Simplicity of reduced groupoid C^*-algebras

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Simple C^*-algebras

Question

When is the (full or reduced) C^*-algebra of an étale groupoid simple?

All this comes from recent arXiv submission [4]

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Sample of known simplicity results

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Long-term goal: to place as many of these as possible under one roof, and to extend/sharpen them if possible

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Example

If (G, X) is a discrete dynamical system then $C_0(X) \subset C_0(X) \rtimes_r G$ is always non-degenerate and regular; it is essential if the action is topologically free (the converse is not true).

General framework for simplicity results

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Let $B \subset A$ be non-degenerate and regular. Then the following conditions are equivalent:

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Proof.

Only tricky bit is showing "A simple" $\implies$ "$B \subset A$ minimal": if $J \subset B$ a non-trivial proper invariant ideal then can show that $L = \overline{\text{span}}\{a_1 b a_2 : a_i \in A, b \in J\}$ is a non-trivial proper ideal of A. □

Groupoid preliminaries

Definition

A *groupoid* is a set G along with a set $G^{(2)} \subset G \times G$ of composable pairs, a composition $G^{(2)} \to G$ which is associative, and an inverse operation $G \to G$ which appropriately cancels composition.

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$$G^{(2)} \ni (\alpha, \beta) \mapsto \alpha\beta \in G$$

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$

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Each (α, α^{-1}) is composable and elements of the form $\alpha\alpha^{-1}$ are *units*, forming unit space $G^{(0)}$.

Units; isotropy

Definition

The map $r : G \to G$ given by $r(\alpha) = \alpha\alpha^{-1}$ maps onto $G^{(0)}$ and is called the *range map*; the corresponding $s : G \to G$ given by $s(\alpha) = \alpha^{-1}\alpha$ is called the *source map*.

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Definition

Let $u \in G^{(0)}$;

$$G_u^u = G(u) := \{\gamma \in G : r(\gamma) = u = s(\gamma)\}$$

is the *isotropy subgroup at u*. Set $\text{Iso}(G) := \cup_u G(u)$.

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Definition

Let $\text{Int Iso}(G)$ be the interior of $\text{Iso}(G) \subset G$

$*$-algebras

If G is étale then for $u \in G^{(0)}$, $r^{-1}(u) = \{\alpha \in G : \alpha\alpha^{-1} = u\}$ and the source fiber $s^{-1}(u)$ are discrete.

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$$(f * g)(\alpha) = \sum_{\alpha=\beta\delta} f(\beta)g(\delta) \qquad f, g \in C_c(G)$$

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Two distinguished C^*-norms: $||\cdot||$ (full), $||\cdot||_r$ (reduced). Complete $C_c(G)$ to get to $C^*(G)$ and $C_r^*(G)$.

Groupoid Inclusions

Two important inclusions

$$C_0(G^{(0)}) \subset C_r^*(\mathrm{Int\,Iso}\,G) \subset C_r^*(G)$$

where $C_0(G^{(0)})$ is abelian and $C_r^*(\mathrm{Int\,Iso}\,G)$ is the reduced C^*-algebra of the étale group bundle Int Iso G.

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where $C_0(G^{(0)})$ is abelian and $C_r^*(\mathrm{Int\,Iso}\,G)$ is the reduced C^*-algebra of the étale group bundle $\mathrm{Int\,Iso}\,G$.
The pure states on $C_0(G^{(0)})$ are the evaluations ev_u for $u \in G^{(0)}$, which have natural (not typically unique) extension to $C_r^*(G)$ via extending the bounded map $C_c(G) \ni f \mapsto f(u)$.

Essential subalgebras in groupoid C^*-algebras

One of the main results of "Cartan subalgebras in C^*-algebras of Hausdorff étale groupoids" (BNRSW, 2016) is the following (omitting the full norm bit):

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Question

For which étale groupoids G is $C_0(G^{(0)}) \subset C_r^*(G)$ *essential?*

A simplicity criterion for groupoid C^*-algebras

For a groupoid G, let
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Proposition

If G is a groupoid, then $C_r^*(G)$ is simple if and only if the inclusion $C_r^*(\text{Int Iso } G) \subset C_r^*(G)$ is minimal. Equivalently, if there are no nontrivial ideals $J \triangleleft C_r^*(\text{Int Iso } G)$ that are invariant under the action of $\mathfrak{N}(G)$.

Technical tools - two seminorms

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(i) Define J_q to be the ideal generated by $C_0(Q \setminus \{q\})$ in A, and let $p_q^{\text{unif}}(a) = ||a + J_q||$ for $a \in A$; $Q_{\text{simple}}^{\text{unif}}$ is set of points where A/J_q is simple.

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If Q_{simple}^{unif} is dense in Q, and the expectation $\mathbb{E}$ is essentially faithful, then $C_0(Q) \subset A$ is essential.

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If $C_0(Q) \subset A$ as above with A separable, and if $\mathbb{E}$ is essentially faithful, then $Q_{\text{r-cont}}$ is dense G_δ.

Simplicity for groupoid C^*-algebras

Theorem (C., Nagy)

Assume that there is a unit $u_0 \in G^{(0)}_{\text{r-cont}}$ *such that* $\operatorname{Int}\operatorname{Iso}(G)_{u_0}$ *is* C^**-simple. Then the following are equivalent:*

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Proof.

That (ii) implies (i) is standard; suppose that (i) is true. Just need to show $C_r^*(\operatorname{Int}\operatorname{Iso}(G)) \subset C_r^*(G)$ is minimal. □

Proof ctd.

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-There is a faithful expectation $\mathbb{E} : C_r^*(\text{Int Iso}(G)) \to C_0(G^{(0)})$

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-Thus $C_0(G^{(0)}) \subset C^*_r(\text{Int Iso}(G))$ is essential
-Any ideal in $C^*_r(G)$ has nontrivial intersection with $C^*_r(\text{Int Iso}(G))$, hence with $C_0(G^{(0)})$.

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-Minimality finishes it.

□

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Let G be a discrete group acting minimally on a locally compact space Q. If there is $q \in Q$ such that the interior stabilizer G_q° is C^*-simple, then $C_0(Q) \rtimes_r G$ is simple.

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(This sharpens a result of Ozawa that required G_q to be C^*-simple.)

The End

Thank You!

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