Graph Traces on Product Graphs

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Overview

1. Directed graphs, product graphs, and their traces

2. Higher-rank graphs and their traces

3. Products of higher rank graphs and their traces

Directed Graphs and Traces

Definition

Let E be a directed graph. A function $g : E^0 \to [0,1]$ is a *graph trace* if
(i) For any regular vertex $v \in E^0$,

$$g(v) = \sum_{e \in E^1, r(e)=v} g(s(e)).$$

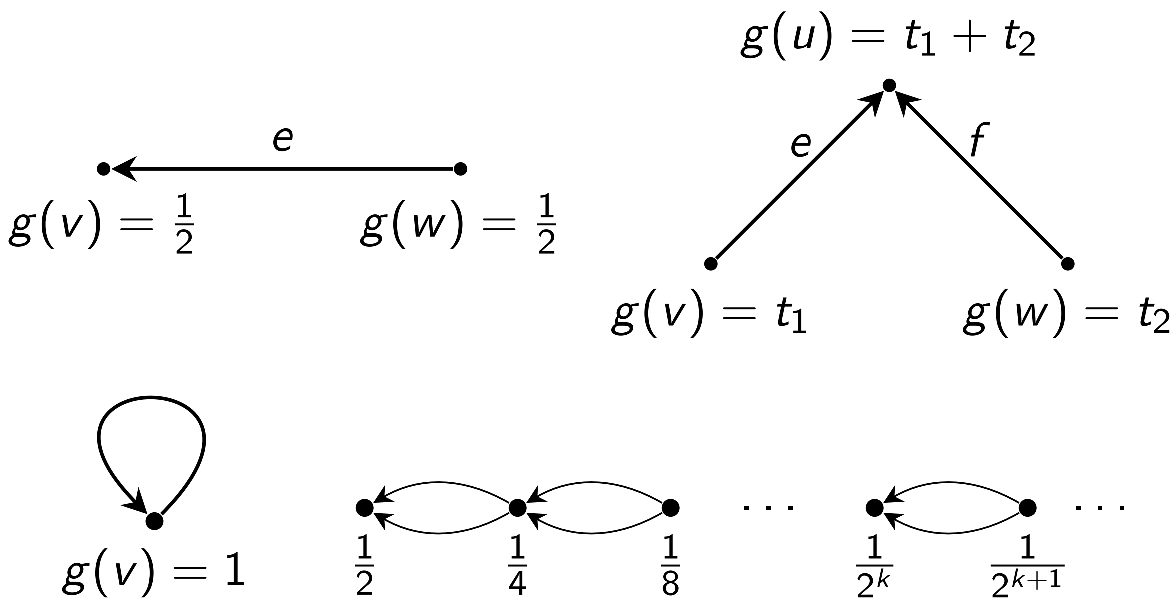
(ii) For any infinite receiver v $\in E^0$ and any finite collection of edges in $r^{-1}(v)$, we have

$$g(v) \geq \sum_{i=1}^{n} g(s(e_i)).$$

(iii)

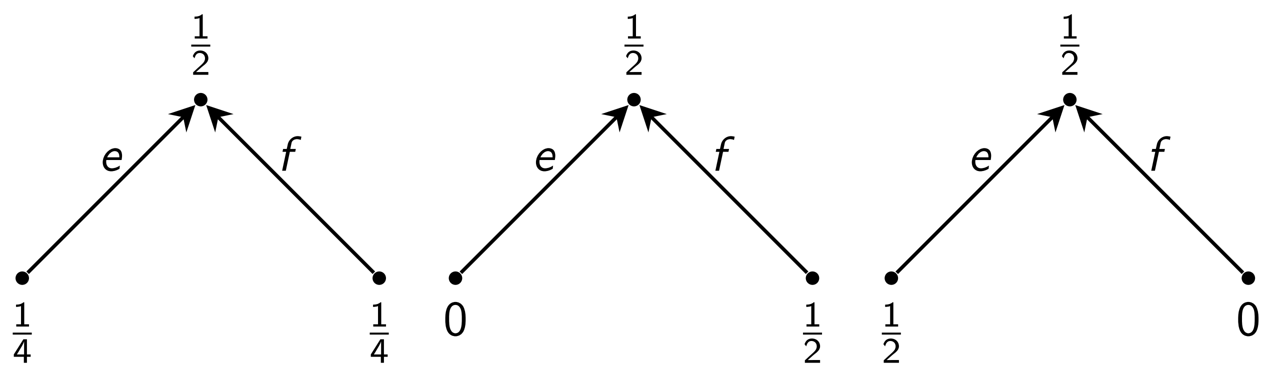
$$\sum_{v \in E^0} g(v) = 1$$

Examples of Directed Graph Traces



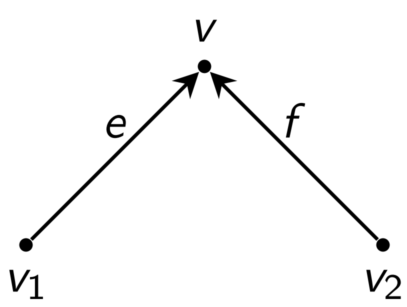
Definition

An *extreme graph trace* is a graph trace which cannot be written as a convex combination of other graph traces. That is, if g is an extreme graph trace and $g = tg' + (1-t)g''$ for graph traces g', g'' and $t \in (0,1)$, then $g' = g'' = g$.

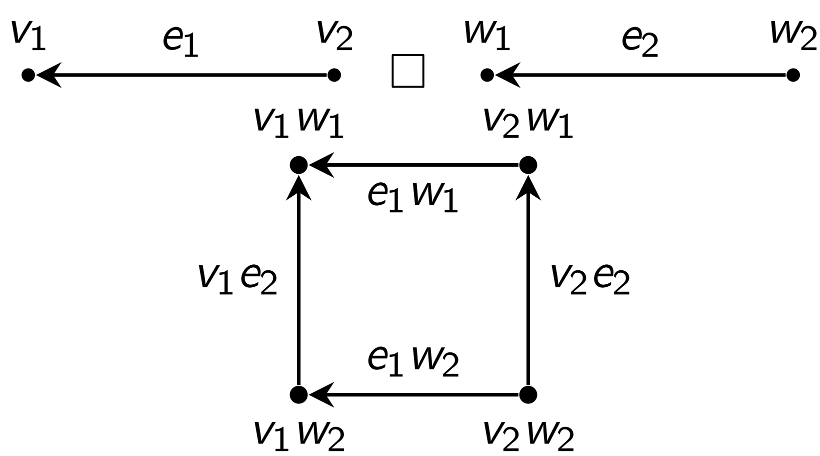


Theorem [Johnson]

Let E be a finite directed graph with no cycles. Then there exists a bijection between the set of all sources of E, S_E, and the set of all extreme traces on E, $ext(T(E))$.



