CONTINUOUS-TRACE k-GRAPH C^*-ALGEBRAS

DANNY CRYTSER

ABSTRACT. A characterization is given for directed graphs that yield graph C^*-algebras with continuous trace. This is established for row-finite graphs with no sources first, and then extended to the general case via a Drinen-Tomforde desingularization. Partial results are given to characterize when higher-rank graphs that yield C^*-algebras with continuous trace.

1. INTRODUCTION

To any directed graph E one can affiliate a graph C^*-algebra $C^*(E)$, generated by a universal family of projections and partial isometries that satisfy certain Cuntz-Krieger relations. Many C^*-algebraic properties of $C^*(E)$ are governed by graph-theoretic properties of E. For example, $C^*(E)$ is an AF algebra exactly when E has no directed cycles. To any graph E there is also affiliated a *path groupoid* G_E, an étale groupoid which models the shift dynamics of infinite paths in E. This groupoid provides an alternate model for $C^*(E)$, in the sense that $C^*(E) \cong C^*(G_E)$. This isomorphism allows for the use of tools from the theory of groupoid C^*-algebras to study graph algebras. In this paper we give an example of this approach, characterizing continuous-trace graph C^*-algebras by applying the main result of [7] to the path groupoid of a directed graph.

The path groupoid is easiest to use if the graph is *non-singular*, in the sense that each vertex is the range of a finite non-empty set of edges. We first work in the non-singular case, and then use desingularization to extend to the general case. Desingularization takes a non-regular graph E and returns a non-singular graph $\tilde{E}$ such that the affiliated graph C^*-algebras are Morita equivalent (so that continuity of trace is preserved). As an application we use a result from [12] to characterize continuous trace AF algebras in terms of their Bratteli diagrams.

In the last section we consider higher-rank graph C^*-algebras with continuous trace. Higher-rank graphs are categories which generalize the category of finite directed paths within a directed graph. They have C^*-algebras defined along the same lines as graph C^*-algebras. We include necessary background on the theory of higher-rank graph algebras. Again, the use of groupoids is crucial. The higher-rank case is more complicated and we are only able to give partial results. In particular, giving a combinatorial description of when the isotropy groups vary continuously for a k-graph path groupoid seems out of reach, so we focus instead on the principal/aperiodic case. We note a simple necessary condition on a higher-rank graph for its associated C^*-algebra to have continuous trace, a corollary of a result from [3].

2. Continuous-trace C^*-algebras; graph algebras; groupoids

For an element a in a C^*-algebra A, and an equivalence class $s = [\pi] \in \hat{A}$, we define the rank of $s(a)$ to be the rank of $\pi(a)$. We say that $s(a)$ is a projection if and only if $\pi(a)$ is a projection.

Definition 2.1. Let A be a C^*-algebra with Hausdorff spectrum $\hat{A}$. Then A is said to have *continuous trace* (or be continuous-trace) if for each $t \in \hat{A}$ there exist an open set $U \subset \hat{A}$ containing t and an element $a \in A$ such that $s(a)$ is a rank-one projection for every $s \in U$.

The reader who is familiar with graph algebras may pass over the following standard definitions.

Definition 2.2. A *(directed) graph* E is an ordered quadruple $E = (E^0, E^1, r, s)$, where the E^0 and E^1 are countable sets called the *vertices* and *edges*, and $r, s : E^1 \to E^0$ are maps called the *range* and *source* maps. The finite path space E^* consists of all finite sequences $e_1 \dots e_n$ in E^1 such that $s(e_i) = r(e_{i+1})$ for $i = 1, \dots, n-1$. The range of the path $e_1 \dots e_n$ is defined to be $r(e_1)$ and its source is $s(e_n)$. If $\mu = e_1 \dots e_n$ is a finite path, then we define the length to be n and write $|\mu| = n$. The vertices are included in the finite path space as the paths of length zero. If $\lambda = e_1 \dots e_n$ and $\mu = f_1 \dots f_m$ are finite paths with $s(\lambda) = r(\mu)$, we can concatenate them to from $\lambda\mu = e_1 \dots e_n f_1 \dots f_m \in E^*$. The *infinite path space* is $E^\infty = \{e_1 e_2 \dots | s(e_i) = r(e_{i+1}) \forall i \geq 1\}$. If $\lambda = e_1 \dots e_n \in E^*$ and $x = f_1 \dots \in E^\infty$, then $\lambda x = e_1 \dots e_n f_1 \dots \in E^\infty$. The range of $x = e_1 e_2 \dots \in E^\infty$ is defined as $r(x) := r(e_1)$. The shift map $\sigma : E^\infty \to E^\infty$ removes the first edge from an infinite path: $\sigma(e_1 e_2 \dots) = e_2 e_3 \dots$. Composing σ with itself yields powers $\sigma^2, \sigma^3, \dots$. A vertex $v \in E^0$ is *regular* if $r^{-1}(v) \subset E^1$ is nonempty and finite; otherwise it is said to be *singular*. A *cycle* in a directed graph is a path $\lambda \in E^* \setminus E^0$ with $r(\lambda) = s(\lambda)$; a *simple cycle* is a cycle λ which does not contain another cycle. An *entrance* to the cycle $\lambda = e_1 \dots e_n$ is an edge e with $r(e) = e_k$ and $e \neq e_k$.

Definition 2.3. Let E be a directed graph. Then the *graph C^*-algebra* of E, denoted $C^*(E)$, is the universal C^*-algebra generated by projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ satisfying the following Cuntz-Krieger relations:

(1) $s_e^* s_e = p_{s(e)}$ for any $e \in E^1$;
(2) $s_e s_e^* \leq p_{r(e)}$ for any $e \in E^1$;
(3) $s_e^* s_f$ for distinct $e, f \in E^1$;
(4) If v is regular, then $p_v = \sum_{r(e)=v} s_e s_e^*$.

