_{i_{\tau(1)}} \otimes x_{i_{\tau(2)}} \otimes \dots x_{i_{\tau(d)}}$ are all basis elements of F' (using the terminology of Lemma 5.1), and as we are working in characteristic zero, we see that for any sequence $I = (i_1 \leq i_2 \leq \dots \leq i_d)$ the element $z_I := \sigma_d(\sum_{\pi \in S_d} x_{i_{\pi(1)}} x_{i_{\pi(2)}} \dots x_{i_{\pi(d)}})$ is a nonzero element modulo F. If $I_1, \dots, I_t$ is a collection of distinct such sequences, then the corresponding elements $z_{I_1}, \dots, z_{I_t}$ will be linearly independent modulo F. This last bit follows because different sequences I_k will contribute different basis tensors $x_{i_{\tau(1)}} \otimes \dots \otimes x_{\tau(n)}$ to the various z_k. (I know this last bit is a little hand-wavy, but it's hard to write down precisely.)

Finally we can use

$$u_1 = \sum_{\bar{i} \in I_d} \alpha_{i_1, \dots, i_d} \left(\sum_{\tau \in S_d} x_{i_{\tau(1)}} x_{i_{\tau(2)}} \dots x_{i_{\tau(d)}} \right).$$

and the fact that the set of all z_I is independent modulo F to see that $\sigma_d(u_1) \notin F$, hence $\sigma_d(u) = \sigma_d(u_1) + \sigma_d(u_2) \notin F$, and $\sigma_d(u) \neq 0$. $\square$

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