# Traces arising from regular inclusions

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# Outline

# Tracial states

If $A$ is a $C^*$-algebra then a *tracial state* on $A$ is a state such that $\phi(xy) = \phi(yx)$ for all $x, y \in A$. We are interested in (1) how to define tracial states on $C^*$-algebras and (2) how to be sure that the methods we use exhaust all possible tracial states. In particular, we are interested in the case of *graph* $C^*$-algebras.

# Tracial states: inclusions + conditional expectations

Our approach to studying tracial states assumes the following initial data:

(1) an inclusion $B \subset A$ of an abelian $C^*$-subalgebra which is *non-degenerate* ($B$ contains an approximate identity for $A$) and *regular* (the multiplicative and self-adjoint set $N(B) = \{n \in A : nBn^* \cup n^*Bn \subset B\}$ spans dense subset of $A$)

(2) a conditional expectation $\mathbb{E} : A \to B$ (a completely positive linear bimodule map), which we will require to have additional properties later on.

For the rest of the talk, unless otherwise specified, we will assume this framework.

# Tracial states: state extensions

If $\phi : B \to \mathbb{C}$ is a state on $B$, then $\phi \circ \mathbb{E}$ is a state extension to $A$.

**Question**

For which states $\phi \in S(B)$ is the extension $\phi \circ \mathbb{E}$ a *tracial state* on $A$?

# Invariant states

## Definition

If $\phi \in S(B)$ and $n \in N(B)$, then $\phi$ is called *n-invariant* if $\phi(nbn^*) = \phi(n^*nb)$ for all $b \in B$. If $N_0 \subset N(B)$, then $\phi$ is *$N_0$-invariant* if it is $n$-invariant for all $n \in N_0$. If $\phi$ is $N(B)$-invariant we will call $\phi$ *totally invariant*

## Example

If $\tau \in T(A)$ is a tracial state, then $\phi = \tau|_B$ is a totally invariant state on $B$.

The upshot of our analysis is that with fairly mild assumptions the converse of the above example is also true.

# Normalization of conditional expectations

**Definition**

Let $E : A \to B$ be a conditional expectation. We say that $\mathbb{E}$ is *normalized* by $n \in N(B)$ if $\mathbb{E}(nan^*) = n\mathbb{E}(a)n^*$ for all $a \in A$. (Similar for $N_0 \subset N(B)$.)

In the cases that we care about, the relevant conditional expectations will be normalized by a set of normalizers that generate $A$.

# Invariant states, part II

## Theorem (C., Nagy '15)

*Suppose that $B \subset A$ is a regular inclusion and $\mathbb{E} : A \to B$ is a conditional expectation which is normalized by $N_0 \subset N(B)$. Then for any $N_0$-invariant state $\phi \in S(B)$, the composition $\phi \circ \mathbb{E}$ is a tracial state when restricted to $C^*(B \cup N_0) \subset A$.*

## Corollary

*Suppose that $\mathbb{E} : A \to B$ is normalized by $N_0 \subset N(B)$ and $\phi$ is a $N_0$-invariant state on $B$, where $N_0$ generates $A$ as a $C^*$-algebra. Then $\phi \circ \mathbb{E}$ is a tracial state on $A$.*

# Parametrizing the trace space

The previous result shows that if we have a conditional expectation which is normalized by $N(B)$, then there is a surjective map

$$\text{res} : T(A) \ni \tau \mapsto \tau|_B \in S_{\text{inv}}(B)$$

from the tracial states on $A$ to the totally invariant states on $B$. In other words, every totally invariant state lifts to a tracial state on the $C^*$-algebra. The restriction map is affine and continuous.

## Question

When is the restriction map injective? That is, for which inclusions $B \subset A$ is a tracial state $\tau \in T(A)$ fully determined by its restriction to $B$?

# The extension property

**Definition**

A non-degenerate inclusion $B \subset A$ is said to have the *extension property* if every pure state $\phi \in P(B)$ has a *unique* extension to a state on $A$ (which must then be pure).

If an inclusion has the extension property one automatically obtains a conditional expectation $\mathbb{E} : A \to B$ so these inclusions fall within our framework.

**Proposition (C., Nagy '15)**

*If $B \subset A$ is a non-degenerate inclusion with the extension property, then the restriction map carrying $T(A)$ to $S_{\text{inv}}(B)$ is injective.*

# Proof of proposition

## Proof.

By a result of Archbold [1], if $B \subset A$ has the extension property, the kernel of the associated conditional expectation $\mathbb{E} : A \to B$ is spanned by the commutators $\{ab - ba : a \in A, b \in B\}$. A tracial state vanishes on any commutator, hence any tracial state factors through the conditional expectation onto $B$, i.e. $\tau = (\tau_B) \circ \mathbb{E}$. Thus the restriction map from $T(A)$ to $S_{\text{inv}}(B)$ is injective. □

## Remark

We do not claim that the tracial state space is non-empty in this case (in fact there are many examples of inclusions with the extension property where $T(A) = \emptyset$).

# Graph $C^*$-algebras

If $E = (E^0, E^1, r, s)$ is a directed graph, then there is an affiliated $C^*$-algebra $C^*(E)$ generated by a family $\{s_e, p_v\}_{e \in E^1, v \in E^0}$ such that

(1) the $p_v$ are mutually orthogonal projections;

(2) the $s_e$ are partial isometries with mutually orthogonal range projections;

(3) $s_e^* s_e = p_{s(e)}$ for all $e \in E^1$;

(4) $s_e s_e^* \leq p_{r(e)}$, and if $r^{-1}(v)$ is finite and non-empty (i.e. $v$ is a *regular vertex*), then $p_v = \sum_{r(e)=v} s_e s_e^*$.