A useful introduction to graph algebras is [9].

Definition 2.4. Let E be a graph and let E^∞ denote its infinite path space. Then the *path groupoid* of E is

$$G_E = \{(x, n, y) \in E^\infty \times \mathbb{Z} \times E^\infty : \text{ there exists } p, q \in \mathbb{N} \text{ such that } \sigma^p x = \sigma^q y \text{ and } p - q = n\}$$

The groupoid operations are $(x, n, y)(y, m, z) = (x, m + n, z)$ and $(x, n, y)^{-1} = (y, -n, x)$. The unit space is identified with E^∞ via the mapping $x \mapsto (x, 0, x)$, so that the range and source maps are given by $r(x, n, y) = x$ and $s(x, n, y) = y$.

Definition 2.5. If G is a groupoid and $x \in G^{(0)}$ is a unit of G, then by $G(x) = \{\gamma \in G : s(\gamma) = r(\gamma) = x\}$ is called the *isotropy group* (or *stabilizer*), of x. The union of all the isotropy groups forms the *isotropy bundle* of G, denoted $\mathrm{Iso}(G)$. (Note: this generally is not a locally trivial bundle.) The groupoid G is called *principal* if $\mathrm{Iso}(G) = G^{(0)}$.

Remark. If $G = G_E$, then the isotropy group of an infinite path x is either trivial ($\sigma^p x = \sigma^q x$ implies $p = q$) or infinite cyclic.

The topology on G_E is generated by basic open sets of the form

$$Z(\alpha, \beta) = \{(\alpha z, |\alpha| - |\beta|, \beta z) \in G_E : r(z) = s(\alpha)\}$$

where $\alpha, \beta \in E^*$ with $s(\alpha) = s(\beta)$. It is noted in [6] that $Z(\alpha, \beta) \cap Z(\gamma, \delta) = \emptyset$ unless $(\alpha, \beta) = (\gamma\epsilon, \delta\epsilon)$ or vice versa. This topology makes G_E into a locally compact étale groupoid, because the restriction of the range map to the basic sets is a homeomorphism, and each basic set is compact. Thus G_E has a canonical Haar system $\{\lambda_x\}_{x \in E^\infty}$ consisting of counting measures on the source fibers. Because G_E is a locally compact groupoid with Haar system, it has an affiliated groupoid C^*-algebra $C^*(G_E)$ ([11]), defined by completing the convolution algebra $C_c(G)$ in a suitable norm. (The following theorem has been modified from its original statement to fit our orientation convention.) The following theorem justifies the use of groupoids as a means to study graph C^*-algebras.

Remark. The topology defined above restricts to the relative product topology on $G_E^{(0)} = E^\infty \subset \prod_{\mathbb{N}} E^1$, if we treat E^1 as a discrete space. We will refer to this topology on E^∞ using basic compact-open sets of the form $Z(\alpha) = \{\alpha x | x \in E^\infty, r(x) = s(\alpha)\}$.

Theorem 2.6 ([6])**.** *For any row-finite graph with no sources E, we have $C^*(E) \cong C^*(G_E)$ via an isomorphism carrying s_e to $\mathbf{1}_{Z(e,s(e))} \in C_c(G)$ and p_v to $\mathbf{1}_{Z(v,v)}$.*

3. Continuous isotropy

To describe graph C^*-algebras with continuous trace, we need to know when the isotropy varies continuously among the units of the path groupoid. First the topology on the set of isotropy groups has to be defined.

Definition 3.1 ([10])**.** Let X be a topological space. Consider the collection $F(X)$ of all closed subsets of X; the *Fell topology* on $F(X)$ is defined by the requirement that a net $(Y_i) \subset F(X)$ converges to $Y \in F(X)$ exactly when

(1) if elements y_i are chosen in Y_i such that $y_i \to z$, then z belongs to Y, and
(2) for any element $y \in Y$ there is a choice of elements $y_i \in Y_i$ (possibly taking a subnet of (Y_i) and relabeling) such that $y_i \to y$.

We say that G has *continuous isotropy* if the isotropy map $G^{(0)} \to F(G)$ defined by $x \mapsto G(x)$ is continuous.

We will not need to handle the Fell topology directly in this paper, thanks to the following result which describes continuous isotropy for graph algebras.

Theorem 3.2 ([4])**.** *Let E be a row-finite graph with no sources. Then G_E has continuous isotropy if and only if no cycle in E has an entrance.*

4. Proper groupoids

Definition 4.1. A *topological groupoid* is a groupoid equipped with a topology that makes the multiplication and inversion operations continuous. A topological groupoid G is *proper* if the orbit map $\Phi_G : G \to G^{(0)} \times G^{(0)}$ given by $g \to (r(g), s(g))$ is proper (where the codomain is equipped with the relative product topology).

Definition 4.2. Let G be a groupoid. Let $\pi_R : G \to G^{(0)} \times G^{(0)}$ be given by $\pi_R(g) = (r(g), s(g))$. Then the *orbit groupoid* of G, denote by $R_G = R$, is the image of π_R, where the groupoid operations are

$$(u, v)(v, w) = (u, w)$$

$$(u, v)^{-1} = (v, u).$$

The unit space of R is identified with the unit space of G. The range and source maps are naturally identified with the projections onto the first and second factors.

Definition 4.3. The topology on $R = R_G$ is the quotient topology induced by the above map $\pi_R : G \to R$.

