STABILITY OF C^*-ALGEBRAS ASSOCIATED TO GRAPHS, k-GRAPHS, AND GROUPOIDS

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Abstract. We give an emended proof of a result in the literature characterizing which graphs yield stable Cuntz-Krieger graph C^*-algebras. We strengthen this result by adding another necessary condition. We characterize stability of C^*-algebras associated to certain higher-rank graph C^*-algebras, as well as étale groupoids.

1. Introduction

A C^*-algebra A is said to be *stable* if $A \cong A \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the C^*-algebra of compact operators on an infinite-dimensional separable Hilbert space. Hjelmborg and Rørdam in [11, Theorem 3.3] characterized stability for C^*-algebras which admit approximate identities consisting of projections. Hjelmborg in [10, Theorem 2.14] used this characterization to characterize stability for the Cuntz-Krieger algebra $C^*(E)$ associated to a locally finite directed graph E. His description introduced two graph-theoretic concepts. A vertex $v \in E^0$ is said to be *left-finite* if one only finitely many vertices lie on the paths whose source is v; such a vertex can form an obstruction to stability of $C^*(E)$ by leading to a unital (and hence non-stable) quotient. A *graph trace* on E is a $\mathbb{R}^+$-valued function on the vertices of E which satisfies Cuntz-Krieger-type relations; when suitably normalized, such a trace induces a tracial state on $C^*(E)$, another obstruction to stability. Tomforde in [25, Theorem 3.2] treated the case of the Cuntz-Krieger graph C^*-algebra $C^*(E)$ associated to an arbitrary directed graph E, showing that $C^*(E)$ is stable if and only if no vertex on a cycle of E is left-finite and E has no bounded graph traces. There is a gap in the proof of [25, Theorem 3.2] which is fixed by the proof of Theorem 3.31, which also adds an additional necessary condition for stability.

A k-graph (or higher-rank graph) is a higher-dimensional generalization of a directed graph, formed from equivalence classes of directed paths within a colored directed graph. To any well-behaved k-graph Λ one can affiliate a universal C^*-algebra $C^*(\Lambda)$ generated by partial isometries satisfying Cuntz-Krieger relations. The analysis of $C^*(\Lambda)$ is closely analogous to that of a graph C^*-algebra; however, k-graphs can present rich combinatorial difficulties not present in graphs. Due to this complexity, we cannot at present characterize when a k-graph gives rise to a stable C^*-algebra. We give necessary conditions (Corollaries 4.13 and 4.16) and a sufficient condition (Theorem 4.17), all inspired by the graph case.

Generalizing even farther we study groupoid C^*-algebras. Again, we are unable to give a condition on a groupoid that is necessary and sufficient for its C^*-algebra to be stable . We give partial results inspired by the graph and k-graph cases.

The layout of the paper is as follows. In Section 2, we record some background results for stable C^*-algebras. In Section 3, we give background on graph C^*-algebras and give a complete characterization of stability for graph C^*-algebras (Theorem 3.31). In Section 4, we give a partial extension of these results to the realm of k-graph C^*-algebras. In Section 5, we extend some of our results to the realm of étale groupoid C^*-algebras.

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2. Stability of C^*-algebras

This section reviews the basic properties of stable C^*-algebras that we will use throughout the paper.

Note. In this section and throughout the paper, we use the term "ideal" to mean a closed, two-sided ideal of a C^*-algebra.

Definition 2.1. A C^*-algebra A is *stable* if $A \cong A \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the C^*-algebra of compact operators on an infinite-dimensional separable Hilbert space.

Remark. Because $\mathcal{K}$ is nuclear, there is no need to specify a tensor norm for $A \otimes \mathcal{K}$.

The following is immediate from the structure of $\mathcal{K}$ and is stated without proof.

Lemma 2.2. *No stable C^*-algebra is unital.*

Proposition 2.3 ([20, Cor. 2.3(ii)])**.** *Any ideal or quotient of a stable C^*-algebra is stable.*

Corollary 2.4. *A stable C^*-algebra has no nonzero unital quotients.*

Definition 2.5. A *tracial state* on a C^*-algebra A is a state $\phi \in S(A)$ such that $\phi(xy) = \phi(yx)$ for all $x, y \in A$. The (possibly empty) set of tracial states on A is denoted by $T(A)$.

Lemma 2.6 ([11, Prop. 5.1])**.** *If A is stable then $T(A) = \emptyset$.*

Recall that if p and q are projections in a C^*-algebra A, then we say that p is *subequivalent* to q, denoted $p \precsim q$, if there is a partial isometry $x \in A$ such that $x^*x = p$ and $xx^* \leq q$. By a *comparison* between two projections we mean such a partial isometry.

Lemma 2.7 ([25, Lemma 3.6],[10],[11])**.** *Let A be a C^*-algebra with increasing approximate identity $(p_n)_{n=1}^{\infty}$ consisting of projections. Then the following are equivalent.*

1. *A is stable.*
2. *For every projection $p \in A$ there exists a projection $q \in A$ such that $p \sim q$ and $p \perp q$.*
3. *For every $n \in \mathbb{N}$ there exists $m > n$ such that $p_n \precsim p_m - p_n$.*

3. Stability of graph algebras

In this section we give background theory on graph C^*-algebra, as well as the notions of left infinite vertices and graph traces. Then we prove a strengthened version (Theorem 3.31) of [25, Thm. 3.2] with an emended proof.

Definition 3.1. A *(directed) graph* consists of a quadruple $E = (E^0, E^1, r, s)$ where E^0 and E^1 are countable sets called, respectively, the *vertices* and *edges* of E, and $r, s : E^1 \to E^0$ are called the *range* and *source* maps. A vertex $v \in E^0$ is called *regular* if it receives a finite and positive number of edges; that is, if $0 < |r^{-1}(v)| < \infty$. If a vertex is not regular then it is said to be *singular*, either a *source* ($|r^{-1}(v)| = 0$) or an *infinite receiver* ($|r^{-1}(v)| = \infty$). A graph E is said to be *row-finite* if no vertex receives infinitely many edges, and E is said to have *no sources* if every vertex receives at least one edge.

A *path* in E is a finite sequence of edges $\lambda = e_1 e_2 \dots e_n$, such that $s(e_i) = r(e_{i+1})$ for $1 \le i \le n-1$. The (finite) *path space* of E, denoted E^*, is the set of all such paths. The range of a path $\lambda = e_1 \dots e_n$ is defined as $r(\lambda) := r(e_1)$ and the source is $s(\lambda) := s(e_n)$. The *length* of $\lambda = e_1 \dots e_n$ is defined to be $|\lambda| = n$. We include the vertices E^0 in E^* as the paths of length zero with $r(v) = v = s(v)$. A *cycle* is a directed path $\lambda \in E^* \setminus E^0$ with $s(\lambda) = r(\lambda)$. (Note that we orient paths as in [17], as opposed to the orientation used in [25].)

Definition 3.2. Let E be a directed graph and let B be a C^*-algebra. A *Cuntz-Krieger E-family* in B is a collection $\{S_e, P_v\}_{e\in E^1, v\in E^0} \subset B$, where the S_e are partial isometries with mutually orthogonal range projections and the P_v are mutually orthogonal projections, satisfying the following *Cuntz-Krieger relations*:

(1) if $e \in E^1$, then $S_e^* S_e = P_{s(e)}$
(2) if $e \in E^1$, then $S_e S_e^* \le P_{r(e)}$;
(3) if $v \in E^0$ is regular, then $\sum_{r(e)=v} S_e S_e^* = P_v$.

Typically we abbreviate $\{S_e, P_v\}_{e\in E^1, v\in E^0}$ as $\{S, P\}$. The C^*-algebra generated by a Cuntz-Krieger family $\{S, P\} \subset B$ is denoted by $C^*(S, P) \subset B$. The *Cuntz-Krieger graph C^*-algebra of E*, denoted $C^*(E)$, is the universal C^*-algebra generated by a Cuntz-Krieger E-family $\{s, p\}$; any other $C^*(S, P)$ is obtained as the quotient of $C^*(E)$ via a unique $*$-homomorphism $\pi : C^*(E) \to C^*(S, P)$ satisfying $\pi(p_v) = P_v$ and $\pi(S_e) = s_e$. (It can be shown that such a C^*-algebra exists for any E and is unique up to isomorphism.)

If $\mu = e_1 \dots e_n$ is a directed path in E, then by s_μ we denote the partial isometry $s_\mu = s_{e_1} \dots s_{e_n}$ in $C^*(E)$.

The following properties are well-known consequences of the Cuntz-Krieger relations and describe the $*$-algebraic structure of $C^*(E)$. We will use them constantly throughout the paper.