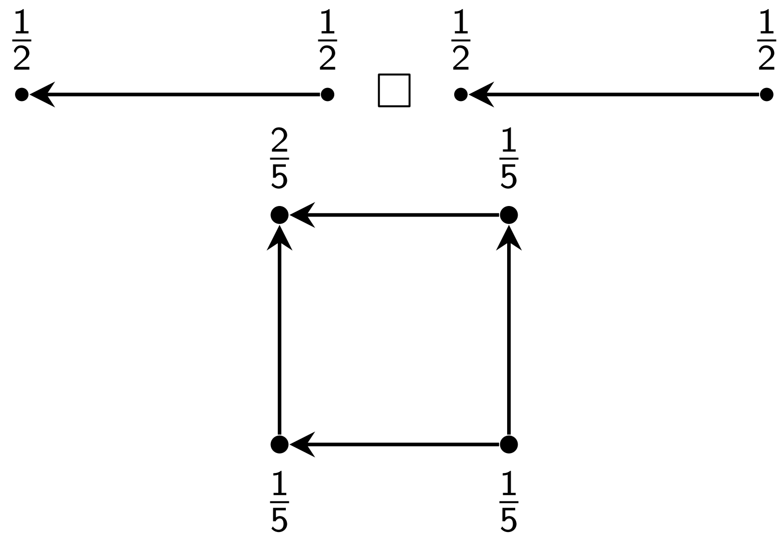
Box Product



Box Product of Directed Graphs

The *box (Cartesian) product* of E with F is the graph $E\square F = (E^0 \times F^0, (E^1 \times F^0) \cup (E^0 \times F^1), r_\square, s_\square)$, where $r_\square, s_\square$ are defined as follows: For all $e \in E^1$, $f \in F^1$, $u \in E^0$, $v \in F^0$:

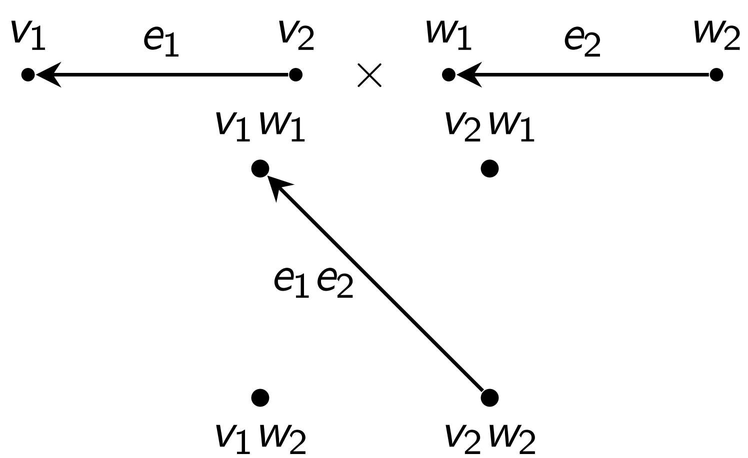
$$r_\square(e, v) = (r_E(e), v) \qquad r_\square(u, f) = (u, r_F(f))$$
$$s_\square(e, v) = (s_E(e), v) \qquad s_\square(u, f) = (u, s_F(f))$$



Box Product

This product operation does not guarantee that the product of graph traces on factor graphs is a trace on the product graph.

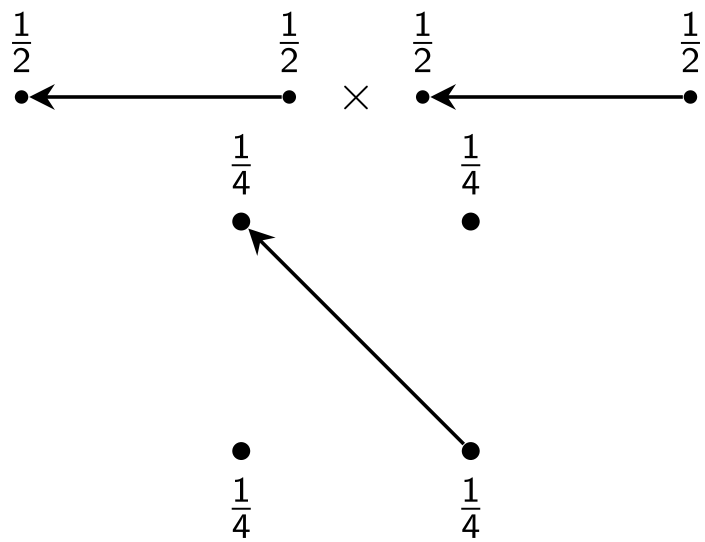
Tensor Product



Tensor Product of Directed Graphs

The *tensor product* of E with F is the graph $E \otimes F = (E^0 \times F^0, E^1 \times F^1, r_\otimes, s_\otimes)$, such that for all $(e, f) \in E^1 \times F^1$ we define:

$$r_\otimes(e, f) = (r_E(e), r_F(f)) \text{ and } s_\otimes(e, f) = (s_E(e), s_F(f)).$$



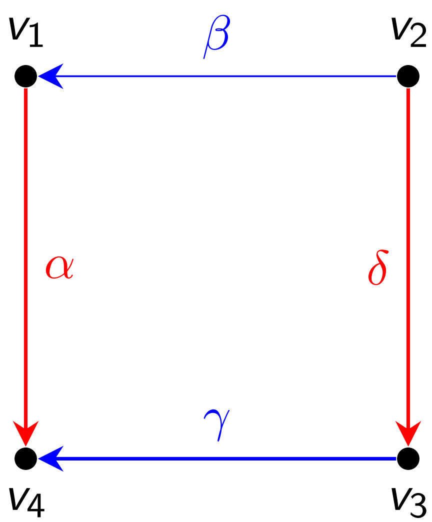
Tensor Product

The tensor product of traces on factor graphs gives a trace on the product graph. This is not necessarily the only way to find traces on the product graph.