Remark. If the groupoid G is principal, in the sense that $G(x) = \{x\}$ for every unit $x \in G^{(0)}$, then the map π_R is a groupoid isomorphism. For an arbitrary topological groupoid it need not be the case that the quotient topology makes R into a topological groupoid. For graph and k-graph groupoids this issue never arises and R will always be a topological groupoid. The following commutative diagram serves to keep the relevant groupoids and spaces distinct. Note that as a set map, Φ_R is just an inclusion. However, R carries a different topology from the product topology on $G^{(0)} \times G^{(0)}$, so we distinguish between the two.

$$\begin{tikzcd}
G \arrow[dr, "\Phi_G"] \arrow[dd, "\pi"'] & \\
& G^{(0)} \times G^{(0)} \\
R \arrow[ur, "\Phi_R"'] &
\end{tikzcd}$$

Remark. In some sources (such as [7]) the orbit groupoid is denoted by G/A, to indicate that it is the quotient of G by the (in this case, abelian) isotropy bundle.

The notation of the following theorem is somewhat different in the original [**?**].

Theorem 4.4 ([**?**])**.** *Let G be a locally compact second-countable groupoid with Haar system and abelian isotropy groups. Then $C^*(G)$ has continuous trace if and only if*

(1) the isotropy map $x \to G(x)$ is continuous; and
(2) R_G is proper.

5. Continuous-trace graph algebras

The path groupoid of a directed graph E is made of infinite paths, and the open sets are described by finite path prefixes. It is unsurprising then that the characterization of proper path groupoids is stated in terms of a certain finiteness condition on paths. **For this section, the standing assumption is that E is a row-finite graph with no sources**.

Definition 5.1. Let v and w be vertices in a directed graph E. An *ancestry pair* for v and w is a pair of paths (λ, μ) such that $r(\lambda) = v$, $r(\mu) = w$, and $s(\lambda) = s(\mu)$. A *minimal ancestry pair* is an ancestry pair (λ, μ) such that $(\lambda, \mu) = (\lambda'\nu, \mu'\nu)$ implies that $\nu = s(\lambda) = s(\mu)$. An ancestry pair$(\lambda, \mu)$ is *cycle-free* if neither path contains a cycle. The graph E has *finite ancestry* if for every pair of vertices (not necessarily distinct) v, v' has at most finitely many cycle-free minimal ancestry pairs.

Remark. It is not necessary that any two vertices have at least one ancestry pair in order for the graph to have finite ancestry.

The following technical lemma is used to study properness of the orbit groupoid.

Lemma 5.2. *Let (α, β) and (γ, δ) be two cycle-free minimal ancestry pairs. Then $\pi_R(Z(\alpha, \beta)) \cap \pi_R(Z(\gamma, \delta)) = \emptyset$ unless there is a simple cycle λ and either factorizations $\alpha = \gamma\alpha'$, $\delta = \beta\delta'$ and $\lambda = \alpha'\delta'$ or factorizations $\gamma = \alpha\gamma'$, $\beta = \delta\beta'$ and $\lambda = \gamma'\beta'$. If (α, β) is a minimal cycle-free ancestry pair, and if no cycle of E has an entrance, then there are finitely many minimal cycle-free ancestry pairs (γ, δ) such that $\pi_R(Z(\alpha, \beta)) \cap \pi_R(Z(\gamma, \delta)) \neq \emptyset$*

Proof. We have that $\pi_R(Z(\alpha, \beta)) = \{(\alpha z, \beta z)\} \subset E^\infty \times E^\infty$ and $\pi_R(Z(\gamma, \delta)) = \{(\gamma w, \delta w)\}$. Suppose that $(\alpha z, \beta z) = (\gamma w, \delta w)$ as pairs of infinite paths. If $|\alpha| \geq |\gamma|$ and $|\beta| \geq |\delta|$, then $\alpha = \gamma\nu$ and $\beta = \delta\nu'$, where ν and ν' are initial segments of the infinite path w. If $\nu = \nu'$, this contradicts minimality of the ancestry pair (α, β). If $\nu \neq \nu'$, then (assuming WLOG that $\nu \subset \nu'$) we can note that $s(\nu) = s(\alpha) = s(\beta) = s(\nu')$, so that ν' contains a cycle, hence that β contains a cycle, contradicting the assumption that (α, β) is a cycle-free ancestry pair.

We must have WLOG that $|\alpha| > |\gamma|$ and $|\beta| < |\delta|$. We can factor $\alpha = \gamma\alpha'$ and $\delta = \beta\delta'$. Then $\lambda = \alpha'\delta'$ is a cycle with range/source equal to $s(\gamma) = s(\delta)$. This cycle must be simple because neither $\alpha' \subset \alpha$ nor $\delta' \subset \delta$ contains a cycle This establishes the first claim in the lemma.

If no cycle has an entry then any vertex is the source of at most one simple cycle. If $(\alpha, beta)$ is a cycle-free minimal ancestry pair, and no cycle of E has an entrance, then there is at most one simple cycle at $s(\alpha) = s(\beta)$. Thus there are only finitely many factorizations of the above form. □

Lemma 5.3. *Let (λ, μ) be an ancestry pair. Then there is a unique minimal ancestry pair (α, β) such that $Z(\lambda, \mu) \subset Z(\alpha, \beta)$ in G_E. If (λ, μ) is cycle-free, then so is (α, β).*

Proof. Let ϵ be maximal (with respect length) such that $(\alpha, \beta) = (\alpha'\epsilon, \beta'\epsilon)$. Then $Z(\alpha, \beta) \subset Z(\alpha', \beta')$. As mentioned in [6], $Z(\lambda, \mu) \cap Z(\gamma, \delta)$ implies that (λ, μ) factors as $(\gamma\epsilon, \delta\epsilon)$ or vice versa. Thus $Z(\alpha, \beta)$ is contained in $Z(\alpha', \beta')$ and not in $Z(\gamma, \delta)$ for any other minimal ancestry pair (γ, δ). The second claim follows from the construction. □

Lemma 5.4. *Let E be a directed graph in which no cycle has an entrance, and let (α, β) be an ancestry pair such that α or β contains a cycle. Then there is a cycle-free minimal ancestry pair (λ, μ) such that $\pi_R(Z(\lambda, \mu))$ contains $\pi_R(Z(\alpha, \beta))$.*