Corollary 3.3 ([17, Corollary 1.14])**.** *Suppose that E is a graph. Let $\mu, \nu \in E^*$. Then the following hold:*

(a) if $|\mu| = |\nu|$ and $\mu \ne \nu$, then $s_\mu s_\mu^* s_\nu s_\nu^* = 0$;

(b) More generally, we always have $s_\mu^* s_\nu = \begin{cases} s_{\mu'}^* & \text{if } \mu = \nu\mu' \\ s_{\nu'} & \text{if } \nu = \mu\nu' \\ 0 & \text{otherwise} \end{cases}$ *;*

(c) if $s_\mu s_\nu \ne 0$, then $\mu\nu$ is a path (that is, $s(\mu) = r(\nu)$) and $s_\mu s_\nu = s_{\mu\nu}$;

(d) if $s_\mu s_\nu^* \neq 0$, then $s(\mu) = s(\nu)$.

The following well-known fact follows directly from Corollary 3.3 (see [14, Prop. 1.4]).

Corollary 3.4. *Let E be a directed graph. If $E^0 = \{v_1, \dots, v_n\}$ is finite, then $C^*(E)$ is unital, with unit $1 = \sum_{k=1}^n p_{v_i}$. If $E^0 = \{v_k\}_{k=1}^\infty$ is infinite, then $C^*(E)$ is non-unital, and if we set $p_n = \sum_{k=1}^n p_{v_k}$ then $(p_n)_{n=1}^\infty$ forms a strictly increasing approximate identity for $C^*(E)$ consisting of projections.*

Definition 3.5. Let E be a directed graph and let $v \in E^0$. Define $L(v) := r(s^{-1}(v)) = \{r(\lambda) : \lambda \in E^*, s(\lambda) = v\}$. We say that v is *left finite* (resp. *left infinite*) if $L(v)$ is finite (resp. infinite). A cycle $\lambda \in E^*$ is *left finite* (resp. *left infinite*) if $s(\lambda)$ is *left finite* (resp. *left infinite*).

The ideal structure of a graph C^*-algebra described by certain sets of vertices.

Definition 3.6. Let E be a directed graph. A subset $H \subset E^0$ is *hereditary* if for any $e \in E^1$, $r(e) \in H$ implies $s(e) \in H$. The subset H is *saturated* if for any regular vertex v, the inclusion $s(r^{-1}(v)) = \{s(e) : r(e) = v\} \subset H$ implies $v \in H$.

The basic example of a saturated and hereditary subset of E^0 is $H_I = \{v \in E^0 : p_v \in I\}$, where I is any ideal of $C^*(E)$. Note that it is trivial to check that the arbitrary intersection of a collection of saturated subsets of E^0 is again saturated (if possibly empty).

Definition 3.7. Let E be a directed graph and let $H \subset E^0$ be any set of vertices. Then the *saturation* $\overline{H}$ is defined to be the smallest saturated subset of E^0 that contains H, that is

$$\overline{H} = \bigcap \{S \subset E^0 : H \subset S, S \text{ saturated}\}.$$

Lemma 3.8. *Let $H \subset E^0$ be any set of vertices.*

(1) *We can write*

$$\overline{H} = \bigcup_{n=0}^{\infty} H_n$$

where $H_0 = H$ and H_k is defined inductively as the set of all regular vertices $v \in E^0 \setminus \bigcup_{n=0}^{k-1} H_n$ such that $\{s(e) : r(e) = v\} \subset \bigcup_{n=0}^{k-1} H_n$.

(2) *If H is hereditary then $\overline{H}$ is a saturated and hereditary subset.*

Proof. The decomposition of $\overline{H}$ is straightforward to verify. As to the second claim, note that for $e \in E^1$, the only way that $r(e)$ can belong to H_n is if $s(e)$ belongs to H_{n-1}, unless $r(e) \in H_0 = H$ in which case the hereditary property of H gives $s(e) \in H_0$, so that $\overline{H}$ is hereditary. □

Lemma 3.9. *Let E be a graph and let $v \in E^0$. Then $H := E^0 \setminus L(v)$ (where $L(v)$ is defined as in Defn. 3.5), is a hereditary subset of E^0. The set H is saturated if v lies on a cycle or v is singular.*

Proof. Checking that H is always hereditary is straightforward. Suppose that v lies on a cycle, and let w be a regular vertex so that $r^{-1}(v) = \{e_1, \dots, e_n\}$ and $s(e_k) \in H$ for $k = 1, \dots, n$. If w were not in H, then there would exist a path from v to w. Unless the path were constant (i.e. a vertex), it would have to contain one of the edges $e_1, \dots, e_n$, so that v could reach the source of such an edge, contradicting our assumptions about $s(e_1), \dots, s(e_n)$. Thus the only way that w could fail to lie in H is if $w = v$. But $v \in L(v) = E^0 \setminus H$ via the constant path of length 0. Thus w must lie in H. So if v lies on a cycle, H is saturated.

Now suppose that v is singular, and let w be a regular vertex receiving edges $e_1, \dots, e_n$ with $s(e_i) \in H$ for $i = 1, \dots, n$. The only way that w could fail to belong to H is if $w = v$; as v is regular and w and singular this is impossible. Thus w belongs to H, and H is saturated.

□

Remark. One can check that the converse of Lemma 3.9 also holds, so that $E^0 \setminus L(v)$ is saturated if and only if v is singular or lies on a cycle.

One can realize certain ideals and quotients of graph C^*-algebras as graph C^*-algebras themselves. Given a saturated and hereditary set of vertices H, we can rather easily write down a description of the ideal generated in $C^*(E)$ by $\{p_v : v \in H\}$ as a graph algebra of a graph E_H, using results from [6] (which were later refined in [21]). The quotient by an ideal I_H is a bit more complicated to describe, as issues can arise where the naive choice of "quotient graph" can lead to relations among the vertex projections that are not present in the quotient $C^*(E)/I_H$. Vaguely speaking, the solution to this problem is to add extra edges to the quotient graph that prevent these relations from arising.

The following definition, originating in [6] and refined in [21], allows us to realize certain ideals as graph C^*-algebras. We don't need the full generality of [21, Def. 4.1], because we will not put any gap projections in our ideals.

Definition 3.10 (cf. [21, Def. 4.1])**.** Let E be a directed graph, let H be a nonempty saturated and hereditary subset of E^0. Let

$$F_1(H) = \{\alpha \in E^* : \alpha = e_1 \dots e_n, s(e_n) \in H, r(e_n) \notin H\}$$

Let $\overline{F_1(H)}$ denote a set of duplicates of $F_1(H)$, i.e. $\overline{F_1(H)} = \{\overline{\alpha} : \alpha \in F_1(H)\}$. Define $\overline{E}_H$ to be a graph with

$$\begin{aligned} \overline{E}_H^0 &= H \cup F_1(H) \\ \overline{E}_H^1 &= \{e \in E^1 : r(e) \in H\} \cup \overline{F_1(H)} \end{aligned}$$

and we extend r and s to $\overline{F_1(H)}$ via $r(\overline{\alpha}) = \alpha \in F_1(H)$ and $s(\overline{\alpha}) = s(\alpha) \in H$.

The following is a weaker version of the result in [21], sufficient for our purposes.

Theorem 3.11 ([21, Thm. 5.1])**.** *Let $H \subset E^0$ be a saturated and hereditary subset, let I_H be the ideal of $C^*(E)$ generated by $\{p_v : v \in H\}$. Then $I_H \cong C^*(\overline{E_H})$.*

Definition 3.12. Let E be a directed graph and let $H \subset E^0$. Call a path $\alpha \in E^*$ *H-minimal* if $s(\alpha) \in H$ and there is no path β with $s(\beta) \in H$ with $\beta\gamma = \alpha$ for some $\gamma \in E^* \setminus E^0$.

The vertices $\{v \in H\}$ are H-minimal paths of length zero. Note that if α and β are distinct H-minimal paths, then $s_\alpha^* s_\beta$ by Corollary 3.3. Thus we obtain the following simple result.

Lemma 3.13. *If we enumerate the H-minimal paths as $\{\alpha_i\}_{i=1}^n$, then $p_n := \sum_{i=1}^n s_\alpha s_\alpha^*$ forms an increasing approximate identity for I_H consisting of projections.*

The following definitions are used to realize the quotient of a graph C^*-algebra as a graph C^*-algebra.