Higher-rank graphs

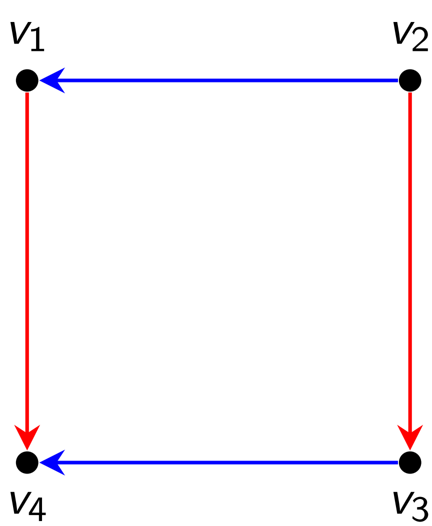
Definition

A *higher-rank graph*, or k-graph,(Λ, d), consists of a category Λ and a *degree functor* $d : \Lambda \to \mathbb{N}^k$ (i.e. $d(\lambda_1\lambda_2) = d(\lambda_1) + d(\lambda_2)$) satisfying the *factorization property*: for any $\lambda \in \Lambda$, if $d(\lambda) = m + n$ for $m, n \in \mathbb{N}^k$, then there exist unique $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m, d(\nu) = n$. For $n \in \mathbb{N}^k$, let Λ^n denote $d^{-1}(n) = \{\lambda \in \Lambda : d(\lambda) = n\}$.

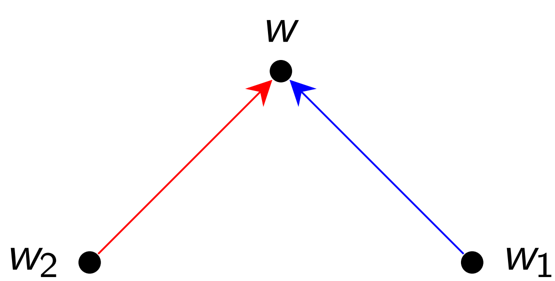


Definition

A k-graph is *locally convex* if whenever a vertex receives different colored edges, the sources of these edges also receive edges of each color other than the one it sends.



Locally Convex



Not Locally Convex

Definition

Let Λ be a (locally convex, row-finite) k-graph, and let Λ^0 be its set of vertices. A function $g : \Lambda^0 \to [0,1]$ is called a *higher-rank graph trace* if

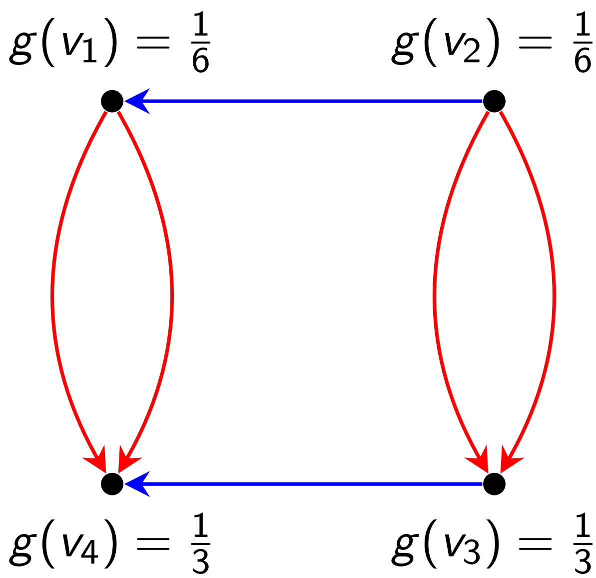
(i) for any vertex $v \in \Lambda^0$ and any degree $n \in \mathbb{N}^k$, we have

$$\sum_{\lambda \in v\Lambda^{\leq n}} g(s(\lambda)) = g(v);$$

(ii)

$$\sum_{v \in \Lambda^0} g(v) = 1$$

.

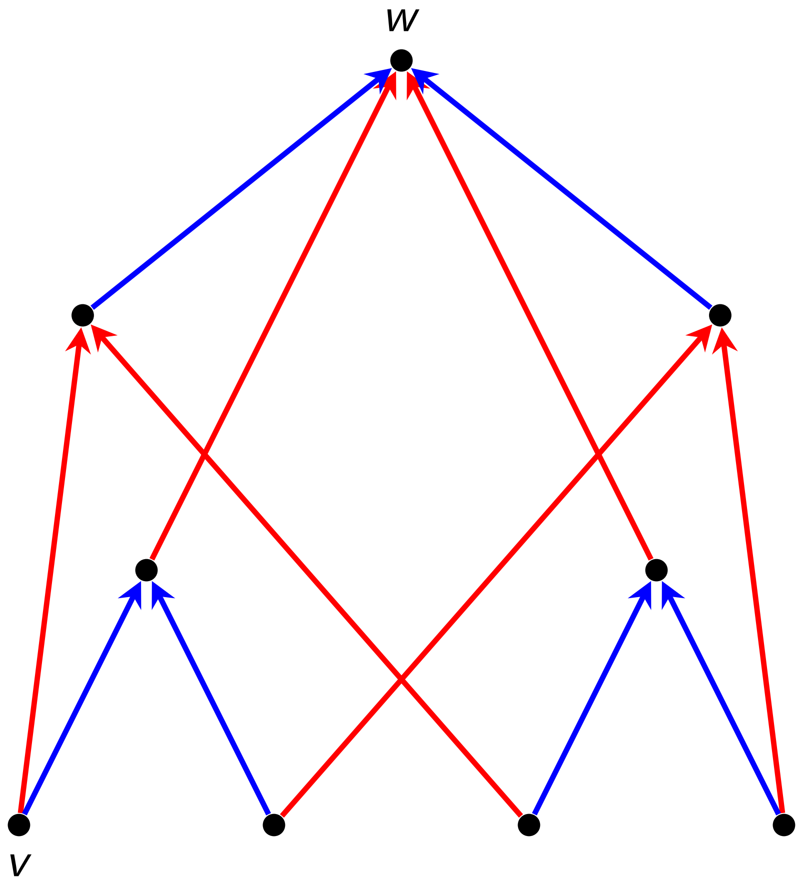

$g(v_1) = \frac{1}{6}$
$g(v_2) = \frac{1}{6}$
$g(v_4) = \frac{1}{3}$
$g(v_3) = \frac{1}{3}$