Proof. Suppose that α contains a cycle, so that $\alpha = \alpha'\lambda\lambda'$, where α' does not contain a cycle, λ is a cycle, and λ' is an initial segment of λ. Let $\lambda = \lambda'\lambda''$. Note that α must factor in this way because of the condition that no cycle has an entrance. Then $\pi_R(Z(\alpha, \beta)) = \{(\alpha x, \beta x) : x \in E^\infty, r(x) = s(\alpha)\}$. Again by the condition that no cycle has an entrance, we have that the only infinite path with range equal to $s(\alpha)$ is $\lambda''(\lambda^\infty)$. Thus $\pi_R(Z(\alpha, \beta)) = \pi_R(Z(\alpha'\lambda', \beta))$. We can perform a similar operation to get rid of any cycles from β, obtaining a cycle free ancestry pair (α', β') such that $\pi_R(Z(\alpha, \beta)) = \pi_R(Z(\alpha', \beta'))$. By the previous lemma we can find a cycle-free minimal ancestry pair (λ, μ) such that $Z(\alpha', \beta') \subset Z(\lambda, \mu)$. The lemma is established on applying π_R to this containment. □

Theorem 5.5. *Let E be a row-finite directed graph with no sources. Then $C^*(E)$ has continuous trace if and only if both*

(1) no cycle of E has an entrance, and
(2) E has finite ancestry.

Proof. Suppose no cycle of E has an entrance and E has finite ancestry. We show that Φ_R is proper. The collection of sets of the form $Z(v) \times Z(w) \subset E^\infty \times E^\infty$ form a compact-open cover for $E^\infty \times E^\infty$. Thus it suffices to prove that $\Phi_R^{-1}(Z(v) \times Z(w))$ is compact for any vertices v and w. It is not hard to see that

$$\Phi_G^{-1}(Z(v) \times Z(w)) = \bigcup_{(\alpha,\beta)} Z(\alpha, \beta),$$

where (α, β) ranges over all minimal ancestry pairs. We can partition these pairs into two families: $\mathcal{C}$, the set of minimal ancestry pairs for v and w containing cycles and $\mathcal{D}$, the set of cycle-free minimal ancestry pairs.

By definition, we have that $\Phi_R^{-1}(Z(v) \times Z(w)) = \pi_R(\Phi^{-1}(Z(v) \times Z(w)))$. Thus we can write

$$\pi_R(\Phi^{-1}(Z(v) \times Z(w))) = \pi_R(\bigcup_{(\alpha,\beta)\in\mathcal{C}} Z(\alpha, \beta) \bigcup \Cup_{(\lambda,\mu)\in\mathcal{D}}.$$

But for each $(\alpha, \beta) \in \mathcal{C}$, we have that $\pi_R(Z(\alpha, \beta)) \subset \pi_R(Z(\lambda, \mu))$ for some $(\lambda, \mu) \in \mathcal{D}$. Thus $\Phi_R^{-1}(Z(v) \times Z(w)) = \cup_{(\lambda,\mu)\in\mathcal{D}}\pi_R(Z(\lambda, \mu))$. Each $\pi_R(Z(\lambda, \mu))$ is compact by the continuity of π_R and compactness of $Z(\lambda, \mu)$, and $\mathcal{D}$ is finite by the assumption that E has finite ancestry. Thus Φ_R is a proper map and the C^*-algebra has continuous trace by [7].

Now suppose that $C^*(E)$ has continuous trace. Then the isotropy groups of G must vary continuously so no cycle of E has an entrance. So we only need to show that E has finite ancestry. Suppose that v and w are two vertices and let $\{(\alpha_k, \beta_k) : k \in \mathcal{A}\}$ be an enumeration of the cycle-free minimal ancestry pairs for v and w. As in the proof of sufficiency, we can write $\Phi_R^{-1}(Z(v) \times Z(w)) = \cup_k \pi_R(Z(\alpha_k, \beta_k))$. By properness of Φ_R we must be able to extract a finite subcover. However for any finite subset $B \subset A$, the set $B' = \{k \in A : \pi_R(Z(\alpha_k, \beta_k)) \cap (\cup_{j\in B}\pi_R(Z(\alpha_j, \beta_j))) \neq \emptyset\}$ is finite by Lemma 1. This implies that A is finite. Thus E has finite ancestry. □

6. Arbitrary graphs

The previous theorem is given only in the context of row-finite graphs with no sources. In this section we will remove the requirement that all graphs be row-finite and have no sources, by use of the Drinen-Tomforde desingularization.

Definition 6.1. A *source* is a vertex that is the range of no edge. An *infinite receiver* is a vertex that is the range of infinitely many edges. A vertex is *singular* if it is a source or an infinite receiver. A *tail* at the vertex v is an infinite chain of edges $e_1, e_2, \ldots$ with $r(e_1) = v$ and $r(e_{i+1}) = s(e_i)$ for $i \geq 1$.

Briefly, the Drinen-Tomforde desingularization adds a tail to each singular vertex. If the singular vertex v is an infinite receiver, it takes all the edges with range v and redirects each to a different vertex on the infinite tail. This produces a new graph $\tilde{E}$ which has no singular vertices. For details, see [2] or [9]. Note that we have reversed the edge orientation of [2], to fit with the higher-rank graphs considered in the next chapter.

Theorem 6.2. [?] *Let E be an (arbitrary) directed graph. Let $\tilde{E}$ be a desingularization for E. Then $C^*(E)$ embeds in $C^*(\tilde{E})$ as a full corner, so that $C^*(E)$ is Morita equivalent to $C^*(\tilde{E})$.*

The basic technical lemma needed is a bijection between finite paths in a singular graph and certain finite paths in its desingularization. (We've omitted the part about infinite paths.)

