What is...a totally positive matrix?

Danny Crytser

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$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ \pi & \sqrt{2} & 3 \end{pmatrix}$$

This is a 3×3 matrix.

Matrices with variable entries

We can put variables in the entries

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Submatrices

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$$\begin{pmatrix} 1 & 1 \\ 3 & 6 \end{pmatrix}$$

How many submatrices?

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More submatrices

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

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$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \underbrace{a, b, c, d, e, f, g, h, i}_{\text{nine of these}}$$

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$$\begin{pmatrix} e & f \\ h & i \end{pmatrix} \qquad \begin{pmatrix} d & f \\ g & i \end{pmatrix} \qquad \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

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$$\begin{pmatrix} e & f \\ h & i \end{pmatrix} \quad \begin{pmatrix} d & f \\ g & i \end{pmatrix} \quad \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

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$$\begin{pmatrix} b & c \\ e & f \end{pmatrix} \quad \begin{pmatrix} a & c \\ d & f \end{pmatrix} \quad \begin{pmatrix} a & b \\ d & e \end{pmatrix}$$

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$$\begin{pmatrix} b & c \\ e & f \end{pmatrix} \quad \begin{pmatrix} a & c \\ d & f \end{pmatrix} \quad \begin{pmatrix} a & b \\ d & e \end{pmatrix}$$

Nineteen submatrices total.

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A specific example

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$$\begin{pmatrix} 1 & 1 \\ 3 & 6 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 1 & 6 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

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Pop quiz: What is the determinant if all the entries are equal?

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You can find the determinant of any square matrix

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$$= aei - afh - bdi + bfg + cdh - ceg$$

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$$\begin{vmatrix} a & b & c & d \\ i & j & k & \ell \\ p & q & r & s \\ w & x & y & z \end{vmatrix}$$

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$$\begin{vmatrix} a & b & c & d \\ i & j & k & \ell \\ p & q & r & s \\ w & x & y & z \end{vmatrix} \ldots \text{ too long! (but there is a way)}$$

Determinant of a 3×3 matrix

$$\begin{aligned}
\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} &= 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 8 & 9 \end{vmatrix} \\
&= 1(45 - 48) - 2(36 - 42) + 3(36 - 40) \\
&= -3 + 12 - 12 \\
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\end{aligned}$$

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For bigger matrices, better to use a computer.

Minors

A *minor* of a matrix A is the determinant of a submatrix.

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Minors of a 3×3 matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei - afh - bdi + bfg + cdh - ceg$$

$$\begin{vmatrix} e & f \\ h & i \end{vmatrix} = ei - fh \qquad \begin{vmatrix} d & f \\ g & i \end{vmatrix} = di - fg \qquad \begin{vmatrix} d & e \\ g & h \end{vmatrix} = dh - eg$$

$$\begin{vmatrix} b & c \\ h & i \end{vmatrix} = ib - hc \qquad \begin{vmatrix} a & c \\ g & i \end{vmatrix} = ai - cg \qquad \begin{vmatrix} a & b \\ g & h \end{vmatrix} = ah - bg$$

$$\begin{vmatrix} b & c \\ e & f \end{vmatrix} = bf - ec \qquad \begin{vmatrix} a & c \\ d & f \end{vmatrix} = af - dc \qquad \begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - bd$$

$$a, b, c, d, e, f, g, h, i$$

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is not positive because its determinant is $1 \times 4 - 2 \times 2 = 0$.

$$C = \begin{pmatrix} -3 & -2 \\ -2 & -3 \end{pmatrix}$$

is ___________ .

Positive 3×3 matrix

The matrix

$$\begin{matrix} -1 & -2 & -3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}$$

is positive

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is positive (you can check that it has determinant 3).

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is totally positive. (Which one?)

Motivation

Reasonable Q: Why would anyone care about TP matrices?

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Official A: “They have a number of uses in physics, mathematics, and engineering.”

Unofficial A: I just think they’re cool to think about.

A totally positive 3×3 matrix

The matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$$

is totally positive

A totally positive 3×3 matrix

The matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$$

is totally positive (here are its 2×2 minors, and its determinant is 1)

$$\begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} = 3 \qquad \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \qquad \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$$

$$\begin{vmatrix} 1 & 1 \\ 3 & 6 \end{vmatrix} = 3 \qquad \begin{vmatrix} 1 & 1 \\ 1 & 6 \end{vmatrix} = 5 \qquad \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2$$

$$\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 \qquad \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2 \qquad \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2$$

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Bad Answer: Find all the minors, see that they are all positive.

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Why is this approach bad?

How do we test TP?

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For example: a 10×10 matrix would require you to compute $\binom{20}{10} - 1 = 184755$ minors. This would take a while...

How do we get a better test?

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Key Idea: minors are related to one another in “ways” that don’t change positivity (adding positive numbers, multiplying positive numbers, and dividing positive numbers).

Reducing the number of tests

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The minors are $\Delta = ad - bc, a, b, c, d$. That's five – not too bad. If I want to check that A is TP, I just need to check that Δ, a, b, c, d are all positive.

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But $\Delta + bc = ad$, so $d = \frac{\Delta + bc}{a}$. Thus if a, b, c, and Δ are positive, so is d.

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But $\Delta + bc = ad$, so $d = \frac{\Delta + bc}{a}$. Thus if a, b, c, and Δ are positive, so is d. We reduced the number of tests to run by 1.