Definition 3.14. Let E be a directed graph and let H be a saturated and hereditary subset of E^0. Define

$$B_H = \{v \in E^0 : |r^{-1}(v)| = \infty \text{ and } 0 < |r^{-1}(v) \cap s^{-1}(E^0 \setminus H)| < \infty\},$$

the set of *breaking vertices* for H. Define for $S \subset B_H$ the ideal $I_{(H,S)}$ generated by $\{p_v : v \in H\} \cup \{p_v^H : v \in S\}$, where

$$p_v^H = p_v - \sum_{\substack{r(e)=v \\ s(e)\notin H}} s_e s_e^*.$$

(The projections p_v^H are referred to as *gap projections*.)

Definition 3.15. Let H be a saturated hereditary subset of E^0 and let $S \subset B_H$, then we define a graph $E_{(H,S)}$ as follows (an apostrophe indicates a duplicate copy of an edge or vertex)

$$E^0_{(H,S)} = E^0 \setminus H \cup \{v' : v \in B_H \setminus S\}$$
$$E^1_{(H,S)} = \{e \in E^1 : s(e) \notin H\} \cup \{e' : s(e) \in B_H \setminus S\}$$

Remark. Note that any cycle $\lambda \in E^*_{(H,S)}$ must belong to E^*.

Proposition 3.16 ([1, Cor. 3.5])**.** *Suppose that $H \subset E^0$ is saturated and hereditary and $S \subset B_H$, and let $I_{(H,S)}$ be defined as in Definition 3.14. Then $C^*(E)/I_{(H,S)} \cong C^*(E_{(H,S)})$.*

In particular, this shows that if H is a *proper* saturated and hereditary subset of E^0, then the quotient $C^*(E)/I_H$ is a nonzero C^*-algebra. The following definition and lemma are needed to lift comparisons between projections in quotient graph C^*-algebras. The idea of the following lemma comes from the proof of [25, Thm. 3.2].

Lemma 3.17. *Let E be a directed graph such that $C^*(E)$ is stable. If $v \in E^0$ lies on a cycle or is singular, then v is left infinite.*

Proof. Let v be as in the statement of the theorem. By Lemma 3.9, we know that $H := E^0 \setminus L(v)$ is saturated and hereditary. The quotient graph $E_{(H,B_H)}$ has non-empty vertex set $L(v)$ by Definition 3.15. The quotient $C^*(E)/I_{(H,B_H)} \cong C^*(E_{(H,B_H)})$ is a nonzero stable C^*-algebra and hence non-unital; this implies that $L(v)$ is infinite. □

A key step in the proof of our main result (Theorem 3.31) involves comparing certain projections. We therefore require a short digression into comparison theory for positive elements.

Definition 3.18 ([19, Prop. 2.4])**.** Let A be a C^*-algebra and let $x, y \in A^+$. We write $x \precsim y$ if $\exists r_j \in A$ such that $r_j y r_j^* \to x$. For $x \in A$ and $y \in M_2(A)$ we write $x \precsim y$ if there exists a sequence $r_j \in M_{1,2}(A)$ of 1×2 matrices such that $r_j y r_j^* \to x$; we write $y \precsim x$ if there is a sequence $r_j \in M_{2,1}$ of 2×1 matrices such that $r_j x r_j^* \to y$.

It is shown in [19] that the relation $\precsim$ on A^+ is transitive and agrees with the usual partial ordering on positive elements; in particular, it agrees with the usual ordering projections ([19, Prop. 2.1]).

Lemma 3.19. *Let e, f be projections in a C^*-algebra A such that $ef = 0$. Then $e \oplus f \precsim e + f$.*

Proof. Let $r_j = [e \quad f]$ and see that $r_j(e \oplus f)r_j^* = e + f$. □

The following technical lemma is adapted from [10, Lemma 2.6]: we relax the hypothesis that the ideal is stable, but it only applies to graph C^*-algebras.

Lemma 3.20. *Let E be a directed graph and let H be a proper saturated hereditary subset of E^0, and let $\pi : C^*(E) \to C^*(E)/I_H$ denote the quotient $*$-homomorphism. Suppose that $e, f \in C^*(E)$ are projections such that $\pi(e) \precsim \pi(f)$. Then there exists a projection $q \in I_H$ such that $e \precsim f \oplus q$. We can choose q to have the form $q = \sum_{i=1}^k s_{\alpha_i} s_{\alpha_i}^*$ for a set of paths $\{\alpha_i\}_{i=1}^k$ with $s(\alpha_i) \in H$ for each $i = 1, \ldots, n$.*

Proof. Let $\{\alpha_i\}_{i=1}^\infty$ be the set of all H-minimal paths as in Definition 3.12, with $p_n = \sum_{i=1}^n s_{\alpha_i} s_{\alpha_i}^*$ the associated approximate unit from Lemma 3.13. By [12, Lemma 4.12], there is a positive element $x \in I(H)^+$ such that $e \precsim f \oplus x$. For each $\epsilon > 0$, define $\varphi_\epsilon : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\varphi_\epsilon(t) = \begin{cases} 0 & t \leq \epsilon, \\ \epsilon^{-1}(t - \epsilon) & \epsilon \leq t \leq 2\epsilon, \\ 1 & t \geq 2\epsilon. \end{cases}$$

By [19, Prop. 2.4], we can find $\delta \in (0, 1/2)$ such that $e \precsim f \oplus \varphi_\delta(x)$. Take a projection $q = p_n$ from the approximate unit such that $||x - p_n x p_n|| < \delta$; then [19, Prop. 2.2] implies that $\varphi_\delta(x) \precsim qxq$. It is trivial to verify that $qxq \precsim q(\frac{x}{||x||})q \precsim q$, so we have

$$e \precsim f \oplus \phi_\delta(x) \precsim f \oplus q,$$

with q in the desired form. □

Definition 3.21 ([10],[25])**.** A *graph trace* on a directed graph E is a function $g : E^0 \to \mathbb{R}^+$ such that

(1) for any $v \in E^0$, we have $g(v) \geq \sum_{r(e)=v} g(s(e))$ (in particular, the sum is always convergent), and
(2) for any regular $v \in E^0$, we have $g(v) = \sum_{r(e)=v} g(s(e))$.

We define the *norm* of g to be $||g|| := \sum_{v \in E^0} g(v)$, and when g has finite norm we say that g is *bounded*. If $||g|| = 1$ then we call g a *normalized* graph trace. The (possibly empty) collection of graph traces on E with norm 1 is denoted by $T(E)$.

Remark. The set of graph traces forms a convex cone and any bounded graph trace can be scaled to obtain a normalized graph trace.

Example 3.22 ([25])**.** If E is a directed graph and τ is a tracial state on $C^*(E)$, then we can define a normalized graph trace g_τ on E via

$$g_\tau(v) = \tau(p_v).$$

That is, any tracial state on $C^*(E)$ induces a graph trace on E. (This process for obtaining graph traces from tracial states is called *restriction*.)

In fact, every graph trace on E arises as the restriction of a tracial state on $C^*(E)$. In other words, we can induce tracial states from graph traces.

Theorem 3.23 ([23, Prop. 3.2],[5, Thm. 4.23])**.** *Let E be a directed graph, and let $g \in T(E)$ be a normalized graph trace. Then there is a unique tracial state $\tau_g \in T(C^*(E))$ such that every $\alpha, \beta \in E^*$*

$$\tau_g(s_\alpha s_\beta^*) = \begin{cases} g(s(\alpha)) & \alpha = \beta \\ 0 & else \end{cases}.$$

In particular, $\tau_g(p_v) = g(v)$.

Remark. The map $g \mapsto \tau_g$ is a left inverse to the map $\tau \mapsto g_\tau$, i.e. $g_{\tau_h} = h$ for any normalized graph trace h. These maps are bijective exactly when every cycle of E is *essentially left infinite*, see [5, Thm. 4.41]. (This is weaker condition to place on E than Condition (K), which implies that the maps are bijective, as noted in [23].)

Combining Theorem 3.23 and Corollary 2.6 we obtain the following corollary, which is basically contained in the proof of [25, Thm. 3.2] and in a more restricted form in [10, Lemma 2.8].

Corollary 3.24. *If E is a directed graph such that $C^*(E)$ is stable, then E has no bounded graph traces.*

Lemma 3.25 ([25, Cor. 3.3])**.** *Let E be a directed graph and let $v \in E^0$ be left infinite. For any finite subset $F \subset E^0$ there is a finite subset $W \subset E^0 \setminus F$ such that $p_v \lesssim \sum_{w \in W} p_w$.*

Proof. If $w \in L(v) = \{r(\lambda) : \lambda \in E^*, s(\lambda) = v\}$, say $r(\lambda) = w$ and $s(\lambda) = v$ then $p_v = s_\lambda^* s_\lambda \sim s_\lambda s_\lambda^* \leq p_w$. Therefore we can take any vertex $w \in L(v) \setminus F$ and set $W = \{w\}$. □

The following well-known fact is included for convenience.