Lemma 6.3. [2]*[Lemma 2.6] Let E be a directed graph and let $\tilde{E}$ be a desingularization. Then there is a bijection*

$$\phi : E^* \to \{\beta \in \tilde{E}^* : s(\beta), r(\beta) \in E^0\}.$$

The map ϕ preserves source and range.

Lemma 6.4. *Let E be a directed graph and let $\tilde{E}$ be a desingularization for E. Then no cycle of E has an entrance if and only if no cycle of $\tilde{E}$ has an entrance.*

Proof. Suppose that no cycle of $\tilde{E}$ has an entrance. Let $\lambda = e_1 \ldots e_n$ be a cycle in E and let $\tilde{\lambda} = \phi(\lambda) = f_1 \ldots f_m$ be the corresponding path in $\tilde{E}$, with $r(\tilde{\lambda}) = r(\lambda)$ and $s(\tilde{\lambda}) = s(\lambda)$. Suppose that e is an edge in E with $r(e) = r(e_k)$ and yet $e \neq e_k$. Then $\tilde{e} = \phi(e)$ is a path in $\tilde{E}$ with $r(\tilde{e}) = r(\phi(e_k))$ and yet $\tilde{e} \neq \tilde{e}_k$ (here we are using the fact that ϕ is a bijection). Thus $\tilde{e}$ is an entrance to the cycle $\tilde{\lambda}$.

Now suppose that no cycle of E has an entrance, and let $\mu = f_1 f_2 \ldots f_n$ be a cycle in $\tilde{E}$. If $s(\mu)$ belongs to E^0, then $\phi^{-1}(\mu)$ is a cycle in E. Furthermore, we know that no vertex of E^0 on $\phi^{-1}(\mu)$ can be singular, because then the cycle $\phi^{-1}(\mu)$ would have an entrance. Thus μ consists solely of edges in E and does not meet any singular vertices or tails. The only edges in $\tilde{E}$ that meet μ are images under ϕ of edges from E, and we know that μ has no entrances in E, so μ has no entrances.

Now we show that, under the assumption that no cycle of E has an entrance, no cycle of $\tilde{E}$ can have source on an infinite tail. Suppose that μ is a cycle in $\tilde{E}$ with source on an infinite tail. Because no infinite tail contains a cycle, we can write $\mu = f_1 \ldots f_k d_1 \ldots d_j$, where $d_1 \ldots d_j$ is the largest path in the infinite tail containing $s(\mu)$ such that $d_1 \ldots d_j$ is contained in μ. Then $r(d_1)$ must be the vertex to which the infinite tail is attached, i.e. $r(d_1) \in E^0$. Consider the cycle $\mu' = d_1 \ldots d_j f_1 \ldots f_k$. This begins and ends in E^0, so it equals $\phi(\lambda)$ for some cycle λ in E. This cycle cannot meet any singular vertices in E (or else it would have an entrance), so it must be the case that $\lambda = \phi(\lambda)$. But $s(\mu)$ belongs to λ, contradicting our assumption that $s(\mu)$ belongs to an infinite tail. Combining this with the previous part shows that if no cycle of E has an entrance, then no cycle of $\tilde{E}$ has an entrance. $\square$

Lemma 6.5. *Let E be a directed graph and let $\tilde{E}$ be a desingularization of E. Then $\tilde{E}$ has finite ancestry if and only if E has finite ancestry.*

Proof. Suppose that E has finite ancestry and let v be a vertex of $\tilde{E}$. We show that v has finitely many cycle free minimal ancestry pairs by defining an injection from minimal ancestry pairs of v to minimal ancestry pairs of some vertex in E.

If v belongs to E, then it must be the case that $s(\alpha) = s(\beta)$ belongs to E^0 for any minimal ancestry pair (α, β). For otherwise $s(\alpha)$ lies on an infinite tail, with only one edge f leaving $s(\alpha)$, and we could factor a common edge $(\alpha, \beta) = (\alpha' f, \beta' f)$. Thus in the case that v belongs to E, we can map $(\alpha, \beta) \mapsto (\phi^{-1}(\alpha), \phi^{-1}(\beta))$. This carries cycle-free minimal ancestry pairs to cycle-free minimal ancestry pairs, so v must have finitely many cycle free minimal ancestry pairs in F.

If v belongs to an infinite tail, let $d_1 \dots d_j$ be the path from v to the singular vertex w to which the infinite tail is attached. Again, if (α, β) is a minimal ancestry pair for v, it must terminate in a vertex of E. Define a map $(\alpha, \beta) \mapsto (\phi^{-1}(d_1 \dots d_j \alpha, d_1 \dots d_j \beta_j))$. As above, this will give us an injection into minimal cycle free ancestry pairs for w in E. Hence v has finitely many minimal cycle free ancestry pairs.

If $\tilde{E}$ has finite ancestry then E trivially has finite ancestry by using ϕ.

$\square$

Theorem 6.6. *Let E be an arbitrary graph. Then $C^*(E)$ has continuous trace if and if both*

(1) no cycle of E has an entrance, and
(2) E has finite ancestry.

Proof. Let us begin by fixing a desingularization $\tilde{E}$ of E. If no cycle of E has an entrance and E has finite ancestry, then Lemmas 8 and 9 tell us that the same is true of $\tilde{E}$. Then Theorem 4 says that $C^*(\tilde{E})$ has continuous trace. Theorem 5 and the well-known fact that the class of continuous-trace C^*-algebras is closed under Morita equivalence then give that $C^*(E)$ has continuous trace.