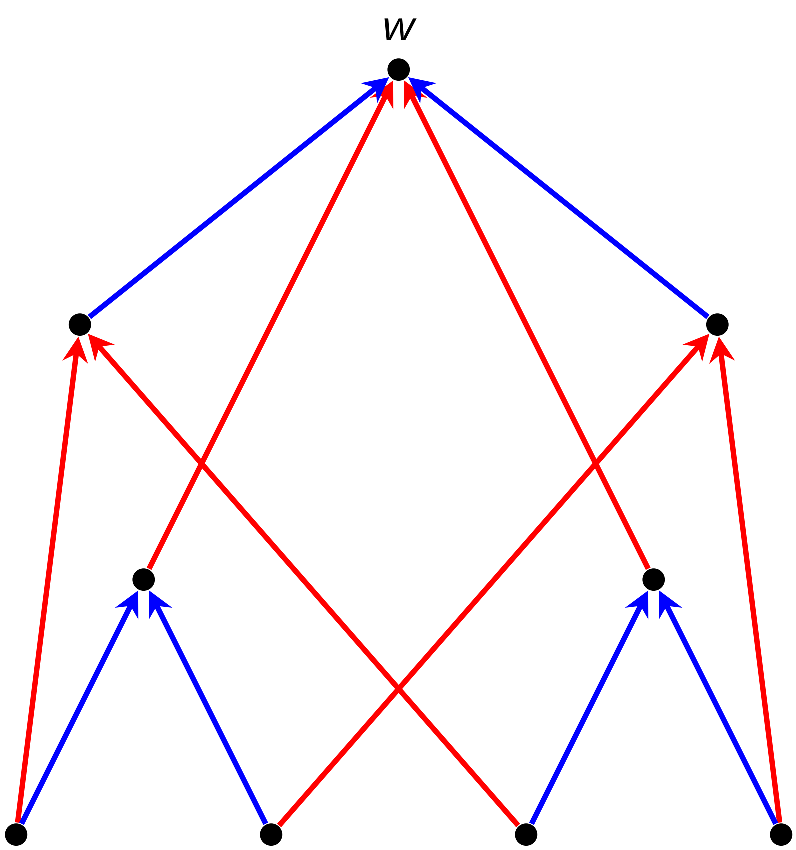
Definition

Let E be a finite graph with no cycles and let v, $w \in E^0$. Then, define the number of finite paths from v as $n(v) = |\{\lambda \in E^* : s(\lambda = v\}|$. Also define the number of paths between v and w as
$n(v, w) = |\{\lambda \in E^* : s(\lambda) = v, r(\lambda) = w\}|$.

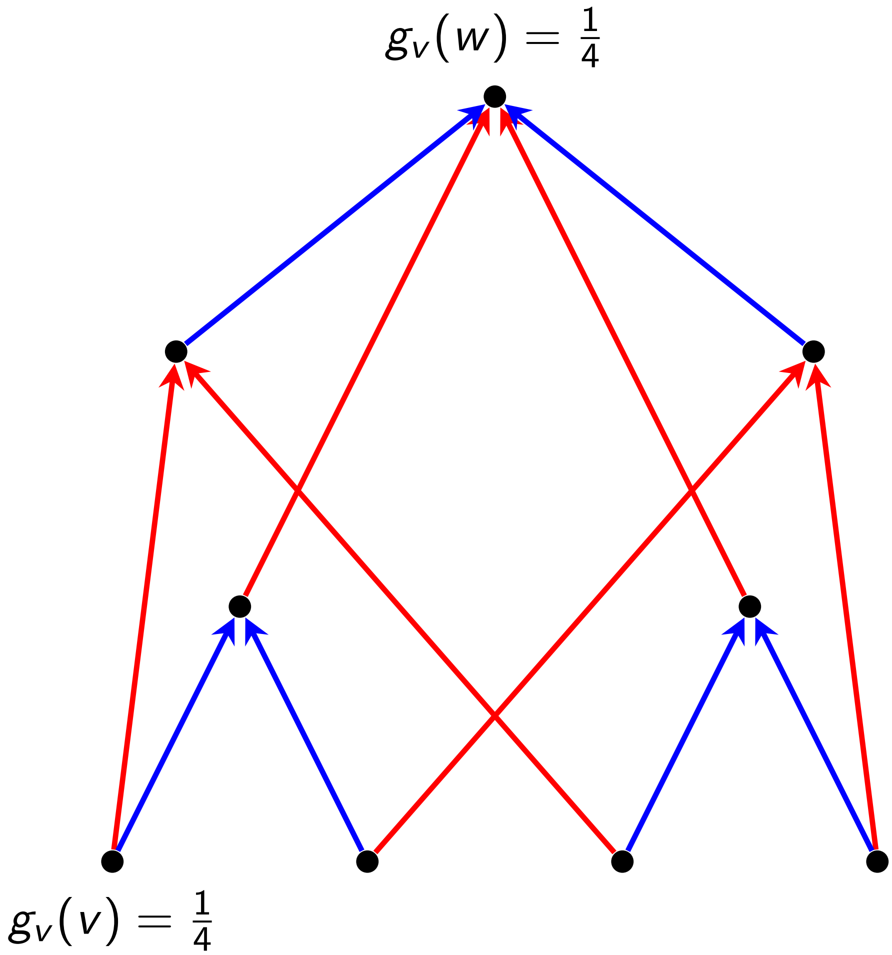
Theorem

If Λ is a finite locally convex k-graph with no cycles, then there is a one to one correspondence between sources and extreme traces defined by
$S_\Lambda \ni v \mapsto g_v \in T(\Lambda)$ where $g_v(w) = \frac{n(v,w)}{n(v)}$.

