# Large inclusions for groupoid and $k$-graph $C^*$-algebras

## Preliminary report

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Often motivated by simplicity criteria and ideal structure problems (see [Zar]).

# What is a pseudo-expectation?

**Definition**

Let $A$ be a $C^*$-algebra. Then an **injective envelope** for $A$ consists of an injective $C^*$-algebra $I(A)$ containing $A$ as a $C^*$-subalgebra the only ucp map $I(A) \to I(A)$ that restricts to the identity on $A$ is the identity map. (It exists and is unique up to isomorphism ([Ham79]).)

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**Definition ([Pit12])**

A **pseudo-expectation** for $B \subset A$ is a ucp map $A \to I(B)$ extending $B \hookrightarrow I(B)$. (Generalizes conditional expectation, always exists.)

Shown in [Pit12] that regular MASA inclusions have unique pseudo-expectations.

# What is a groupoid

Groupoid = "category where every morphism is invertible". So if $\alpha, \beta \in G$ then $\alpha\beta$ may or may not be defined. Can always cancel, e.g. $\alpha^{-1}(\alpha\beta) = \beta$.

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### Example

If discrete group $G \curvearrowright X$ a topological space, then $G \times X$ is a groupoid with $(h, g.x)(g, x) = (hg, x)$. (Call this $G \ltimes X$, transformation groupoid.)

# Groupoid $C^*$-algebras

Let $G$ be an étale locally compact Hausdorff second countable groupoid with *compact* unit space. Construct reduced groupoid $C^*$-algebra $C^*_r(G)$ out of convolution algebra $C_c(G)$ as in construction of group $C^*$-algebra.

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Example (Graph algebras)

If $E$ is a graph with path groupoid $G_E$, then these correspond to the diagonal $C^*(s_\lambda s_\lambda^*) \subset C^*(E)$ and the abelian core $M_E \subset C^*(E)$ (Nagy-Reznikoff).

Main questions for this work in progress:

### Question

*When do the inclusions $C(G^{(0)}) \subset C_r^*(G)$ and $C_r^*(\mathrm{Int\,Iso}\, G) \subset C_r^*(G)$ associated to an étale groupoid $G$ have the various* **largeness properties** *on the first slide?.*

Specifically, when does $C_r^*(\mathrm{Int\,Iso}\, G) \subset C_r^*(G)$ have the faithful unique pseudo-expectation property?

# The inclusion $C(G^{(0)}) \subset C_r^*(G)$

**Proposition (C.)**

*The inclusion $C(G^{(0)}) \subset C_r^*(G)$. Following are equivalent:*

(a) *has a unique conditional expectation*

(b) *has a unique pseudo-expectation*

(c) $\text{Int Iso}\, G = G^{(0)}$

**Proof.**

Easy proof, combining results of Renault, Pitts, and Zarikian. Avoids considering properly outer actions as in Zarikian. □

# The inclusion $C_r^*(\operatorname{Int}\operatorname{Iso} G) \subset C_r^*(G)$

The inclusion $C_r^*(\operatorname{Int}\operatorname{Iso} G) \subset C_r^*(G)$ is always regular (noted by many, including [CN]).

**Proposition ([BNR+16, Prop. 4.1])**

*The inclusion $C_r^*(\operatorname{Int}\operatorname{Iso} G) \subset C_r^*(G)$ has a conditional expectation if and only if* Int Iso $G$ *is closed in* $G$. *The expectation is given on* $C_c(G)$ *by* $f \mapsto f|_{\operatorname{Int}\operatorname{Iso} G}$.

This conditional expectation is unique using an argument of [Zar18]

**Question**

*What about pseudo-expectations?*

The inclusion $C_r^*(\operatorname{Int}\operatorname{Iso} G) \subset C_r^*(G)$ is essential ([BNR+16]), leads us to hope for unique faithful pseudo-expectation.

# Maximal abelian subalgebras and pseudoexpectations

Theorem ([BNR$^+$16, Thm. 4.3])

*Suppose that* Int Iso $G$ *is abelian. If* Int Iso $G$ *is closed or if there exists a countable discrete abelian group* $H$ *and a continuous 1-cocycle* $c : G \to H$ *that is injective on each* $G_x^x$, *then* $C_r^*(\text{Int Iso } G) \subset C_r^*(G)$ *is a MASA*

Any essential MASA has unique faithful pseudo-expectation [Pit12], we obtain.

Proposition (C.)

*In either of the above cases, there is a unique pseudo-expectation from* $C_r^*(G) \to I(C_r^*(\text{Int Iso } G))$, *which is faithful*

Hope that the above is **always** true, not just for MASA $C_r^*(\text{Int Iso } G)$.

# The End

Thank You!

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