Essential inclusions

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Graph algebras

$E = (E^0, E^1, r, s)$ directed graph. $C^*(E) = C^*(\{s_e, p_v\})$ such that

(i) p_v mutually orth. projections;

(ii) $s_e^* s_e = p_{s(e)}$;

(iii) if $F \subset r^{-1}(v)$ finite, then $\sum_{e \in F} s_e s_e^* \leq p_v$;

(iv) if $r^{-1}(v)$ is finite and nonempty, then $p_v = \sum_{e : r(e) = v} s_e s_e^*$

Moreover $C^*(E)$ maps $*$-homomorphically onto any other C^*-algebra generated by such a family.

Uniqueness problem for graph algebras

Question

Let $A = C^*(E)$ be the graph C^*-algebra of E and let $\pi : A \to B$ be a $*$-homomorphism. What is an easy criterion for determining if π is injective?

An obvious necessary condition (assuming that one has already shown that $p_v \neq 0$ for all $v \in E^0$) is that $\pi(p_v) \neq 0$ for all v. Most answers to the above question try to impose conditions that make this condition sufficient as well.

Uniqueness theorems for graph algebras

Theorem (Kumjian, Pask, Raeburn, Fowler, etc.)

If every cycle of E has an entrance, then a repn $\pi : C^*(E) \to B$ is faithful if and only if $\pi(p_v) \neq 0$ for all $v \in E^0$.

What if Condition (L) is not satisfied? (For a path $\lambda = e_1 \dots e_n \in E^*$, let $s_\lambda = s_{e_1} \dots s_{e_n}$ be the associated partial isometry.)

Theorem (Szymanski, Nagy-Reznikoff)

Let E be a graph and let $\pi : C^*(E) \to B$ be a $*$-homomorphism. Then π is faithful if and only if both

(a) *$\pi(p_v) \neq 0$ for all $v \in E^0$, and*

(b) *the spectrum of $\pi(s_\lambda)$ contains $\mathbb{T}$ for each entryless cycle λ*

Uniqueness theorems for graph algebras, ctd.

Definition

If E is a directed graph, then $M(E) \subset C^*(E)$ is the C^*-subalgebra generated by $\{s_\alpha s_\alpha^* : \alpha \in E^*\} \cup \{s_\alpha s_\lambda s_\alpha^* : \lambda \text{ entryless cycle }\}$, called the *abelian core*

Theorem ([7])

If E is a directed graph and $\pi : C^*(E) \to B$ is a $*$-homomorphism, then π is faithful if and only if $\pi|_{M(E)}$ is faithful.

A generalization of this for k-graphs by Brown-Nagy-Reznikoff exists as well, with the abelian core replaced by the *cycline subalgebra*.

Uniqueness problem for groupoid algebras

If G is an étale (+ locally compact, 2nd countable, Hausdorff) groupoid, the vector space $C_c(G)$ is a $*$-algebra with

$$f * g(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta) \qquad f^*(\gamma) = \overline{f(\gamma^{-1})}$$

Give $C_c(G)$ the *reduced norm* $||\cdot||_r$, coming from the pre-$C_c(G^{(0)})$ structure given by $E : C_c(G) \to C_c(G^{(0)})$, and complete to obtain $C_r^*(G)$. We get an inclusion $C_0(G^{(0)}) \subset C_r^*(G)$.
We can ask: given a representation $\pi : C_r^*(G) \to B$, what conditions are necessary + sufficient to ensure that π is isometric?

Uniqueness theorems for groupoid algebras

An étale groupoid is *topologically free* if the set of units u such that $G_u^u = r^{-1}(u) \cap s^{-1}(u) = \{u\}$ forms a dense subset of $G^{(0)}$.

Theorem ([3])

Let G be a second countable, Hausdorff, étale topologically free groupoid. Then a $*$-homomorphism $\pi : C_r^*(G) \to B$ is isometric if and only if the restriction to $C_0(G^{(0)})$ is isometric.

This generalizes an existing result of Archbold-Spielberg for crossed products $C(X) \rtimes G$, with $C(X)$ playing the role of $C_0(G^{(0)})$.

Uniqueness theorems for groupoid algebras, ctd.

The isotropy bundle of G is $\{\gamma \in G : r(\gamma) = s(\gamma)\} \subset G$. The interior of the isotropy bundle is an étale groupoid and there is a natural inclusion $C^*(\mathrm{IntIso}\, G) \subset C^*(G)$ and for the reduced algebras as well.

Theorem ([5])

Suppose that G is a locally compact Hausdorff étale groupoid. Let π be a representation of $C_r^*(G)$. Then π is isometric if and only if the restriction to $C_r^*(\mathrm{IntIso}\, G)$ is isometric.

Essential inclusions

Definition

A C^*-subalgebra $0 \subsetneq B \subset A$ is called *essential* whenever a repn $\pi : A \to B(H)$ is injective on B, it follows that π is injective on all of A. Equivalently, B is essential if, whenever J is a closed two-sided ideal of A such that $J \cap B = \{0\}$, it follows that $J = \{0\}$.