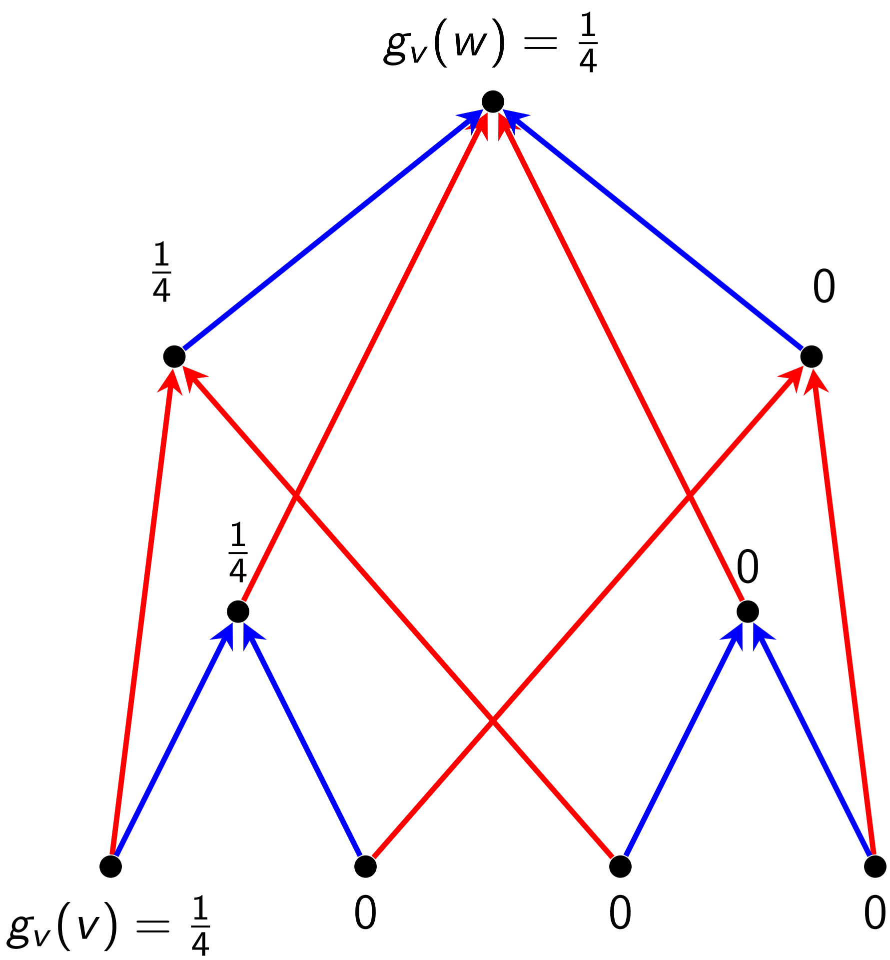

w
v



$g_v(v) = \frac{1}{4}$



$g_v(w) = \frac{1}{4}$
$g_v(v) = \frac{1}{4}$


$g_v(w) = \frac{1}{4}$
$\frac{1}{4}$
0
$\frac{1}{4}$
0
$g_v(v) = \frac{1}{4}$
0
0
0

Product of Higher Rank Graphs

Definition

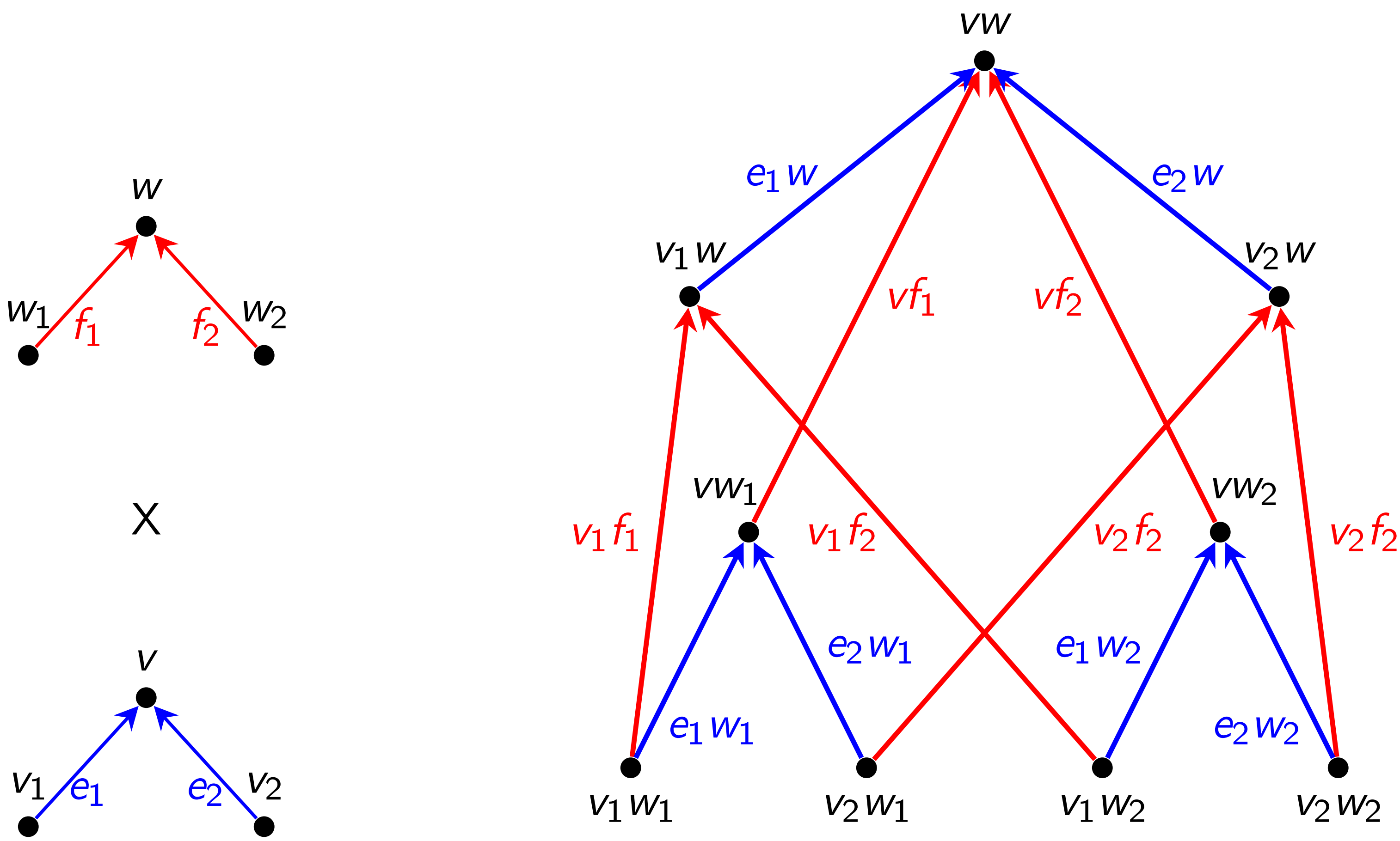
Let Λ be a k-graph and let Π be an ℓ-graph. The product of Λ and Π, denoted $\Lambda \times \Pi$, is simply the Cartesian product of Λ and Π, equipped with the following structure:

(i) $d(\lambda, \pi) = (d(\lambda), d(\pi))$

(ii) $r(\lambda, \pi) = (r(\lambda), r(\pi))$ and likewise for the source map.

(iii) $(\lambda, \pi)(\lambda', \pi') = (\lambda\lambda', \pi\pi')$ whenever both compositions in the factor graphs are defined.

