Simplicity of reduced groupoid C^*-algebras

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GPOTS 2018

Simple C^*-algebras

Question

When is the (full or reduced) C^*-algebra of an étale groupoid simple?

All this comes from recent arXiv submission [4]

Simplicity results

Sample of known simplicity results

(i) If (A, G, α) is a minimal and topologically free C^*-dynamical system, then $A \rtimes_r G$ simple [1]

(ii) Full groupoid C^*-algebra $C^*(G)$ is simple if and only if G is minimal and topologically free and $C^*(G) = C_r^*(G)$ [2]

(iii) If G is a discrete group then $C_r^*(G)$ is simple if and only if G acts freely on its Furstenberg boundary [5]

(iv) If (G, X) minimal discrete dynamical system and G_x is C^*-simple for some $x \in X$, then $C(X) \rtimes_r G$ is simple [6]

Long-term goal: to place as many of these as possible under one roof, and to extend/sharpen them if possible

Inclusions

Different flavors of inclusions $B \subset A$ of C^*-algebras:

(i) $B \subset A$ is *non-degenerate* if B contains an approx. unit for A;

(ii) $B \subset A$ is *essential* if any nonzero ideal $J \trianglelefteq A$ must have $J \cap B \neq 0$

(iii) $B \subset A$ is *regular* if the normalizer $N_A(B) = \{n \in A : nBn^* \cup n^*Bn \subset B\}$ has dense span in A.

Example

If (G, X) is a discrete dynamical system then $C_0(X) \subset C_0(X) \rtimes_r G$ is always non-degenerate and regular; it is essential if the action is topologically free (the converse is not true).

General framework for simplicity results

Definition

An ideal $J \triangleleft B \subset A$ is *invariant* if $N_A(B) \subset N_A(J)$; the inclusion $B \subset A$ is *minimal* if there are no non-trivial invariant ideals in B.

Proposition (C., Nagy)

Let $B \subset A$ be non-degenerate and regular. Then the following conditions are equivalent:

(i) *A is simple;*

(ii) *$B \subset A$ is essential and minimal*

Proof.

Only tricky bit is showing "A simple" $\implies$ "$B \subset A$ minimal": if $J \subset B$ a non-trivial proper invariant ideal then can show that $L = \overline{\text{span}}\{a_1 b a_2 : a_i \in A, b \in J\}$ is a non-trivial proper ideal of A. □

Groupoid preliminaries

Definition

A *groupoid* is a set G along with a set $G^{(2)} \subset G \times G$ of composable pairs, a composition $G^{(2)} \to G$ which is associative, and an inverse operation $G \to G$ which appropriately cancels composition.

$$G^{(2)} \ni (\alpha, \beta) \mapsto \alpha\beta \in G$$

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$

$$\alpha^{-1}(\alpha\beta) = \beta = (\beta\gamma)\gamma^{-1}$$

Each (α, α^{-1}) is composable and elements of the form $\alpha\alpha^{-1}$ are *units*, forming unit space $G^{(0)}$.

Units; isotropy

Definition

The map $r : G \to G$ given by $r(\alpha) = \alpha\alpha^{-1}$ maps onto $G^{(0)}$ and is called the *range map*; the corresponding $s : G \to G$ given by $s(\alpha) = \alpha^{-1}\alpha$ is called the *source map*.

Definition

Let $u \in G^{(0)}$;

$$G_u^u = G(u) := \{\gamma \in G : r(\gamma) = u = s(\gamma)\}$$

is the *isotropy subgroup at u*. Set $\mathrm{Iso}(G) := \cup_u G(u)$.

Topological groupoids

Definition

A *topological groupoid* is a groupoid endowed with a topology that makes the composition and inverse maps continuous. A groupoid is *étale* if the map r above is a local homeomorphism. (Equivalent to existence of bisections.)

(Assume all locally compact, second countable, Hausdorff, and étale.)

Definition

Let $\text{Int}\,\text{Iso}(G)$ be the interior of $\text{Iso}(G) \subset G$

$*$-algebras

If G is étale then for $u \in G^{(0)}$, $r^{-1}(u) = \{\alpha \in G : \alpha\alpha^{-1} = u\}$ and the source fiber $s^{-1}(u)$ are discrete.

Definition

Let G be an étale The $*$-algebra structure on $C_c(G)$ is defined by

$$(f * g)(\alpha) = \sum_{\alpha=\beta\delta} f(\beta)g(\delta) \qquad f, g \in C_c(G)$$

$$f^*(\alpha) = \overline{f(\alpha^{-1})}$$

Two distinguished C^*-norms: $||\cdot||$ (full), $||\cdot||_r$ (reduced). Complete $C_c(G)$ to get to $C^*(G)$ and $C_r^*(G)$.

Groupoid Inclusions

Two important inclusions

$$C_0(G^{(0)}) \subset C_r^*(\mathrm{Int\,Iso}\,G) \subset C_r^*(G)$$

where $C_0(G^{(0)})$ is abelian and $C_r^*(\mathrm{Int\,Iso}\,G)$ is the reduced C^*-algebra of the étale group bundle $\mathrm{Int\,Iso}\,G$.
The pure states on $C_0(G^{(0)})$ are the evaluations ev_u for $u \in G^{(0)}$, which have natural (not typically unique) extension to $C_r^*(G)$ via extending the bounded map $C_c(G) \ni f \mapsto f(u)$.