Lemma 3.26. *Let p, p', q, q' be projections in a C^*-algebra A with $p \perp q$ and $p' \perp q'$. If $p \lesssim p'$ and $q \lesssim q'$, then $p + q \lesssim p' + q'$.*

The following definition makes it easier to refer to certain paths.

Definition 3.27. If E is a directed graph and $v \in E^0$, let $vE^* = \{\lambda \in E^* : r(\lambda) = v\}$ and $E^*v = \{\lambda \in E^* : s(\lambda) = v\}$. We write wE^*v for $wE^* \cap E^*v$.

Lemma 3.28. *Suppose that E is a directed graph in which every cycle is left infinite and that $T(E) = \emptyset$. Then every singular vertex in E^0 is left infinite.*

Proof. Let $w \in E^0$ be a singular vertex which is not left infinite, so that $L(w)$ is a finite set; we derive a contradiction. First, note that the hypotheses imply w does not lie on any cycle.

Claim: there is a vertex $v \in L(w)$ which is singular and such that E^*v is finite. If E^*w is finite we are done. Otherwise consider the (finite) set of vertices $\{r(e) : s(e) = w\}$. At least one of these must receive infinitely many edges, say w'. If E^*w' is finite, we are done; otherwise repeat the operation, obtaining a new singular vertex w'' (note that $w', w'', \ldots$ all belong to $L(w)$). The process can never repeat, because that would entail the existence of a directed cycle among the vertices of $L(w)$, contradicting left finiteness of w. We cannot repeat the process forever, because w is assumed to be left finite. Thus we eventually find a singular vertex $v \in L(w)$ so that E^*v is finite.

Now we can define a function $g : E^0 \to \mathbb{N}$ given by setting $g(z) = |zE^*v|$. Note that $g \in \ell^1(E^0)$, with $||g||_1 = |E^*v|$, and it is not difficult to check that g satisfies the Cuntz-Krieger relations for graph traces. This contradicts the assumption that there are no bounded graph traces on E. □

Lemma 3.29 ([10, Lemma 2.3])**.** *Let E be a directed graph, and let $E^0 = \{v_0, v_1, \ldots, \}$ be an enumeration of the vertices of E, with approximate identity of projections $(p_n)_{n=0}^\infty$ as in Definition 3.4. Then $C^*(E)$ is stable if and only if for any $F \subset E^0$, there exists a finite set $W \subset E^0 \setminus F$ such that $\sum_{v\in F} p_v \lesssim \sum_{w\in W} p_w$.*

Proof. Apply Corollary 3.4 and Lemma 2.7. □

Remark. As pointed out in [25], an induction argument shows that $C^*(E)$ is stable as long as we can, for every $v \in E^0$ and finite $F \subset E^0$, find some finite $W \subset E^0 \setminus F$ such that $p_v \lesssim \sum_{w\in W} p_w$.

Here is the characterization of stable graph C^*-algebras given in [25, Theorem 3.2].

Theorem 3.30 ([25, Thm. 3.2])**.** *Let E be a directed graph. Then the following are equivalent.*

(a) $C^*(E)$ is stable.
(b) $C^*(E)$ has no nonzero unital quotients and no tracial states.
(c) E has no left finite cycles and $T(E) = \emptyset$.
(d) E has no left finite cycles and no nonzero bounded graph traces.
(e) For any $v \in E^0$ and any finite $F \subset E^0$, there exists finite $W \subset E^0 \setminus F$ such that $p_v \lesssim \sum_{w\in W} p_w$.
(f) For any finite $V \subset E^0$ there exists finite $W \subset E^0 \setminus V$ such that $\sum_{v\in V} p_v \lesssim \sum_{w\in W} p_w$.

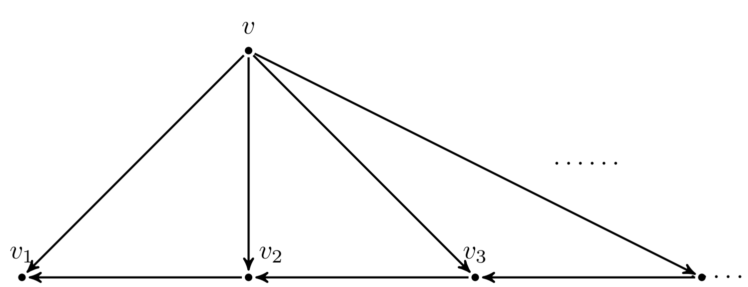

FIGURE 1.

The proof of this theorem in [25] contains a gap. Let $H = H(E) \subset E^0$ be the set of left infinite vertices and let $\overline{H}$ denote its saturation. There is a small error in the proof that for any $v \in \overline{H}$ and any finite $F \subset E^0$ there exists finite $W \subset E^0 \setminus F$ with $p_v \lesssim \sum_{w\in W} p_w$. Specifically the comparison constructed is backwards: $xpx^* = q$ implies that $q \lesssim p$, not vice versa. This will be addressed in the proof of Theorem 3.31 below.

There's also an issue with the incorrect statement of [25, Lemma 3.8], which we have already fixed with Lemma 3.20. The graph in Figure 1 shows that one cannot assume that the ideal generated by the will satisfy the hypotheses of [10, Lemma 2.6]

The search for a proof that avoids these problems eventually lead us to consider singular vertices, which ended up adding a necessary condition that a graph must satisfy in order to yield a stable C^*-algebra: all singular vertices must be left infinite. Here our the strengthened characterization of stable graph C^*-algebras.

Theorem 3.31. *Let E be a directed graph. The following are equivalent.*

1. *$C^*(E)$ is stable.*
2. *$C^*(E)$ has no nonzero unital quotients and no tracial states.*
3. *$C^*(E)$ has no left finite cycles and no bounded graph traces.*
4. *E has no left finite cycles, no left finite singular vertices, and no bounded graph traces.*
5. *for any vertex $v \in E^0$ and any finite $F \subset E^0$, there exists finite $W \subset E^0 \setminus F$ such that $p_v \lesssim \sum_{w \in W} p_w$.*
6. *for any finite $V \subset E^0$, and any finite $F \subset E^0$, there exists finite $W \subset E^0 \setminus F$ such that $\sum_{v \in V} p_v \lesssim \sum_{w \in W} p_w$.*

Proof. (1) $\implies$ (2): Apply Corollary 2.4 and Corollary 2.6.

(2) $\implies$ (3): Apply Corollary 3.17 and Corollary 3.24.

(3) $\implies$ (4): Apply Lemma 3.28.

(4) $\implies$ (5): We must show that, for any $v \in E^0$ and finite $F \subset E^0$, there is a finite set $W \subset E^0 \setminus F$ such that $p_v \lesssim \sum_{w \in E} p_w$. We adapt the proof of [25, Thm 3.2]. Let H be the set of left infinite vertices and $\overline{H}$ be its saturation.

Case I: $v \in \overline{H}$. As in the proof of [25, Thm. 3.2], we establish this by induction on $k = \min\{n : v \in H_n\}$. If $k = 0$, then Lemma 3.25 shows that this is possible. Suppose inductively that we can, for any $v' \in \cup_{n=0}^{k-1} H_n$ and for any finite $F' \subset E^0$, find a finite $W \subset E^0 \setminus F'$ with $p_{v'} \lesssim \sum_{w \in W} p_w$. Because $v \in H_k$ it must be the case we can list the edges with range v as $e_1, \ldots, e_j$, and $v_i := s(e_i) \in \bigcup_{n=0}^{k-1} H_n$ for each $i = 1, \ldots, j$. Use the inductive assumption to find a finite set $W_1 \subset E^0 \setminus F$ with $s_{e_1} s_{e_1}^* \sim p_{v_1} \lesssim \sum_{w \in W_1} p_w$. Repeat this, finding W_i a finite subset of $F \cup W_1 \ldots W_{i-1}$ and $p_{v_i} \lesssim \sum_{w \in W_i} p_w$ for $i = 1, \ldots, j$. Thus there are partial isometries $x_1, \ldots, x_j$ with $x_i^* x_i = s_{e_i} s_{e_i}^*$ and $x_i x_i^* \leq \sum_{w \in W_i} p_{w_i}$. Set $W = W_1 \sqcup \ldots \sqcup W_j$, a finite subset of $E^0 \setminus F$. The pairwise disjointness of the sets W_i and Lemma 3.26 ensures that $x = \sum_{i=1}^j x_i$ is a partial isometry with $x^* x = \sum_{i=1}^j s_{e_i} s_{e_i}^* = p_v$ and $x x^* \leq \sum_{w \in W} p_w$. Thus in the first case, the condition (5) holds.