Note that with this degree map, the vertex set of $\Lambda \times \Pi$ is just $\Lambda^0 \times \Pi^0$.


w
w_1
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vf_1
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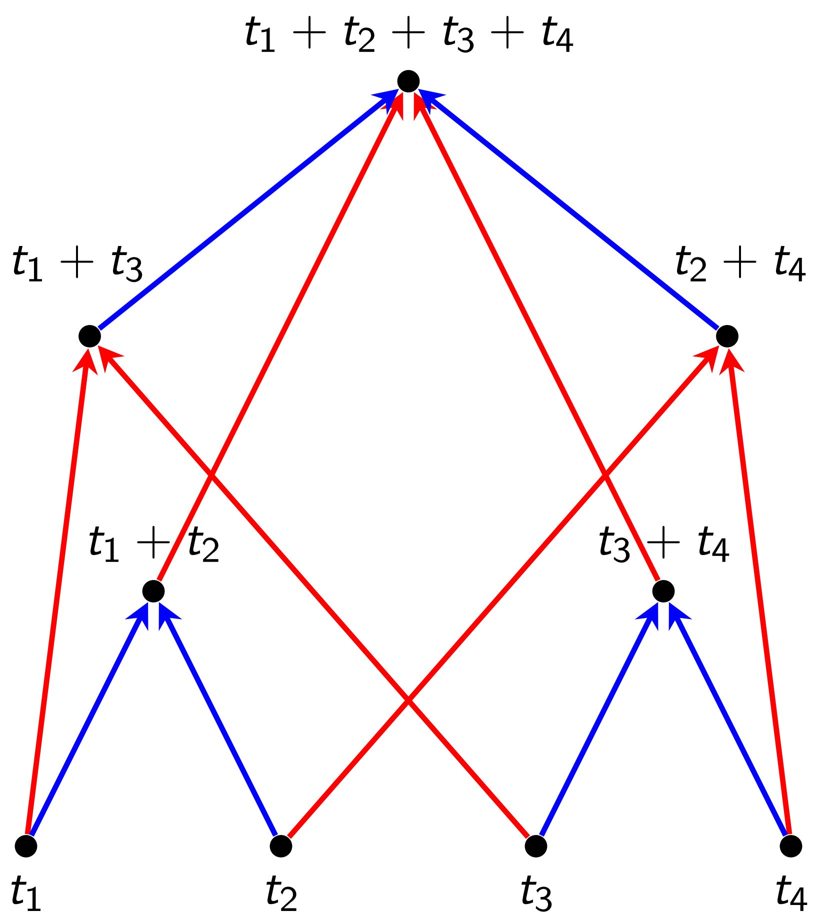
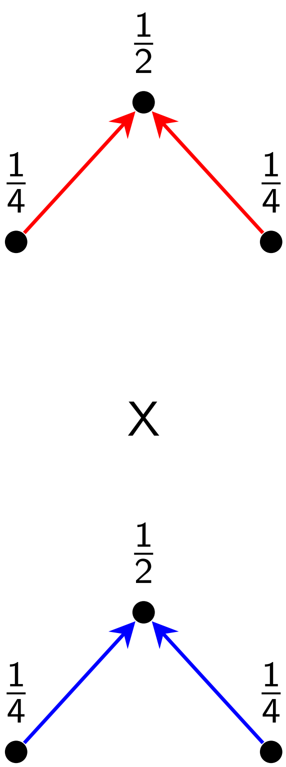
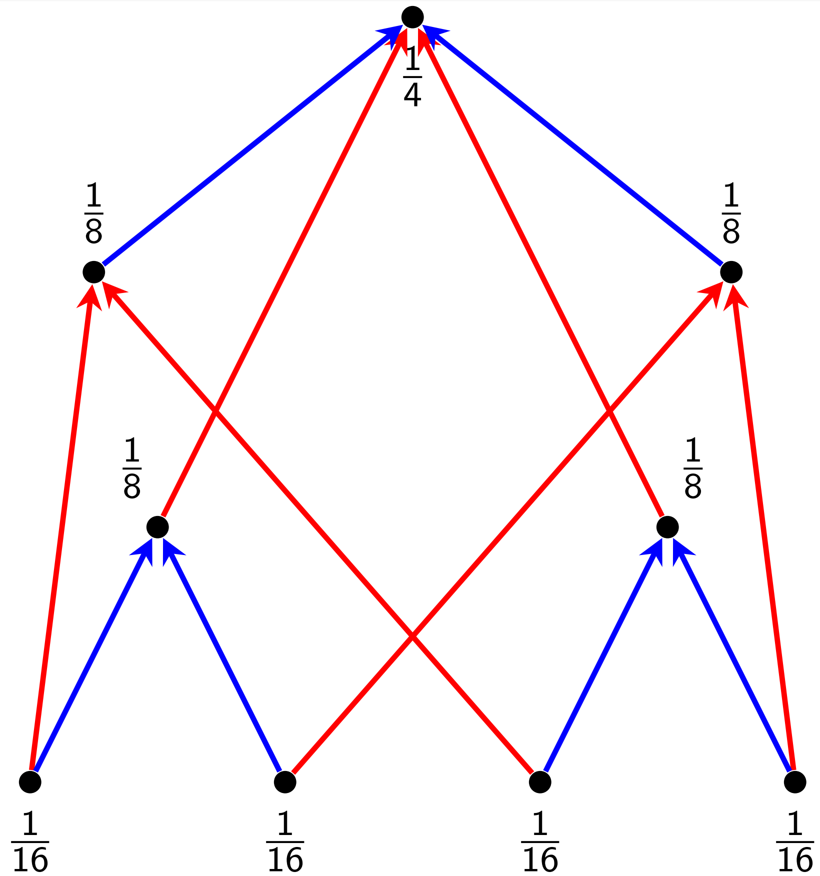