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SURPRISING FACT:

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SURPRISING FACT: the nine "solid minors" which touch either the first row, first column, or both, are enough — if these are positive all the others are positive (!!!!)

Reducing the number of tests

What about 3×3 matrices? A 3×3 matrix has $\binom{6}{3} - 1 = 19$ minors.

SURPRISING FACT: the nine "solid minors" which touch either the first row, first column, or both, are enough — if these are positive all the others are positive (!!!!)

In this case we have cut the number of tests in half!

Example of testing a 3×3 matrix

Let's test the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

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(Its determinant is $5 - 3 - 1 = 1$.)

How much better is the good approach?

Number of minors required for testing a $n \times n$ matrix to see if its TP

n	bad approach	good approach
1	1	1

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4	69	16

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1	1	1
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3	19	9
4	69	16
5	251	25

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9	48619	81
10	184755	100

Reducing the number of tests

How do we know the "important" minors to look at?

Reducing the number of tests

How do we know the"important" minors to look at?
Idea: Certain diagrams encode the set of "essential minors".

Double-wiring diagram

A double-wiring diagram with n nodes:

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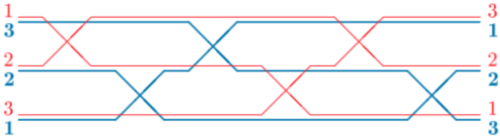

Figure: a 3×3 double wiring diagram

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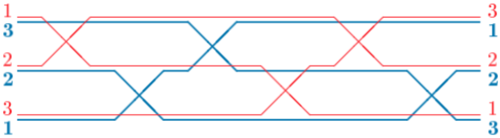

Figure: a 3×3 double wiring diagram

(1) there are n red nodes and n blue nodes, listed in reverse order on two sides

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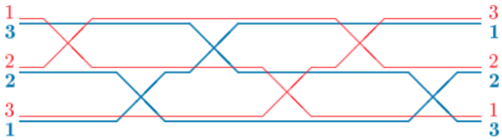

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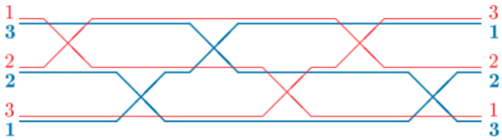

Figure: a 3×3 double wiring diagram

(1) there are n red nodes and n blue nodes, listed in reverse order on two sides

(2) each node on the left is connected to the corresponding node on the left

(3) each pair of wires (of the same color!) cross exactly once

Chamber minors

Every region of the diagram can be labeled with symbols corresponding to the red and blue lines passing underneath it.

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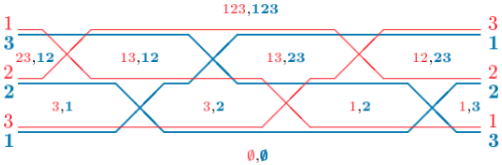

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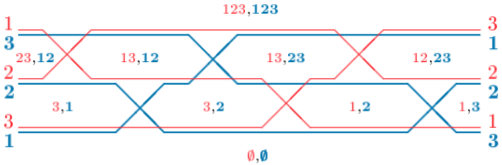

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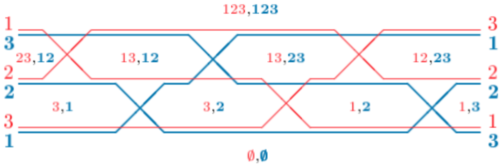

To read a minor from a chamber, rows equal red numbers and columns according to the blue numbers. For example, $12, 23$ corresponds to the minor that has rows 1 and 2 and columns 1 and 2 (of original 3×3 matrix) :

$$12, 23 \mapsto \begin{vmatrix} b & c \\ e & f \end{vmatrix} = bf - ec$$

Surprising Facts

Surprising Facts

Fomin and Zelevinsky (1999): If you draw any double wiring diagram, and if P is a matrix so that all the n^2 chamber minors of P are positive, then P is totally positive.

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This is a builds on previous investigations by Gasca-Pena, Cryer, and Fekete.

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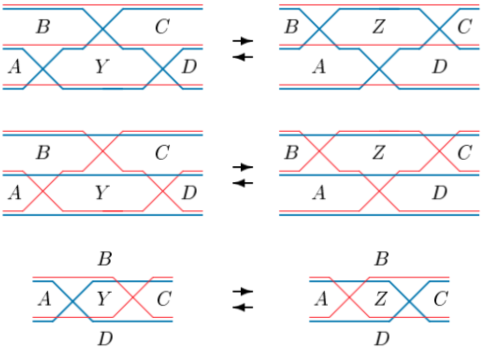

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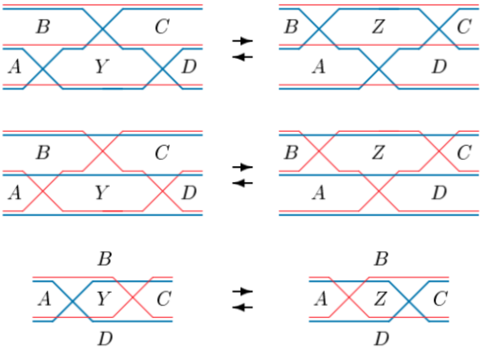

Swap the chamber minor Y for $Z = \frac{AB+CD}{Y}$. Positivity is preserved!

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2. The "good" approach uses n^2 tests, which are chosen according to *any* double wiring diagram
3. You can use any double wiring diagram you like to choose the tests, because any diagram can be mutated into any other diagram, without affecting positivity of the minors.

The End

Thank You!