Case II: $v \notin \overline{H}$. This follows the same as in the proof of [25, Thm. 3.2], with a few adjustments. Let I_H be the ideal of $C^*(E)$ generated by $\{p_v : v \in H\}$ and let $\pi : C^*(E) \to C^*(E)/I_H \cong C^*(E_{(H,\emptyset)})$ be the quotient $*$-homomorphism (where the isomorphism is Theorem 3.11). Note that any vertex v which is the range of a cycle $\lambda \in E^*$ must belong to H, by our assumption that every cycle is left infinite. Thus the graph $E \setminus H$ has no cycles and so $C^*(E_{(H,\emptyset)})$ is AF by [7, Cor. 2.13]. Any tracial state on $C^*(E_{(H,\emptyset)}))$ would give a tracial state on $C^*(E)$ by composing with π, and this in turn would restrict to a graph trace on E. Thus [2, Thm. 4.10] implies that $C^*(E_{(H,\emptyset)})$ is a stable AF algebra.

Enumerate $E^0 \setminus \overline{H}$ as $\{v_k\}_{k=1}^{\infty}$, with $v = v_1$. Set $q_n = \sum_{i=1}^n \pi(p_{v_i})$ and notice that $(q_n)_{n=1}^{\infty}$ is an approximate unit for $C^*(E)/I_H$ consisting of projections. Let

$m = \max\{k : v_k \in F\}$, where we set $m = 1$ if $F \subset \overline{H}$. By stability of $C^*(E)/I_H$ and Lemma 2.7 we can find $n > m$ such that $q_m \precsim q_n - q_m$. In other words, $\pi(p_v) \precsim \pi(\sum_{k=m+1}^{n} p_{v_k})$; now Lemma 3.20 provides us with set $\{\alpha_i\}_{i=1}^{n}$ of H-minimal paths such that

$$p_v \precsim \left(\sum_{k=m+1}^{n} p_{v_k}\right) + \left(\sum_{i=1}^{j} s_{\alpha_i} s_{\alpha_i}^*\right)$$

For each $i = 1, \ldots, j$ we have $s_{\alpha_i} s_{\alpha_i}^* \sim s_{\alpha_i}^* s_{\alpha_i} = p_{s(\alpha_i)}$, and as $s(\alpha_i) \in \overline{H}$ we can use Case I to find $W_1, \ldots, W_j$ finite sets in E^0 so that $W_i \cap (F \cup \{v_{m+1}, \ldots, v_n\} \cup W_1 \cup \ldots W_{i-1}) = \emptyset$ and $s_{\alpha_i} s_{\alpha_i}^* \precsim \sum_{w \in W_i} p_w$. Finally set $W = \{v_{m+1}, \ldots, v_n\} \cup W_1 \cup \ldots \cup W_j$ and we have $p_v \precsim \sum_{w \in W} p_w$ with $W \cap F = \emptyset$.

(5) $\Longrightarrow$ (6): This follows using the exact same argument as in [25, Thm. 3.2].

(6) $\Longrightarrow$ (1): This follows from Lemma 3.29. □

Remark. Another proof of the (4) $\Longrightarrow$ (5) part of Theorem 3.31 could use the fact that the ideal $I_{(H,B_H)}$ is itself isomorphic to a slightly different graph C^*-algebra as in [21]. The present approach seems to involve the least machinery.

A *sink* in a directed graph is a vertex $v \in E^0$ which is not the source of any edge. The first part of the following corollary, which is the same as [25, Cor. 3.3], is one of the most direct ways to tell if a given directed graph yields a stable C^*-algebra.

Corollary 3.32 ([25, Cor. 3.3])**.** *If E is a directed graph and every vertex of E is left infinite, then $C^*(E)$ is stable. If E has no sinks and $C^*(E)$ is stable, then every vertex of E is left infinite.*

Remark. As pointed out in [25], taking E to be an infinite chain of edges terminating in a sink gives an example of a directed graph whose C^*-algebra is stable (in fact, $C^*(E) \cong \mathcal{K}$), yet none of whose vertices are left infinite.

4. Stability of k-graph algebras

The rest of the paper will center around extending parts of Theorem 3.31 to "combinatorial" C^*-algebras beyond graph C^*-algebras. So far a complete generalization has eluded us because the "(4) $\Longrightarrow$ (5)" part of Theorem 3.31 uses facts about AF graph C^*-algebras that don't readily generalize. In the present section we consider k-graphs (also known as *higher-rank graphs*), which were introduced in [13] as higher-dimensional generalizations of directed graphs.

Note. The semigroup $\mathbb{N}^k$ is a category with one object and composition given by coordinate-wise addition. For an element $m \in \mathbb{N}^k$ we denote the coordinates by m_i for $i = 1, \ldots, k$. The standard basis elements in $\mathbb{N}^k$ are denoted by $e_1, e_2, \ldots, e_k$, so that $m = \sum_{i=1}^{k} m_i e_i$. (We regard all categories as "arrows-only" so that the objects are precisely the identity morphisms.)

Definition 4.1. A *k-graph* (also called a *higher-rank graph*) consists of a countable category Λ along with a *degree functor* $d : \Lambda \to \mathbb{N}^k$ which satisfies the *factorization property*: if $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ satisfy $d(\lambda) = m + n$, then there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$, and $\mu\nu = \lambda$.

For $n \in \mathbb{N}^k$ we write $\Lambda^n := d^{-1}(n) = \{\lambda \in \Lambda : d(\lambda) = n\}$ for the paths of degree n in Λ. Using the factorization property, one can see that the set of objects (that is, identity morphisms) of Λ is precisely $\Lambda^0 = d^{-1}(0)$, which we refer to as the set of *vertices* of Λ. The range and source maps $r, s : \Lambda \to \Lambda^0$ satisfy $r(\lambda)\lambda = \lambda = \lambda s(\lambda)$ for all $\lambda \in \Lambda$. For $v \in \Lambda^0$, we set $v\Lambda = \{\lambda \in \Lambda : r(\lambda) = v\}$ and for $n \in \mathbb{N}^k$ let $v\Lambda^n$ denote $v\Lambda \cap \Lambda^n$.

Remark. In analogy with directed graphs the morphisms in a k-graph are often called *paths*. Directed graphs can be identified with 1-graphs, where the vertices are the morphisms of degree 0, the edges are the morphisms of degree 1, and the other paths in E^* are the morphisms of degree > 1. (For details, see [17, Ch.10].)

The following restrictions on a k-graph ensure the existence of a non-trivial associated C^*-algebra, and although they are not the most general such conditions, (see e.g. [17, Ch. 10]), they are comparatively simple to state and give a wide variety of examples.

Definition 4.2. Let Λ be a k-graph. We say that Λ is *row-finite* if, for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, the set $v\Lambda^n$ is finite.

We say that Λ *has no sources* if, given any $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ the set $v\Lambda^n$ is non-empty.

Definition 4.3. If Λ is a row-finite k-graph with no sources and B is a C^*-algebra then a *Cuntz-Krieger Λ-family in B* is a collection of partial isometries $S = \{S_\lambda : \lambda \in \Lambda\} \subset B$ satisfying:

(i) $\{S_v : v \in \Lambda^0\}$ are mutually orthogonal projections;
(ii) $S_\lambda S_\mu = S_{\lambda\mu}$ if $s(\lambda) = r(\mu)$;
(iii) $S_\lambda^* S_\lambda = S_{s(\lambda)}$ for every $\lambda \in \Lambda$;
(iv) $S_v = \displaystyle\sum_{\{\lambda\in\Lambda^n : r(\lambda)=v\}} S_\lambda S_\lambda^*$ for every $v \in \Lambda^0$ and every $n \in \mathbb{N}^k$.

The universal C^*-algebra generated by a Λ-family is called the *k-graph C^*-algebra* of Λ, and is denoted by $C^*(\Lambda) = C^*(\{s_\lambda : \lambda \in \Lambda\})$.

The existence of a nonzero Cuntz-Krieger for arbitrary Λ as above is shown in [13, Prop. 2.11]. For $v \in \Lambda^0$ we denote s_v by p_v, in analogy with the graph case.