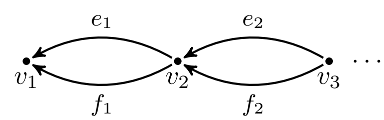
Now suppose that $C^*(E)$ has continuous trace. Then $C^*(\tilde{E})$ has continuous trace as in the previous part of the proof. By Theorem 4, we see that no cycle of $\tilde{E}$ has an entrance and $\tilde{E}$ has finite ancestry. Lemmas 8 and 9 again give us that E satisfies the same conditions. $\square$

Corollary 6.7. *If E is a graph with no cycles, then $C^*(E)$ has continuous trace if and only if E has finite ancestry.*

This is useful for studying AF algebras. Drinen showed that every AF algebra arises as the C^*-algebra of a locally finite pointed directed graph ([1]). Tyler gave a useful complementary result, showing that if E is Bratteli diagram for an AF algebra A, then there is a Bratteli diagram KE for A such that (treating the diagrams as directed graphs) $C^*(KE)$ contains A and $C^*(E)$ as complementary full corners ([12]). Thus in particular A and $C^*(E)$ are Morita equivalent.

Corollary 6.8. *Let A be an AF algebra and let E be a Bratteli diagram for A, treated as a directed graph. Then A has continuous trace if and only if E has finite ancestry.*

Example 6.9. Let $A = \bigotimes_{n=1}^{\infty} M_2(\mathbb{C})$ be the UHF algebra of type 2^∞. The familiar Bratteli diagram for A (after decoration with labels) is



This graph fails to have finite ancestry: for each k, we have the cycle-free minimal ancestry pair $(f_1e_2f_3\dots e_{2k}, e_1f_2e_3\dots f_{2k})$ for v_1, v_1. Thus it does not have continuous trace. (We also know that the spectrum isn't Hausdorff, because A is simple but has infinitely many pure states by Glimm's construction.)

7. Higher-rank graphs

In this section we partially extend the results of the previous section to the realm of higher-rank graphs. We have not completely described which higher-rank graph C^*-algebras have continuous trace. However, we do characterize the *aperiodic* higher-rank graphs which yield continuous-trace C^*-algebras. The jump in combinatorial complexity from the graph to the k-graph case is noteworthy. In addition, we provide some negative results regarding the *generalized cycles* of [3]. In particular, a generalized cycle with entry causes the affiliated vertex projection to be infinite, which cannot happen if the algebra has Hausdorff spectrum.

Remark. The semigroup $\mathbb{N}^k$ is treated as a category with a single object, 0.

Definition 7.1. A *higher-rank graph* (or *k-graph*) is a countable category Λ equipped with a *degree functor* $d : \Lambda \to \mathbb{N}^k$ which satisfies the following factorization property: if $d(\lambda) = m + n$ for some $m, n \in \mathbb{N}^k$, then $\lambda = \mu\nu$ for some unique μ, ν such that $d(\mu) = m$ and $d(\nu) = n$. The *vertices* Λ^0 of Λ are identified with the objects. The elements of Λ are referred to as *paths*. For fixed degree $n \in \mathbb{N}^k$, the paths of degree n are denoted by Λ^n. We refer to paths of degree 0 as *vertices* in the k-graph (each path in Λ) has a well-defined range vertex and source vertex.

We can affiliate a C^*-algebra to a higher-rank graph but some additional hypotheses have to be added in order to ensure the result is not trivial. The hypotheses we use are not the weakest set which defines a meaningful C^*-algebra; however, they allow us to use the groupoid machinery easily.

Definition 7.2. A higher-rank graph is *row-finite* if each vertex $v \in \Lambda^0$ and degree $n \in \mathbb{N}^k$, there are only finitely many paths of degree n with range v. It is said to have *no sources* if for all $v \in \Lambda^0$ and n there is some path λ with $d(\lambda) = n$ and $r(\lambda) = v$.

Definition 7.3. Let Λ be a row-finite k-graph with no sources. Then the higher-rank graph C^*-algebra of Λ, denoted $C^*(\Lambda)$, is the universal C^*-algebra generated by a family of partial isometries $\{s_\lambda\}_{\lambda\in\Lambda}$ satisfying:

(1) $\{s_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections;
(2) if $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$, then $s_\lambda s_\mu = s_{\lambda\mu}$;
(3) $s_\lambda^* s_\lambda = s_{s(\lambda)}$;

(4) for any $v \in \Lambda^0$ and any degree $n \in \mathbb{N}^k$, we have $s_v = \sum_{\lambda \in \Lambda^n : r(\lambda) = v} s_\lambda s_\lambda^*$.

Just as in the graph case, we study continuous-trace higher-rank graph C^*-algebras by studying an affiliated groupoid. The following k-graph is used to define infinite paths in k-graphs.

Definition 7.4. Let Ω_k be the category of all pairs $\{(m,n) : m \leq n\}$, where $m \leq n$ if $m_i \leq n_i$ for $i = 1, \ldots, k$. The composition is given by $(m,n)(n,p) = (m,p)$. The degree functor is given by $d(m,n) = n - m$. The objects are all pairs of the form (m,m).

Definition 7.5. An *infinite path* in a k-graph Λ is a degree preserving functor $x : \Omega_k \to \Lambda$. The collection of infinite paths in Λ is denoted Λ^∞.

Definition 7.6. Let Λ be a k-graph and let x be an infinite path in Λ. For any $p \in \mathbb{N}^k$, we define $\sigma^p x$ to be the infinite path given by $\sigma^p x(m,n) = x(m+p, n+p)$. The graph Λ is *aperiodic* if for every infinite path x in Λ and degrees $p, q \in \mathbb{N}^k$, $\sigma^p x = \sigma^q x$ implies that $p = q$. The *range* of an infinite path $x \in \Lambda^\infty$ is defined to be $x(0,0)$. If $\lambda \in \Lambda$ and $x \in \Lambda^\infty$ with $s(\lambda) = r(x)$, then there is a unique path $y = \lambda x \in \Lambda^\infty$ such that $\sigma^{d(\lambda)} y = x$ and $y(0, d(\lambda)) = \lambda$.