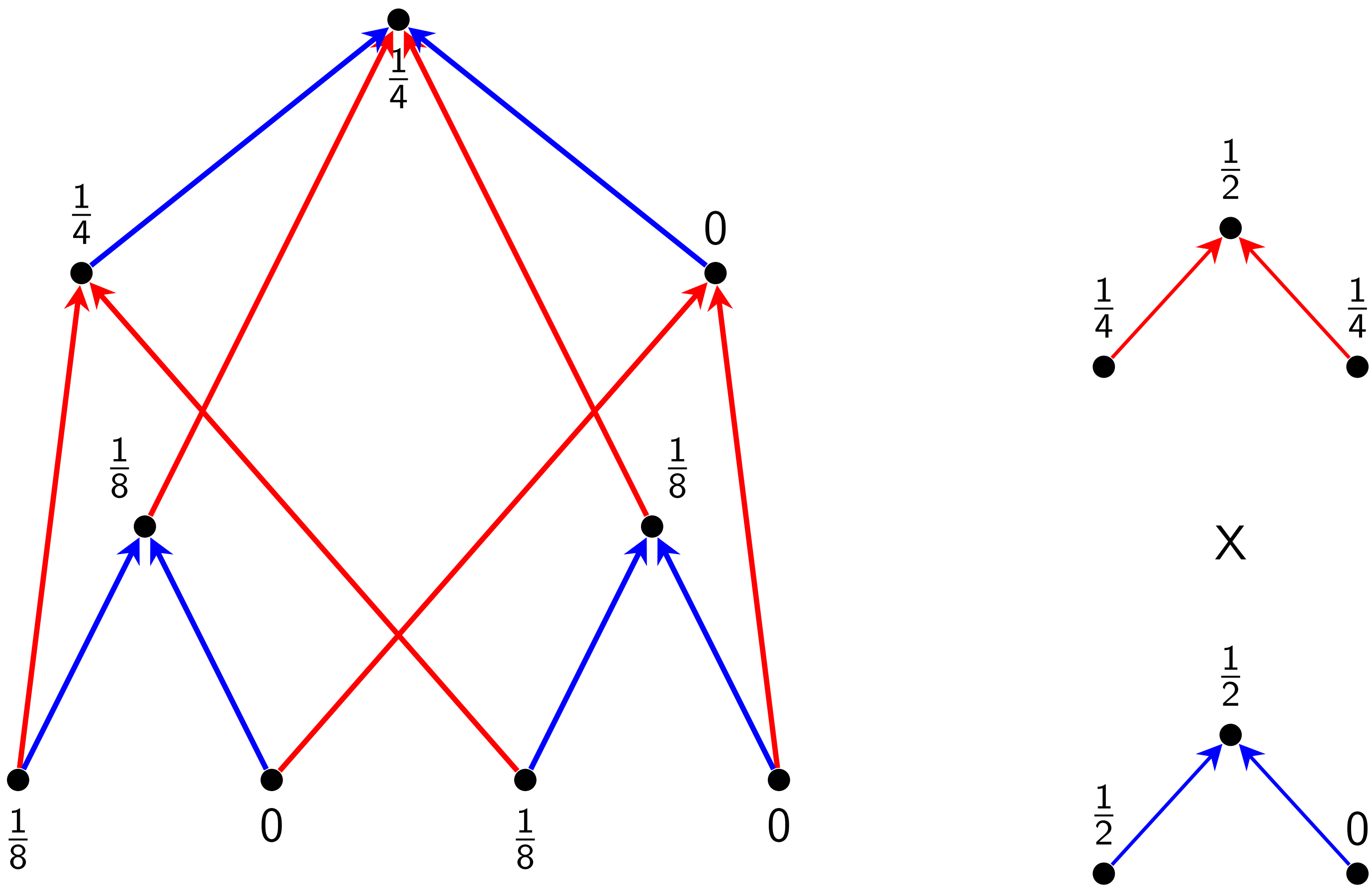
If you assign values to t_1, t_2, t_3, and t_4, then the graph trace values at the remaining vertices are as shown. Note that these graph traces must also satisfy the relation: $t_1 + t_2 + t_3 + t_4 = \frac{1}{4}$.

Proposition

Let g_λ be a graph trace on Λ and g_π be a graph trace on Π. Then $g_\lambda g_\pi(vw) = g_\lambda(v)g_\pi(w)$ is a graph trace on $\Lambda \times \Pi$.






$\frac{1}{4}$
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$\frac{1}{2}$
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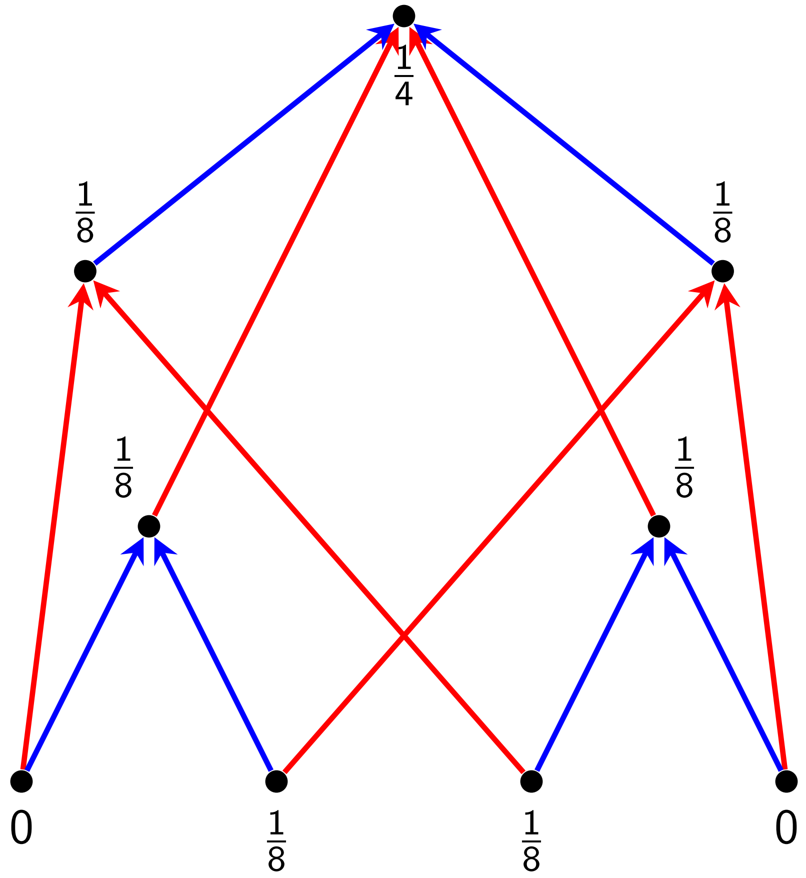
$\frac{1}{4}$
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0
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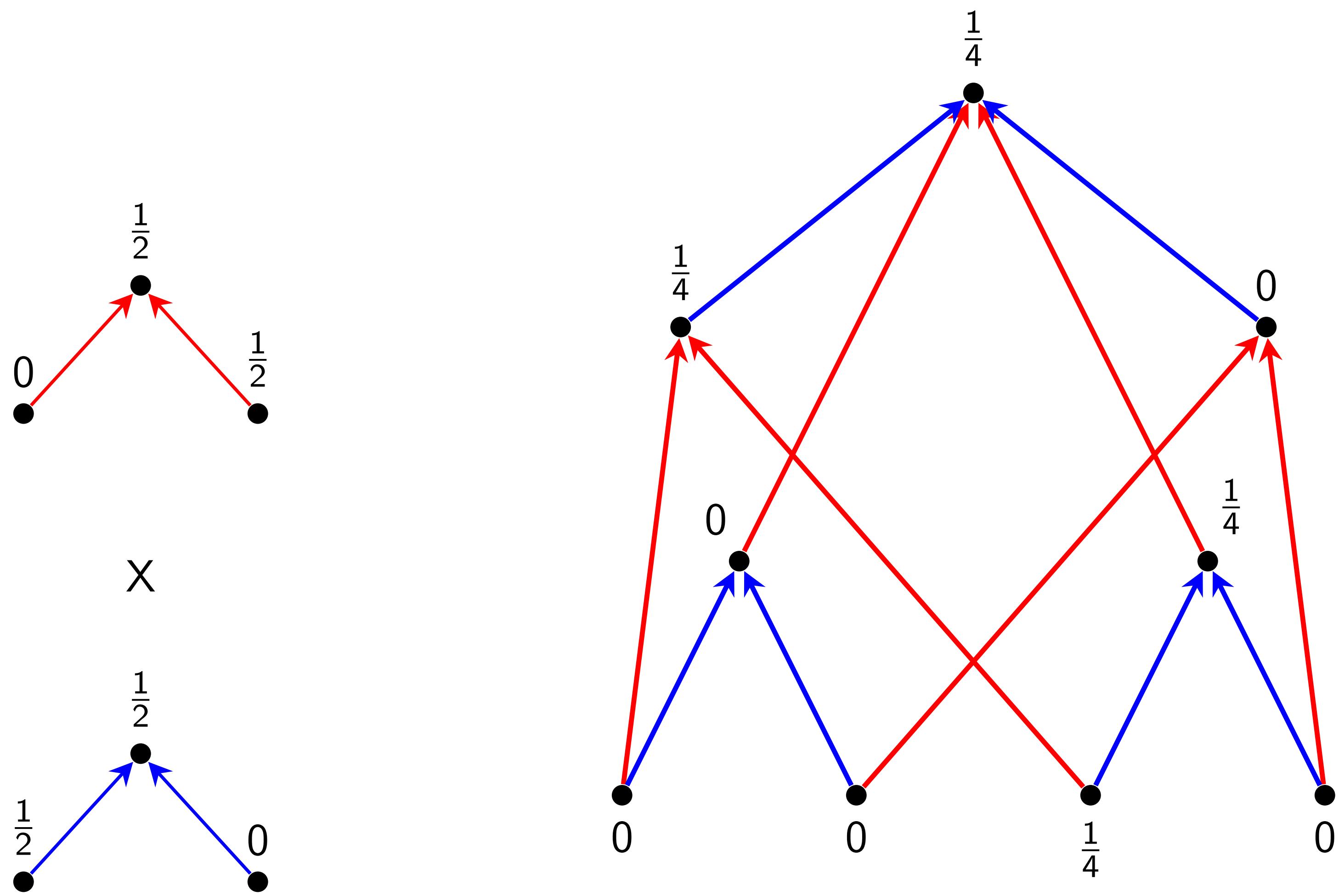
0