Definition 4.4. For $m, n \in \mathbb{N}^k$ denote $m \vee n \in \mathbb{N}^k$ by $(m \vee n)_i = \max\{m_i, n_i\}$ for $i = 1, \ldots, k$. If $\lambda, \mu \in \Lambda$ have $r(\lambda) = r(\mu)$, then a *minimal common extension* of λ and μ is an element $\nu \in \Lambda$ such that there exist $\alpha, \beta \in \Lambda$ with $\nu = \lambda\alpha = \mu\beta$ and $d(\nu) = d(\lambda) \vee d(\mu)$. Let $\Lambda^{\min}(\lambda, \mu)$ be the set of all ordered pairs $(\alpha, \beta) \in \Lambda \times \Lambda$ such that $\lambda\alpha = \mu\beta$ is a minimal common extension of λ and μ.

The k-graph analogue of Corollary 3.3 is the following, which justifies the equation $C^*(\Lambda) = \overline{\operatorname{span}}\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda\}$.

Lemma 4.5 ([17, Lemma 10.6])**.** *Let Λ be a row-finite k-graph with no sources, and let $S = \{S_\lambda\}$ be a Cuntz-Krieger Λ-family. Then for every $\lambda, \mu \in \Lambda$,*

$$S_\lambda^* S_\mu = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} S_\alpha S_\beta^*.$$

Corollary 4.6. *The k-graph C^*-algebra $C^*(\Lambda)$ is unital if and only if Λ^0 is finite; otherwise the sequence $\{\sum_{i=1}^n p_{v_i}\}_{n=1}^\infty$ (where $\{v_i\}_{i=1}^\infty = \Lambda^0$) forms an approximate identity consisting of projections, just as in Corollary 3.4.*

Definition 4.7. Let Λ be a row-finite k-graph with no sources. A subset $H \subset \Lambda^0$ is said to be *hereditary* if $r(\lambda) \in H$ implies $s(\lambda) \in H$. A subset $H \subset \Lambda^0$ is said to be *saturated* if $s(v\Lambda^n) \subset H$ implies $v \in H$ for any $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. If $H \subset \Lambda^0$ then we let I_H denote the ideal of $C^*(\Lambda)$ generated by $\{p_v : v \in H\}$.

Theorem 4.8 ([17, Ch. 10])**.** *If Λ is a row-finite k-graph with no sources and $H \subset \Lambda^0$ is saturated and hereditary then $\Lambda \setminus \Lambda H := (\Lambda^0 \setminus H, s^{-1}(\Lambda^0 \setminus H), r, s)$ is a row-finite k-graph with no sources and $C^*(\Lambda \setminus \Lambda H) \cong C^*(\Lambda)/I_H$.*

Definition 4.9. Let Λ be a row-finite k-graph with no sources. A *k-graph trace* on Λ is a function $g : \Lambda^0 \to [0, \infty)$ such that $g(v) = \sum_{\lambda \in v\Lambda^n} g(s(\lambda))$ for all $v \in \Lambda^0$ and all $n \in \mathbb{N}^k$. A k-graph trace g is *bounded* if $\sum_{v \in \Lambda^0} g(v) < \infty$, in which case the sum $\sum g(v)$ is referred to as the *norm* of g. The (possibly empty) collection of k-graph traces on Λ with norm 1 is denoted by $T(\Lambda)$.

Example 4.10. Just as in Example 3.22, we can obtain k-graph traces on Λ by restricting tracial states on $C^*(\Lambda)$.

Producing tracial states on k-graph algebras requires a different approach from that used by Tomforde in [24]. In [8, Lemma 2.1] it is shown that any $g \in T(\Lambda)$ which is *faithful* (in the sense that $g(v) > 0$ for all v) is the restriction of a faithful tracial state τ_g on $C^*(\Lambda)$.

Lemma 4.11. *Let Λ be a row-finite k-graph with no sources and let g be a k-graph trace on Λ. Then $H = \{v \in \Lambda^0 : g(v) = 0\}$ is a saturated and hereditary subset of H.*

Proof. This is essentially the same as [24, Lemma 3.7] and follows from the definition of a k-graph trace. □

Proposition 4.12. *Let Λ be a row-finite k-graph with no sources and let g be a k-graph trace on Λ with norm 1. Then there is a tracial state τ_g on $C^*(\Lambda)$ such that $\tau_g(p_v) = g(v)$ for each $v \in \Lambda^0$.*

Proof. Let H be the zero set of g. By the preceding lemma H is a saturated and hereditary subset of Λ^0. The restriction of g to $\Gamma(\Lambda \setminus \Lambda H)$ is faithful by construction of H. Thus by [8, Lemma 2.1] it lifts to a faithful tracial state τ on the quotient $C^*(\Lambda)/I_H$. If q is the quotient map $C^*(\Lambda) \to C^*(\Lambda)/I_H$, then $\tau \circ q$ is the desired tracial state on $C^*(\Lambda)$. □

Remark. One could obtain the previous result using an approach similar to [5, Thm. 4.26]; this would require generalizing the proof of [5, Thm. 4.19] to k-graphs as a preliminary step, so we use the present approach instead.

Corollary 4.13. *Let Λ be a row-finite k-graph with no sources. If $C^*(\Lambda)$ is stable, then $T(\Lambda) = \emptyset$.*

Proof. The same argument as Corollary 3.24, with Proposition 4.12 playing the role of Theorem 3.23. □

Definition 4.14. Let Λ be a k-graph. A *cycle* in Λ is a path $\lambda \in \Lambda \setminus \Lambda^0$ such that $r(\lambda) = s(\lambda)$. Given $v \in \Lambda^0$, we define

$$L(v) = r(\Lambda v) = \{w \in \Lambda^0 : w = r(\lambda) \text{ for some } \lambda \in s^{-1}(v)\}.$$

We say that v is *left finite* (resp. *left infinite*) if $L(v)$ is finite (resp. infinite). A cycle λ is called *left finite* (resp. left infinite) if $s(\lambda)$ is left finite (resp. left infinite).

Lemma 4.15. *Let Λ be a row-finite k-graph with no sources. Suppose that $v = s(\lambda)$ for some cycle $\lambda \in \Lambda$; then $H = \Lambda^0 \setminus L(v)$ is a (possibly empty) saturated and hereditary subset.*

Proof. The same as Lemma 3.9. ☐

Corollary 4.16. *Suppose that Λ is a row-finite k-graph with no sources and $C^*(\Lambda)$ is stable. Then every cycle in Λ is left infinite.*

Proof. The same as the proof of Lemma 3.17, with Theorem 4.8 playing the role of Proposition 3.16. ☐

Corollaries 4.13 and 4.16 describe necessary conditions for a k-graph Λ to yield a stable C^*-algebra. We have been unable to show that any Λ with no left finite cycles and with $T(\Lambda) = \emptyset$ is stable. The main difficulty is that there is not yet a criterion to decide when a k-graph Λ yields an AF k-graph C^*-algebra (see [8] for a detailed account), which prevents the proof of Theorem 3.31 from generalizing.

However, we have the following *sufficient* condition.

Theorem 4.17. *Let Λ be a row-finite k-graph with no sources. Suppose that every vertex $v \in \Lambda^0$ is left infinite. Then $C^*(\Lambda)$ is stable.*

Proof. Let $\Lambda^0 = \{v_1, v_2, \dots\}$ denote the vertex set of Λ. By Corollary 4.6, we have an approximate identity $(p_n)_{n=1}^\infty$ with $p_n = \sum_{i=1}^n p_{v_i}$. Note that for any path $\lambda \in \Lambda$ we have, just as in the graph case, that

$$p_{s(\lambda)} = s_\lambda^* s_\lambda \sim s_\lambda s_\lambda^* \leq p_{r(\lambda)}.$$

Then the same reasoning as in Lemma 3.29, (and the remark that follows) and Lemma 3.25 gives the desired result. ☐

Remark. The converse to Theorem 4.17 does not hold, due to the example in the comment following [25, Cor. 3.3].

Question. If Λ is a row-finite k-graph Λ with no sources, such that $T(\Lambda) = \emptyset$ and every cycle of Λ is left infinite, is $C^*(\Lambda)$ stable?

5. Stability of groupoid and inverse semigroup C^*-algebras

In this section, we generalize the main results of Section 3 to the context of C^*-algebras of étale groupoids. We only include a limit introduction to groupoid C^*-algebras; the interested reader should consult [18] and [3] for a more detailed background.

We can generalize the notion of a left infinite vertex (in the sense of Definition 3.5) to the context of groupoids in *two* ways, which do not seem to be equivalent. Definition 5.13 is the analogue of left infinite vertex that implies every tracial state vanishes on a suitable set of units (we call these groupoids *weakly left infinite*). Definition 5.22 is the groupoid analogue of "left infinite vertex" that is suitable to produce comparisons between projections as in Lemma 3.29 (we call these groupoids *strongly left infinite*). Checking that strongly left infinite groupoids are also weakly left infinite is routine; however, we have not proven the converse or found a weakly left infinite groupoid that is not strongly left infinite.