Aperiodic k-graphs are the higher-dimensional analogues of graphs without cycles. Now we can define the higher-rank version of the path groupoid. As noted in [5], by the no sources assumption we know that for every vertex $v \in \Lambda^0$, there is at least one $x \in \Lambda^\infty$ with $r(x) = v$.

Definition 7.7 ([5])**.** Let Λ be a k-graph. Then the *path groupoid* of Λ is

$$G_\Lambda = \{(x,n,y) \in \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty : \text{ there exists } p, q \in \mathbb{N}^k \text{ such that } \sigma^p x = \sigma^q y \text{ and } p - q = n\}.$$

The groupoid operations are given by $(x,n,y)(y,m,z) = (x, m+n, z)$ and $(x,n,y)^{-1} = (y, -n, x)$.

The topology on G_Λ is defined in the same way as the topology on G_E, for E a graph. The basic open sets are

$$Z(\alpha, \beta) = \{(x,n,y) \in G_\Lambda : \sigma^{d(\alpha)}(x) = \sigma^{d(\beta)}(y), d(\alpha) - d(\beta) = n\}.$$

The topology on G_Λ generated by these sets makes it into a locally compact étale groupoid over unit space Λ^∞. (See [5]Prop. 2.8.)

Theorem 7.8 ([5]Cor. 3.5)**.** *Let Λ be a row-finite k-graph with no sources and let G_Λ be its path groupoid. Then $C^*(\Lambda) \cong C^*(G_\Lambda)$.*

Therefore we can decide questions about $C^*(\Lambda)$ by studying G_Λ. Deciding when the path groupoid G_Λ has continuously varying isotropy groups is substantially harder than the graph case. Therefore we restrict to the principal case, in which all the isotropy groups are assumed to be trivial and therefore vary continuously. We modify our definition of ancestry pair to this situation.

Definition 7.9. Let Λ be an aperiodic row-finite k-graph with no sources and let $v, w \in \Lambda^0$ be two vertices. Then an *ancestry pair* for v, w is a pair $(\lambda, \mu) \in \Lambda \times \Lambda$ such that $r(\lambda) = v$, $r(\mu) = w$, and $s(\lambda) = s(\mu) = w$. An ancestry pair (λ, μ) is *minimal* if $(\lambda, \mu) = (\lambda' \nu, \mu' \nu)$ implies $\nu = s(\lambda)$. We say that Λ has *finite ancestry* if each pair of vertices has at most finitely many minimal ancestry pairs.

The following lemma is used to show that finite ancestry is necessary for a k-graph to yield a C^*-algebra with continuous trace.

Lemma 7.10. *Let Λ be an aperiodic row-finite k-graph with no sources. Let (α, β) and (γ, δ) be two distinct minimal ancestry pairs in Λ. Then $Z(\alpha, \beta) \cap Z(\gamma, \delta) = \emptyset$.*

Proof. Claim: It suffices to show that if $Z(\alpha, \beta) \cap Z(\gamma, \delta) \neq \emptyset$, then either $d(\alpha) \geq d(\gamma)$ and $d(\beta) \geq d(\delta)$, or $d(\gamma) \geq d(\alpha)$ and $d(\delta) \geq d(\beta)$. For suppose that $(\alpha z, n, \beta z) = (\gamma w, n, \delta w)$, $d(\alpha) \geq d(\gamma)$ and $d(\beta) \geq d(\delta)$. Then $\alpha z = \gamma\alpha' z = \gamma w$, so that $\alpha' z = w$. We also have $\beta z = \delta\beta' z = \delta w$, so that $\beta' z = w$. If $d(\alpha') = d(\beta')$, then this shows that $\alpha' = \beta'$, so that $(\alpha, \beta) = (\gamma\alpha', \beta\alpha')$, contradicting minimality. If $d(\alpha') \neq d(\beta')$, then $\sigma^{d(\alpha')} w = z = \sigma^{d(\beta')} w$, so that w is periodic, against hypothesis. The other case follows by symmetry. This establishes the claim.

If the intersection is nonzero then as above we have $(\alpha z, n, \beta z) = (\gamma w, n, \delta w)$ for some $z, w \in \Lambda^\infty$. Moreover, $n = d(\alpha) - d(\beta) = d(\gamma) - d(\delta)$, so that $d(\alpha) - d(\gamma) = d(\beta) - d(\delta)$. Thus if $d(\alpha) \geq d(\gamma)$, we also have $d(\beta) - d(\delta)$, and vice versa (and vice versa!), reducing to the claim. Thus we assume that $d(\alpha)_1 - d(\gamma)_1 = d(\beta)_1 - d(\delta)_1 > 0$ and $d(\gamma)_2 - d(\alpha)_2 = d(\delta)_2 - d(\beta)_2 > 0$ (*). From the equation $\alpha z = \gamma w$ and the inequality $d(\alpha)_1 > d(\gamma)_1$, we obtain $\alpha(d(\gamma)_1, d(\alpha))z = \gamma(d(\gamma)_1, d(\gamma))w$; call this path x_1. Similarly we have $\beta(d(\beta)_1, d(\beta))z = \delta(d(\delta_1), d(\delta))w$. The conditions (*) imply that $d(\alpha) - d(\gamma)_1 + d(\delta) - d(\delta)_1 = d(\beta) - d(\delta)_1 + d(\gamma) - d(\gamma)_1$.