To a directed path $\alpha = e_1 \ldots e_n$, we associate a partial isometry $s_\alpha = s_{e_1} \ldots s_{e_n}$.

# The abelian core

## Definition

A *cycle* is a path $\lambda = e_1 \dots e_n$ in $E$ with $r(e_1) = s(e_n)$. An *entry* to $\lambda$ is a path $f_1 \dots f_k$ with $r(f_1) = r(e_k)$ and $f_1 \neq e_k$ for some $k$. The *abelian core* $\mathcal{M}(E)$ is the $C^*$-subalgebra of $C^*(E)$ generated by $G_{\mathcal{M}}(E) = \{s_\alpha s_\alpha^*\}_\alpha \cup \{s_\alpha s_\lambda s_\alpha^* : \lambda \text{ a cycle without entry}\}$.

It is shown in [3] that there is a conditional expectation $\mathbb{E}$ from $C^*(E)$ onto $\mathcal{M}(E)$. It is easy to verify that $\mathcal{M}(E) \subset C^*(E)$ is regular (all the generators of $C^*(E)$ are normalizers of $\mathcal{M}(E)$).

# Tracial states on graph $C^*$-algebras

## Definition

A *graph trace* on $E$ is a function $g : E^0 \to [0, \infty)$ such that

(1) if $v$ is a vertex and $\{e_1, \dots, e_n\} \subset r^{-1}(v)$, then $\sum_{i=1}^n g(s(e_i)) \leq g(v)$;

(2) if $v$ is a regular vertex, then $g(v) = \sum_{r(e)=v} g(s(e))$.

A graph trace is *bounded* if it is $\ell^1$ and *normalized* if $||g||_1 = 1$.

## Example

If $\tau$ is a tracial state on $C^*(E)$ then $g_\tau(v) = \tau(p_v)$ defines a normalized graph trace on $E$.

Tomforde showed that the map $\tau \mapsto g_\tau$ is surjective onto the normalized graph traces, using states on $K$-theory.

# The tracial state space

The map $\tau \to g_\tau$ is not always injective – there are examples of graph $C^*$-algebras where many tracial states correspond to the same graph trace.

**Question**

What additional structure needs to be added to parametrize all the tracial states? When is the map $\tau \mapsto g_\tau$ injective?

# The tracial state space, ctd.

Of special interest are graphs which have no entries to cycles. We found a natural operation to remove all the entries to cycles in a graph (corresponds to taking a quotient of $C^*(E)$), which we call tightening $E \mapsto E_{\text{tight}}$. There is a surjective $*$-homomorphism $\rho_{\text{tight}} : C^*(E) \to C^*(E_{\text{tight}})$.

# Cyclically tagged graph traces

## Definition

A *cyclically tagged graph trace* is a pair $(g, \mu)$, where $g$ is a normalized graph trace and $\mu$ is a function from the set of vertices with nonzero $g$-value lying on cycles without entries to the set of probability measures on $\mathbb{T}$. It is *consistent* if whenever $v$ and $w$ are on the same cycle, then $\mu(v) = \mu(w)$. The space of consistent cyclically tagged graph traces is denoted by $T_1^{\mathrm{CCT}}(E)$.

## Theorem (C., Nagy '15)

*For any* $(g, \mu) \in T_1^{\mathrm{CCT}}(E)$ *there is a corresponding tracial state* $\tau_{(g,\mu)}$ *on* $C^*(E)$. *Moreover, the map*

$$T_1^{\mathrm{CCT}}(E_{\mathrm{tight}}) \ni (g, \mu) \mapsto \tau_{(g,\mu)} \circ \rho_{\mathrm{tight}} \in T(C^*(E))$$

*is an isomorphism.*

# When is $\tau \mapsto g_\tau$ injective

Tomforde noted that if $E$ satisfies condition (K), then the map $\tau \mapsto g_\tau$ is injective. However this is not necessary.

## Definition

Two (finite) paths $\lambda$ and $\mu$ are *incomparable* if neither one contains the other as initial prefix. A vertex $v$ is *essentially left infinite* if there is an infinite set $\{\lambda_k\}$ of finite paths that are pairwise incomparable and such that $s(\lambda_k) = v$ for all $k$.

## Theorem (C., Nagy '15)

*For a directed graph $E$ the following are equivalent:*

(i) *the map $\tau \mapsto g_\tau$ is injective;*

(ii) *the source of each cycle in $E$ is essentially left infinite.*

# When is $\tau \mapsto g_\tau$ injective, ctd.

**Proof.**

(ii) $\Rightarrow$ (i): If a vertex $v$ is essentially left infinite, then any bounded graph trace $g$ must vanish on $v$. Thus if the source of each cycle is essentially left infinite, there are no measures to consider (after passing to the tightening) and the map $\tau \mapsto g_\tau$ is injective.

(i) $\Rightarrow$ (ii): If $v$ is the source of a cycle and $v$ is *not* essentially left infinite, then we can define a (non-normalized but bounded) graph trace $g$ on $E$ by $g(w) = |\{\text{paths } v \to w\}|$. Thus there is a normalized graph trace $g$ on $E$ which does not vanish at $v$, and if we take any non-Lebesgue probability measure for $\mu_v$, the tagging $(g, \mu_v)$ is consistent.

□

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Thank you!