ALTHOUGH THE CUNTZ GROUPOID SEEMS WEAKLY LEFT INFINITE BUT NOT STRONGLY LEFT INFINITE

Definition 5.1. A *groupoid* consists of set G along with a collection $G^{(2)} \subset G \times G$ of *composable pairs* and a partially defined *composition* operation $G^{(2)} \to G$, written $(\alpha, \beta) \mapsto \alpha\beta$, along with an involutive inverse function $G \to G$, written $\alpha \to \alpha^{-1}$, such that

(i) The composition is associative, that is $(\alpha\beta)\gamma$ and $\alpha(\beta\gamma)$ are defined and equal whenever (α, β) and (β, γ) belong to $G^{(2)}$.

(ii) For each $\alpha \in G$ we have $(\alpha, \alpha^{-1}) \in G^{(2)}$ and $\alpha^{-1}(\alpha\beta) = \beta$ and $(\alpha\beta)\beta^{-1} = \alpha$ whenever $(\alpha, \beta) \in G^{(2)}$.

Definition 5.2. A *topological groupoid* consists of a groupoid G equipped with a topology that makes the composition and inversion functions continuous (where $G^{(2)}$ is given the relative product topology).

Remark. The inversion function is in fact a homeomorphism from G to itself.

All topological groupoids we mention are assumed to be locally compact, Hausdorff, and second countable.

Definition 5.3 ([18])**.** Let G be a groupoid. The *unit space* of G, denoted $G^{(0)}$ is the set $\{u \in G : (u, u) \in G^{(2)}$ and $u^2 = u\}$. The map $\alpha \mapsto \alpha^{-1}\alpha$ has range equal to $G^{(0)}$ and is called the *source* map $s : G \to G^{(0)}$; the map $\alpha \mapsto \alpha\alpha^{-1}$ is called the range map, $r : G \to G^{(0)}$.

A topological groupoid G is called étale if r is a local homeomorphism when considered as a map from G into itself.

Remark. Two elements α, β are composable (i.e. $(\alpha, \beta) \in G^{(2)}$) if and only if $s(\alpha) = r(\beta)$, in which case $r(\alpha\beta) = r(\alpha)$ and $s(\alpha\beta) = s(\beta)$. We have $r(\alpha) = s(\alpha^{-1})$ and vice versa.

A subset $B \subset G$ is called a *bisection* (also known as a G-set) if $r|_B$ and $s|_B$ are both injective. A topological groupoid G is étale if and only if there exists a basis for the topology of G consisting of open bisections.

An immediate consequence of a groupoid G being étale is that, for each $u \in G^{(0)}$, the range fiber $r^{-1}(u) = \{\gamma \in G : r(\gamma) = u\}$ is discrete in the relative topology, as is the source fiber $s^{-1}(u)$.

Definition 5.4. Let G be an étale groupoid and consider the vector space of continuous complex-valued functions on G with compact support. For $f, g \in C_c(G)$ we define their convolution product $f * g \in C_c(G)$ via

$$f * g(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta),$$

where the fact that the supports are compact ensures the sum is finite. For $f \in C_c(G)$ define the involution $f^* \in C_c(G)$ via

$$f^*(\gamma) = \overline{f(\gamma^{-1})},$$

Remark. It is not difficult to show that the operations above endow $C_c(G)$ with the structure of a $*$-algebra. Furthermore, for each $f \in C_c(G)$ there exists a constant $K > 0$ so that if $\pi : C_c(G) \to B(H)$ is any $*$-representation of $C_c(G)$ as bounded operators on a Hilbert space H, then $||\pi(f)|| \leq K$.

Definition 5.5 (see, e.g. [3])**.** Let G be an étale groupoid; for each unit $u \in G^{(0)}$ let $H_u = \ell^2(s^{-1}(u))$. Define $\pi_r : C_c(G) \to B(H)$ via

$$\pi_r(f)(\delta_u) = f * \delta_u$$

Now set $H = \oplus_{u \in G^{(0)}} H_u$; then the *left-regular representation* of G is $\pi_r : C_c(G) \to B(H)$ given by $\pi(f) = \oplus_u \pi_u(f)$. It can be shown that π_r is in fact a non-degenerate $*$-representation of $C_c(G)$.

We define the *reduced norm* on $C_c(G)$ via $||f||_r := ||\pi_r(f)||$; we define the *universal norm* on $C_c(G)$ via

$$||f|| = \sup\{||\pi(f)|| : \pi \text{ a non-degenerate } *\text{-repn of } C_c(G) \text{ on some Hilbert space}\}.$$

Define $C_r^*(G)$ to be the completion of $C_c(G)$ in $|| \cdot ||_r$; define $C^*(G)$ to be the completion of $C_c(G)$ in $|| \cdot ||$. We let $\pi_r : C^*(G) \to C_r^*(G)$ denote the quotient $*$-homomorphism induced by the inequality $|| \cdot ||_r \leq || \cdot ||$.

Remark. If G is an étale groupoid, then $G^{(0)}$ forms an open subset of G and we can embed $C_c(G^{(0)})$ in $C_c(G)$ by setting each function to 0 outside on $G \setminus G^{(0)}$. This extends to an inclusion of $C_0(G^{(0)})$ into both $C^*(G)$ and $C_r^*(G)$; restricting $\pi_r : C^*(G) \to C_r^*(G)$ to $C_0(G^{(0)})$ gives the identity map.

We also have a map $C_c(G) \to C_c(G^{(0)})$ given by $f \mapsto f|_{G^{(0)}}$; this extends via continuity to faithful expectations $\mathbb{E} : C^*(G) \to C_0(G^{(0)})$ and $\mathbb{E}_r : C_r^*(G) \to C_0(G^{(0)})$. Only the latter is necessarily faithful.

The following is a well-known fact about groupoid C^*-algebras. Our approximate identities are assumed to be positive and bounded of norm 1.

Lemma 5.6. *If G is an étale groupoid, then any approximate identity for $C_0(G^{(0)})$ is an approximate identity for $C^*(G)$ and $C_r^*(G)$.*

Corollary 5.7. *Let G be an étale groupoid. The following are equivalent:*
(1) *$C_r^*(G)$ is unital;*
(2) *$C^*(G)$ is unital;*
(3) *$G^{(0)}$ is compact.*

Definition 5.8. Let G be a groupoid and let $X \subset G^{(0)}$; we say that X is *invariant* if $r^{-1}(X) = s^{-1}(X)$. Equivalently, if $r(\gamma) \in X$ implies that $s(\gamma) \in X$.

If G is a groupoid and X is a non-empty *invariant* subset. Define $G_X := r^{-1}(X)$ to be the *reduction* of G to X.

It is a fact (see [18, Prop. 4.5]) that if Y is an open invariant subset of $G^{(0)}$ and $F := G^{(0)} \setminus Y$, then $G|_Y$ and $G|_F$ are étale and we obtain a short exact sequence of full C^*-algebras:

$$0 \to C^*(G_Y) \to C^*(G) \to C^*(G_F) \to 0.$$

For the reduced C^*-algebras, we always have the inclusion $C_r^*(G_Y) \hookrightarrow C_r^*(G)$ contained in the kernel of $C_r^*(G) \to C_r^*(G|_F)$, but this kernel can properly include $C *_r (G|_Y)$.

This discussion gives us the following useful corollary.

Corollary 5.9. *Let G be an étale groupoid.*
(i) *If $C^*(G)$ is stable, then so is $C_r^*(G)$;*
(ii) *If $C_r^*(G)$ is stable, then there are no non-empty invariant compact open subsets of $G^{(0)}$, and no proper invariant open co-compact subsets of $G^{(0)}$*

Proof. The proof of (i) comes from the fact that stability descends to quotients, see Prop. 2.3. □

Question. If G is an étale groupoid and $C_r^*(G)$ is stable, is it necessarily true that $C^*(G)$ is stable?

Definition 5.10. Let G be an étale groupoid and let μ be a Radon probability measure on $G^{(0)}$. We say that μ is *invariant* if $\mu(s(B)) = \mu(r(B))$ for every open bisection $B \subset G$. (Such measures were called *totally balanced* in [5], to avoid overloading the word "invariant," but "invariant" seems to be more widely used.)