Proposition

Let g be a graph trace on $\Lambda \times \Pi$. Then define $g_1 : \Lambda^0 \to [0,1]$ and $g_2 : \Pi^0 \to [0,1]$ by

$$g_1(v) = \sum_{w \in \Pi^0} g(vw) \qquad g_2(w) = \sum_{v \in \Lambda^0} g(vw).$$

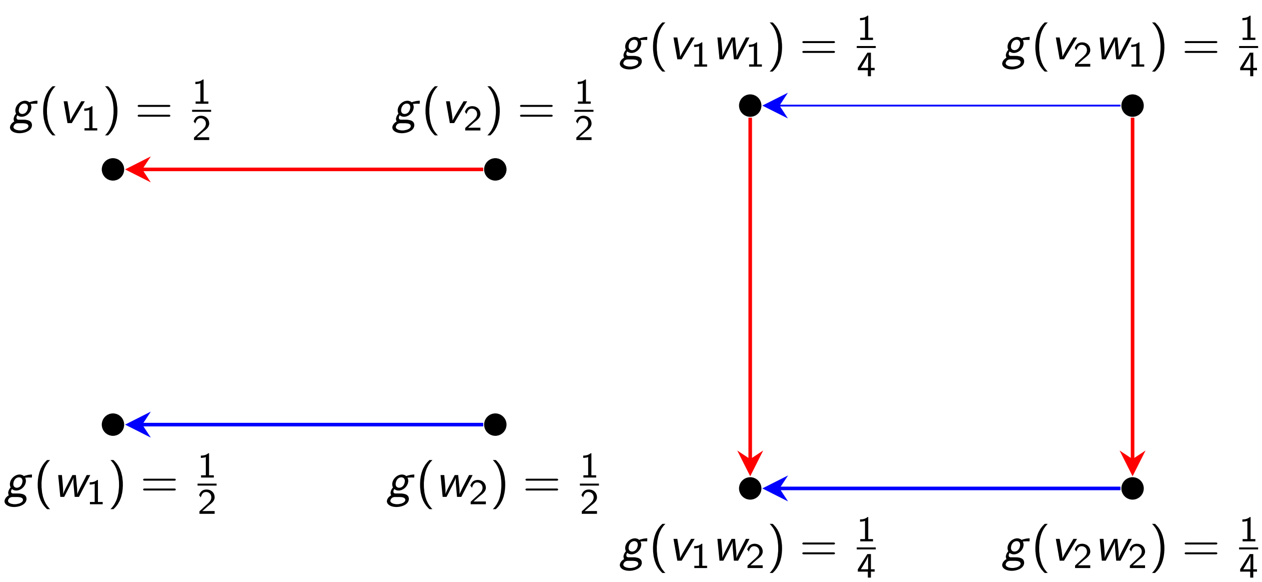
Then g_1 is a graph trace on Λ and g_2 is a graph trace on Π.



Note: $g_1(v_1) = \sum_{w \in \Pi^0} g(v_1 w) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Definition

A graph trace g on a product higher-rank graph $\Lambda \times \Pi$ is a *product trace* if $g = g_\Lambda g_\Pi$ where g_Λ is a graph trace on Λ and g_Π is a graph trace on Π.



Extreme traces on the product graph can be understood using extreme traces on the factor graphs, as shown by these propositions.

Proposition

If g is a trace on $\Lambda \times \Pi$ and g_1 is extreme, then $g = g_1 g_2$.

Proposition

The product trace $g_\Lambda g_\Pi$ is extreme if and only if g_Λ and g_Π are extreme.

Conjecture

Let Λ and Π be higher-rank graphs. Then every extreme trace on $\Lambda \times \Pi$ is a product trace.

The End

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