Essential subalgebras in groupoid C^*-algebras

One of the main results of "Cartan subalgebras in C^*-algebras of Hausdorff étale groupoids" (BNRSW, 2016) is the following (omitting the full norm bit):

Theorem ([3, Thm. 3.1])

Let G be a locally compact Hausdorff groupoid.

(i) $C_r^*(\mathrm{Int\,Iso}(G)) \hookrightarrow C_r^*(G)$ *via an injective $*$-homomorphism ι_r extending the inclusion of $C_c(\mathrm{Int\,Iso}(G))$;*

(ii) *Letting $M_r = \iota_r(C_r^*(\mathrm{Int\,Iso}(G)))$; then $M_r \subset C_r^*(G)$ is essential.*

Question

For which étale groupoids G is $C_0(G^{(0)}) \subset C_r^*(G)$ essential?

A simplicity criterion for groupoid C^*-algebras

For a groupoid G, let
$\mathfrak{N}(G) = \{n \in C_c(G) : \text{supp}(n) \subset \text{ bisection}\} \subset N_{C_r^*(\text{Int Iso } G)}(C_r^*(G))$

Proposition

If G is a groupoid, then $C_r^*(G)$ is simple if and only if the inclusion $C_r^*(\text{Int Iso } G) \subset C_r^*(G)$ is minimal. Equivalently, if there are no nontrivial ideals $J \triangleleft C_r^*(\text{Int Iso } G)$ that are invariant under the action of $\mathfrak{N}(G)$.

Technical tools - two seminorms

Let $C_0(Q) \subset A$ be a central non-degen. inclusion with a conditional expectation $\mathbb{E} : A \to C_0(Q)$. Let $q \in Q$.

(i) Define J_q to be the ideal generated by $C_0(Q \setminus \{q\})$ in A, and let $p_q^{\text{unif}}(a) = ||a + J_q||$ for $a \in A$; $Q_{\text{simple}}^{\text{unif}}$ is set of points where A/J_q is simple.

(ii) Set $p_q^{\mathbb{E}}(a) = |\operatorname{ev}_q \circ \mathbb{E}(a)|$

(iii) Set $Q_{\text{r-cont}}$ be the set of points where these two seminorms agree

Theorem (C., Nagy)

If Q_{simple}^{unif} is dense in Q, and the expectation $\mathbb{E}$ is essentially faithful, then $C_0(Q) \subset A$ is essential.

Theorem (C., Nagy)

If $C_0(Q) \subset A$ as above with A separable, and if $\mathbb{E}$ is essentially faithful, then $Q_{\text{r-cont}}$ is dense G_δ.

Simplicity for groupoid C^*-algebras

Theorem (C., Nagy)

Assume that there is a unit $u_0 \in G^{(0)}_{\text{r-cont}}$ such that $\operatorname{Int}\operatorname{Iso}(G)_{u_0}$ is C^*-simple. Then the following are equivalent:

(i) *G is minimal;*

(ii) *$C^*_r(G)$ is simple*

Proof.

That (ii) implies (i) is standard; suppose that (i) is true. Just need to show $C^*_r(\operatorname{Int}\operatorname{Iso}(G)) \subset C^*_r(G)$ is minimal. □

Proof ctd.

Proof.

-The orbit $r(Gu_0)$ is dense in $G^{(0)}$
-Any $u \in r(Gu_0)$ has isomorphic stabilizer
-Can show that each of these belongs to $G^{(0)}_{\text{r-cont}}$; thus $G^{(0)}_{\text{r-cont}}$ is dense
-There is a faithful expectation $\mathbb{E} : C_r^*(\text{Int Iso}(G)) \to C_0(G^{(0)})$
-Thus $C_0(G^{(0)}) \subset C_r^*(\text{Int Iso}(G))$ is essential
-Any ideal in $C_r^*(G)$ has nontrivial intersection with $C_r^*(\text{Int Iso}(G))$, hence with $C_0(G^{(0)})$.
-Minimality finishes it.

□

Group crossed products

You can show that in the case of a group G acting on a space Q, the associated transformation groupoid $G \times Q$ has $Q_{\text{r-cont}} = Q$.

Theorem (C., Nagy)

Let G be a discrete group acting minimally on a locally compact space Q. If there is $q \in Q$ such that the interior stabilizer G_q° is C^*-simple, then $C_0(Q) \rtimes_r G$ is simple.

(This sharpens a result of Ozawa that required G_q to be C^*-simple.)

The End

Thank You!

Bibliography I

R.J. Archbold and J. Spielberg.
Topologically free actions and ideals in discrete C^*-dynamical systems.
Proc. Edin. Math. Soc., 37:119–124, 1993.

J. Brown, L.O. Clark, C. Farthing, and A. Sims.
Simplicity of algebras associated to étale groupoids.
Semigroup Forum, 88:433–452, April 2014.

J. Brown, G. Nagy, S. Reznikoff, A. Sims, and D. Williams.
Cartan subalgebras in C^*-algebras of Hausdorff étale Groupoids.
Integral Equations and Operator Theory, 85:109–126, 2016.

Bibliography II

D. Crytser and G. Nagy.
Simplicity criteria for groupoid C^*-algebras.
preprint, arXiv:1805.11173.

M. Kalantar and M. Kennedy.
Boundaries of reduced C^*-algebras of discrete groups.
Journal f́ur die reine und angewandte Mathematik, 2017, 2014.

N. Ozawa.
Lecture on the Furstenburg boundary and C^*-simplicity.
http://www.kurims.kyoto-u.ac.jp/~narutaka/notes/yokou2014.pdf, December 2014.