Now iterating the shift we obtain

$$\sigma^{d(\delta)-d(\delta)_1} z = \sigma^{d(\alpha)-d(\gamma)_1+d(\delta)-d(\delta)_1} x_1 = \sigma^{d(\beta)-d(\delta)_1+d(\gamma)-d(\gamma)_1} x_1 = \sigma^{d(\beta)-d(\delta)_1} w.$$

Similarly we obtain

$$\sigma^{d(\alpha)-d(\alpha)_2} w = \sigma^{d(\gamma)-d(\gamma)_2} z.$$

Now we can write

$$\sigma^{d(\delta)-d(\delta)_1} \sigma^{d(\alpha)-d(\alpha)_2} w = \sigma^{d(\delta)-d(\delta)_1} \sigma^{d(\gamma)-d(\gamma)_2} z$$

$$= \sigma^{d(\gamma)-d(\gamma)_2+d(\delta)-d(\delta)_1} z = \sigma^{d(\gamma)-d(\gamma)_2} \sigma^{d(\beta)-d(\delta)_1} w.$$

Now by aperiodicity we must have $d(\delta) - d(\delta)_1 + d(\alpha) - d(\alpha)_2 = d(\gamma) - d(\gamma)_2 + d(\beta) - d(\delta)_1$. Compare second coordinates in the degrees; we must have $d(\delta)_2$ on the left and $d(\beta)_2$ on the right. This gives $d(\delta)_2 = d(\beta)_2$, a contradiction. □

Theorem 7.11. *Let Λ be an aperiodic row-finite k-graph with no sources. Then $C^*(\Lambda)$ has continuous trace if and only Λ has finite ancestry.*

Proof. We don't need to check that the isotropy groups in G_Λ vary continuously because Λ is aperiodic. If $C^*(\Lambda)$ is continuous trace, then $G_\Lambda \cong R_\Lambda$ must be proper. Thus for any v, w we must have $\Phi^{-1}(Z(v) \times Z(w)) = \cup_{(\alpha,\beta)\in A} Z(\alpha, \beta)$, where A is the set of minimal ancestry pairs for (α, β). We have that the union is disjoint by Lemma 7, so A must be finite. Hence Λ has finite ancestry.

Suppose that Λ has finite ancestry. Then as above we get that $\Phi^{-1}(Z(v) \times Z(w)) = \cup_{(\alpha,\beta)\in A} Z(\alpha, \beta)$ is compact. Thus G_Λ is proper and $C^*(\Lambda)$ has continuous-trace. □

Desingularization is in general more complicated for higher-rank graphs, so it seems unlikely that this could be extended to higher-rank graphs with sources. The following is modified somewhat from cite[Aidan Evans]

Definition 7.12 (Aidan and Evans paper)**.** Let Λ be a row-finite graph with no sources. Then a pair $(\lambda, \mu) \in \Lambda \times \Lambda$ is a *generalized cycle* if $r(\lambda) = r(\mu)$ and $s(\lambda) = s(\mu)$ and $Z(\lambda) \subset Z(\mu)$. We say that a generalized cycle (λ, μ) *has an entrance* if (μ, λ) is *not* a generalized cycle. (That is, if $Z(\lambda) \subsetneq Z(\mu)$.)

Lemma 7.13 ([3]Cor. 3.8)**.** *If Λ contains a generalized cycle with entrance, then $C^*(\Lambda)$ contains an infinite projection.*

The following simple observation is probably not new, but is proven here for ease of reference.

Lemma 7.14. *If A is a C^*-algebra containing an infinite projection, then A does not have continuous trace.*

Proof. Let p be a projection in A with a proper subprojection q such that $p \sim q$. Take an irreducible representation $\pi : A \to B(H)$ such that $\pi(p - q) \neq 0$. Then $\pi(q) < \pi(q)$ are equivalent projections in $B(H)$. All compact projections are finite rank, so it cannot be the case that the range of π lies within the compacts. As every irreducible representation of a C^*-algebra with continuous trace has range within the compact operators ([8]Theorem 6.1.11) , we see that A does not have continuous trace. □

Corollary 7.15. *If Λ is a row-finite k-graph with no sources that contains a generalized cycle with entrance, then $C^*(\Lambda)$ does not have continuous trace.*

References

1. D. Drinen. Viewing AF algebras as graph algebras. *Proc. Amer. Math. Soc.*, 128(7):1991–2000, 1999.
2. D. Drinen and M. Tomforde. The C^*-algebras of arbitrary graphs. *Rocky Mountain J. Math.*, 35:105–135, 2005.
3. D.G. Evans and A. Sims. When is the Cuntz-Krieger algebra of a higher-rank graph approximately finite-dimensional? *J. Funct. Anal.*, 263(1):183–215, 2012.
4. G. Goehle. Groupoid C^*-algebras with Hausdorff spectrum. *Bull. Aust. Math. Soc.*, 88:232–242, 2013.
5. A. Kumjian and D. Pask. Higher-rank graph C^*-algebras. *New York J. Math.*, 6:1–20, 2001.
6. A. Kumjian, D. Pask, I. Raeburn, and J. Renault. Graphs, groupoids, and Cuntz-Krieger algebras. *J. Funct. Anal.*, 144:505–541, 1997.
7. P. Muhly, J. Renault, and D. Williams. Continuous-trace groupoid C^*-algebras, III. *Transactions of the American Mathematical Society*, 348(9):3621–3641, 1996.
8. G.K. Pedersen. *C^*-Algebras and their Automorphism Groups.* Academic Press, 1979.
9. I. Raeburn. *Graph Algebras.* CBMS Lecture Notes. American Mathematical Society, 2005.
10. I. Raeburn and D.P. Williams. *Morita Equivalence and Continuous-Trace C^*-Algebras*, volume 60 of *Mathematical Surveys and Monographs.* American Mathematical Society, 1998.
11. J. Renault. *A Groupoid Approach to C^*-Algebras.* Lecture Notes in Mathematics. Springer Verlag, 1980.
12. Jason Tyler. Every AF-algebra is Morita equivalent to a graph algebra. *Bull. Aust. Math. Soc.*, 69:237–240, 2004.