For a Radon probability measure μ on $G^{(0)}$, let ϕ_μ be the corresponding state $f \mapsto \int_{G^{(0)}} f d\mu$ on $C_0(G^{(0)})$. Recall that $\mathbb{E}_r$ is the conditional expectation of $C_r^*(G)$ onto $C_0(G^{(0)})$ defined by extending the restriction map $C_c(G) \to C_c(G^{(0)})$.

Theorem 5.11 ([5, Thm. 2.5])**.** *If μ is a measure on $G^{(0)}$, then the state $C_c(G) \ni f \mapsto \phi_\mu(\mathbb{E}_r(f))$ is a tracial state if and only if μ is invariant.*

Corollary 5.12. *Let G be an étale groupoid. If $C_r^*(G)$ is stable, then there are no invariant Radon probability measures on $G^{(0)}$.*

Proof. Follows immediately from Lemma 2.6 and Theorem 5.11. ☐

It is helpful to be able to recognize when a groupoid has no invariant probability measures. Vaguely speaking, the idea is that if every open set $U \subset G^{(0)}$ can be "spread around" a lot, then an invariant probability measure should vanish on U.

Definition 5.13. Let G be an étale groupoid and let $U \subset G^{(0)}$ be a non-empty open set. We say that U is *weakly left infinite* if, for any n, there exist open bisections $B_1, \ldots, B_n$ such that $B_n^{-1} B_m = \emptyset$ if $m \neq n$ (equivalently, if $r(B_n) \cap r(B_m) = \emptyset$ if $n \neq m$) and such that $s(B_n) = U$ for all n. We call the groupoid G *weakly left infinite* if $G^{(0)}$ has an open cover consisting of weakly left infinite sets.

Theorem 5.14. *Let G be a weakly left infinite étale groupoid. Then G has no invariant probability measures.*

Proof. Let $G^{(0)} = \cup_{n=1}^\infty X_n$, where each X_n is a left infinite open subset of $G^{(0)}$. Suppose that μ is an invariant Radon probability measure on $G^{(0)}$. We will prove that each X_n has $\mu(X_n) = 0$.

Let $m \in \mathbb{N}$; we will show that $\mu(X_n) \leq \frac{1}{m}$. By Definition 5.13, we can find open bisections $B_1, \ldots, B_m$ with mutually disjoint ranges such that $s(B_i) = X_n$ for all $i = 1, \ldots, m$. The set $Y = r(B_1) \sqcup r(B_2) \sqcup \ldots \sqcup r(B_m)$ has measure $\mu(Y) = m \cdot \mu(X_n)$, by the assumption that μ is invariant. Thus $m \cdot \mu(X_n) \leq 1$, because μ is a probability measure; this shows that $\mu(X_n) \leq \frac{1}{m}$ for all $m \in \mathbb{N}$. Thus $\mu(X_n) = 0$ for each n, and we have $\mu(G^{(0)}) \leq \sum \mu(X_n) = 0$, a contradiction. Thus G has no invariant probability measures. ☐

Example 5.15. (This example refers to the path groupoid defined in [15, Defn. 2.3].) Let E be a directed graph and G_E its path groupoid. If v is a left infinite vertex, then the open set $U = Z(v)$ is a weakly left infinite set in $G_E^{(0)}$, as can be checked. In fact in this example we have an infinite sequence of bisections $Z(w_n, v)$ where $w_n \in L(v)$ and $Z(w_n, v) \cap Z(w_m, v) = \emptyset$ if $m \neq n$, somewhat stronger than in Definition 5.13.

If every vertex is left infinite (in the sense of Definition 3.5, then we meet the hypotheses of Theorem 5.14, so that G_E has no invariant probability measures. (There are weaker hypotheses that ensure there are no invariant probability measures on the path groupoid, as evidenced in Theorem 3.31.)

Corollary 5.16. *Let G be a weakly left infinite étale groupoid. Then $T(C^*(G)) = T(C^*_r(G)) = \emptyset$.*

Proof. Any tracial state τ on $C^*(G)$ corresponds to an invariant Radon probability measure on $G^{(0)}$ by Theorem 5.11. Then Theorem 5.14 establishes the corollary. □

Definition 5.17. Let G be a groupoid. A *subgroupoid* is a nonempty subset $H \subset G$ that forms a groupoid when given $H^{(2)} = G^{(2)} \cap (H \times H)$ and operations given by restricting the operations on G. We say that H is *wide* if $H^{(0)} = G^{(0)}$.

Proposition 5.18 ([4, Appendix A])**.** *Let G be an étale groupoid and let H be an open wide subgroupoid. Then the $*$-homomorphism $\iota_0 : C_c(H) \to C_c(G)$ given by*

$$\iota_0(f)(\gamma) = \begin{cases} f(\gamma) & \gamma \in H \\ 0 & \textit{otherwise} \end{cases}$$

extends to a C^*-inclusion, $C^*(H) \subset C^*(G)$.

Remark. The analogous result for the reduced norm should follow from a consideration of the regular representation of G.

So far we have not found any interesting examples where the following proposition applies, but it seems worth noting.

Proposition 5.19. *Let G be an étale groupoid and let H be an open wide subgroupoid of G. If $C^*(H)$ is stable, then $C^*(G)$ is stable.*

Proof. Because H is wide, $C^*(H)$ contains $C_0(G^{(0)})$, which contains an approximate identity for $C^*(G)$ as in Lemma 5.6. Stability of $C^*(G)$ then follows from [11, Prop. 4.4]. □

Definition 5.20 ([22, Defn 3.5])**.** An étale groupoid G is called *ample* if there exists a basis for the topology on G consisting of *compact* open bisections.

Lemma 5.21. *Let G be an ample étale groupoid. Then $C^*_r(G)$ has an approximate identity p_n consisting of projections in $C_c(G)$.*

Proof. Follows from the assumption that our groupoids are second countable. □

Remark. The path groupoids of [15] and [13], as well as the groupoids of germs constructed in [16] and [9], are all ample.

Definition 5.22. Let G be an ample étale groupoid. We say that G is *strongly left infinite* if there exists a collection $\{F_n\}_{n\geq 1}$ of non-empty, pairwise disjoint, compact open subsets of $G^{(0)}$ such that for each $n \geq 1$ the set

$$L(F_n) = \{j \in \mathbb{N} : \exists \text{ open bisection } B \text{ such that } s(B) = F_n \text{ and } r(B) \subset F_j\}$$

is infinite.

Remark. If a groupoid is strongly left infinite, then it is weakly left infinite in the sense of Definition 5.13. We are unsure if the converse is true.

If E is a directed graph, then the condition of G_E being strongly left infinite is weaker than every vertex in the graph being left infinite. For example, the graph consisting of an infinite chain of vertices terminating in a sink has no left infinite vertices, but yields a strongly left infinite path groupoid (informally, you can "go left as well as right" when working with the groupoid).

Proposition 5.23. *Let G be an ample étale groupoid and suppose that G is strongly left infinite. Then $C^*(G)$ is stable (and hence $C^*_r(G)$ is stable as well).*

Proof. Let $\{F_n\}$ be a cover for $G^{(0)}$ as in Definition 5.22. Let p_n be the characteristic function of $\cup_{k=1}^n F_n$; then $p_n \in C_c(G^{(0)})$ and $(p_n)_{n=1}^\infty$ forms an increasing approximate identity of projections for $C^*(G)$. Note that if $m > n$, then $p_m - p_n$ is the characteristic function of $F_{n+1} \cup \ldots \cup F_m$.

By Lemma 2.7, we must find for each $n \in \mathbb{N}$ an $m > n$ such that $p_n \lesssim p_m - p_n$. We begin by finding a compact open bisection B_1 such that $s(B_1) = F_1$ and $r(B_1) \subset F_{n_1}$ is not contained in $F_1 \cup F_2 \ldots \cup F_n$. Inductively find for each $k = 2, \ldots, n$ a compact open bisection B_k such that $s(B_k) = F_k$ and $r(B_k) \subset F_{n_k}$ is not contained in $F_1 \cup F_2 \cup \ldots \cup F_n \cup F_{n_1} \cup \ldots \cup F_{n_{k-1}}$. Note that $B_j \cap B_k = \emptyset$ if $j \neq k$. Set v to be the indicator function of $B_1 \cup B_2 \cup \ldots \cup B_n$, so that $v \in C_c(G)$. Furthermore v^*v is the indicator function of $F_1 \cup \ldots \cup F_n$, i.e. $v^*v = p_n$. If we define m to be the maximum of $\{n_1, \ldots, n_k\}$, then we also have $vv^* \leq p_m - p_n$ (by construction). Now an application of Lemma 2.7 finishes the proof. □

Question. If G is an ample étale groupoid such that $C^*(G)$ is stable, is G strongly left infinite?

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