The previous results show that the abelian core $M(E)$ is an essential C^*-subalgebra of $A = C^*(E)$, and that $C_r^*(\mathrm{IntIso}\, G)$ is essential in $C_r^*(G)$.

Common elements of uniqueness theorems

Many of the proofs of the previous results have a common element: they depend upon the existence of a *set of states on the subalgebra* which have *unique extensions to the larger algebra*. The following general uniqueness result from [5] captures some of this

Theorem

Let A be a C^*-algebra, $M \subset A$ a C^*-subalgebra, and S a collection of states on M such that

(a) *each $\phi \in S$ has unique extension to $\tilde{\phi} \in S(A)$, and*

(b) *the direct sum $\oplus_{\phi \in S} \pi_{\tilde{\phi}}$ of the GNS repns is faithful on A*

Then a representation ρ of A is injective if and only if $\rho|_M$ is injective. [That is, M is essential.]

Uniqueness cells

The set of points with unique extension is dense and $\bigoplus \pi_{\tilde{\phi}}$ is faithful, which is to say $\ker \bigoplus \pi_{\tilde{\phi}} \subset \{0\}$. Idea: set aside faithfulness of repns, replace with controlling size of joint GNS kernel.

Definition

Suppose that we have a (nonempty) $\mathcal{C} \subset S(B/B \cap J) \subset S(B)$ such that $\ker \bigoplus_{\tilde{\phi}} \pi_{\phi} \subset J$ for any choice of extensions $(\tilde{\phi})_{\phi \in \mathcal{C}}$. Then we call $\mathcal{C}$ a J-uniqueness cell for B.

Note: you can expand a J-uniqueness cell to include more states and it will remain a J-uniqueness cell.

Dominant ideals

Definition

Let $B \subset A$ be an inclusion and let $J \subset A$ be an ideal. We say that J is *B-dominant* if for all ideals $K \subset A$, the $K \cap B \subset J$ implies $K \subset J$.

Proposition (C.-Nagy)

Let J be an ideal of A and let $B \subset A$. Then there is a J-uniqueness cell for B if and only if J is B-dominant.

Proof: $\exists$ uniqueness cell implies B-dom.

Suppose that $\mathcal{C}$ is a J-uniqueness cell for B. Let $K \subset A$ be an ideal such that $K \cap B \subset J$. Let $\pi : A \to A/K$. Notice that there is a well-defined map $\mathcal{E} : \pi(B) \to B/J$, and $\sigma \circ \mathcal{E}$ is a state on $\pi(B)$. Take an extension of this state η_σ to $\pi(A)$, and finally set $\phi_\sigma = \eta_\sigma \circ \pi$. Note that $\phi_\sigma(b) = \eta_\sigma(\pi(b)) = \sigma \circ \mathcal{E} \circ \pi(b) = \sigma(b)$, so $(\phi_\sigma)_{\sigma \in \Sigma}$ is a set of extensions. The joint kernel must equal J, but the joint kernel contains $K = \ker \pi$, so $K \subset J$. This proves that J is B-dominant.

Proof: B-dom. implies $\exists$ uniqueness

Now suppose that J is B-dominant. Let $\Sigma = S(B/B \cap J) \subset S(B)$. Let $(\phi_\sigma)_{\sigma \in \Sigma}$ be a set of extensions to all of A. For each ϕ_σ, let π_σ be the associated GNS repn of A, and let K be the joint GNS kernel. Let $b \in B \cap K$. It must be the case that $b \in J$, because otherwise some state in $S(B/B \cap J)$ would be nonzero on b. Thus $B \cap K \subset J$, and then B-dominance of J shows that $K \subset J$. This shows that Σ is a J-uniqueness cell.

Essential subalgebras

Corollary

Let $B \subset A$; then B is essential iff 0 is B-dominant iff there is a set of states on B any extension of which to A is joint GNS-faithful.

We can strengthen this:

Corollary

Let $D \subset M \subset A$ (D non-deg.) and $D \subset Z(A)$. For $\omega \in P(D)$ let $M_\omega = M/M \cdot \ker \omega$. Assume that

(1) *there is some $\Omega_0 \subset P(D)$ with only joint GNS faithful lifts, and;*

(2) *M_ω is simple for all $\omega \in \Omega_0$.*

Then D is essential in M.

Groupoid C^*-algebras

If G is a 2CLCHÉ groupoid, set $A = C_r^*(G)$, $H = \mathrm{IntIso}\, G$, $M = C_r^*(H) \subset A$, $D = C_0(G^{(0)})$. There is a faithful expectation $E : A \to D$. In [5] it is shown that M is essential in A.

Corollary

Let W be the set of units u such that H_u is C^*-simple. If W is dense, then $C_r^*(G)$ is simple. In particular, if G is minimal and there is a unit u with C^*-simple isotropy, then $C_r^*(G)$ is simple.

Compare with Ozawa (see also [4], [6], [1]):

Theorem

If G is a discrete group acting minimally on some compact X, and if there is a point $x \in X$ with C^*-simple isotropy group G_x, then $C(X) \rtimes_r G$ is simple.

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Thank you!