Traces on graphs and higher-rank graphs

Amy Isvik, Nate Kolo, Maria Ross

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1 Introduction

This report examines directed graphs, which consist of vertices and oriented edges between them. Every directed graph E has an associated *graph C*-algebra*, $C^*(E)$, as defined by Kumjian and Pask in [5]. Such a C^*-algebra is generated by a collection of partial isometries and projections satisfying relations defined via the graph E. The study of graph algebras focuses on understanding graph-theoretic conditions on a graph E which control algebraic or analytic behavior of $C^*(E)$.

We will study the properties of well-known graph product operations, such as the tensor product, on directed graphs and higher rank graphs. Higher rank graphs, or k-graphs, introduced by Alex Kumjian and David Pask are the multidimensional analogue to directed graphs. Products on k-graphs are of particular interest due to the fact that the tensor product of the C*-algebras of 1-graphs correspond to the C*-algebra of their product 2-graph, as shown by Johnston and Reynolds.

This report also focuses on graph traces, functions from the vertices of a graph to the natural numbers, as introduced by Hjelmborg, Tomforde, and Johnson. The set of these traces on directed graphs is isomorphic to the set of tracial states on the associated C*-algebra. Extreme graphs traces, which are graph traces that cannot be written as a convex combinations of other graph traces, can be used to determine the set of all graph traces on the graph and to determine the space of tracial states on a graph C*-algebra. Johnson introduces a simple way to discover the extreme traces of a finite, loopless directed graph, highlighting the correlation between extreme traces and sources of a graph. This result is not easily generalized to k-graphs, as the behavior of their graph traces is more complicated.

This report studies how these two ideas interact by exploring how taking graph products of directed and higher rank graphs affects the collection of graph traces on a graph. We examine closely how these graph products affect the convex structure of the collection of graph traces, paying close attention to the extreme traces which facilitate this structure. Minimal results were found using the Cartesian product; however, the number of extreme traces on the product graph is the number of extreme traces on the factor graph multiplied together. Stronger relations between traces on the factor graph and product graph can be found using the tensor product, such as the ability to combine traces from the factor graphs into traces on the product graphs. However, the majority of these relations are found in and this work focuses on locally convex higher-rank graphs, where traces on the factor graphs can be combined to create traces on the product graph, traces on product graphs can be projected onto factor graphs, and where products of extreme traces on factor graphs are extreme traces on product graphs. We speculate that the converse is also true, that extreme traces on the product graphs can be projected onto extreme traces on the factor graphs, but have only confirmed this conjecture for the case of products of 1-graphs.

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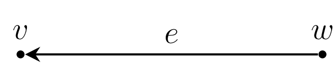
2 Directed graphs, product graphs, and their traces

Definition 2.1 (Raeburn)**.** A *directed graph*, G, consists of a vertex set, G^0, and a set of edges, G^1, along with mappings $s, r : G^1 \to G^0$. For each edge, $e \in G^1$, $s(e)$ is the source of e and $r(e)$ denotes the range of e. (We say that e is an edge from $s(e)$ to $r(e)$.)

Example 2.2. Below is an example of a directed graph, G with $G^0 = \{v, w\}$ and $G^1 = \{e\}$, with $r(e) = v$ and $s(e) = w$.



Definition 2.3 (Johnston, Raeburn)**.** A vertex $v \in E^0$ is considered a *source* if $r^{-1}(v) = \emptyset$; that is, if it receives no edges. For a directed graph E, the (possibly empty) set of sources in E is denoted by S_E. A vertex v is an *infinite receiver* if $r^{-1}(v)$ is infinite. A vertex which is either a source nor an infinite receiver is called *singular*; nonsingular vertices are called *regular*.

Definition 2.4. Let E be a directed graph. A directed path in E consists of a finite sequence of edges $\lambda = e_1 \dots e_n$ such that $s(e_k) = r(e_{k+1})$ for $k = 1, \dots, n-1$. The range of λ is defined to be the range of e_1, and the source of λ is defined to be the source of e_n. The length of λ is equal to n, and denoted by $|\lambda|$. The collection of all finite paths in E is denoted by E^*, and we include E^0 in E^* as the paths of length 0. A *cycle* is a finite path of positive length such that $r(\lambda) = s(\lambda)$. An *entrance* to a cycle $\lambda = e_1 \dots e_n$ is an edge f with $r(f) = r(e_k)$ for some k such that $f \neq e_k$.

Definition 2.5 (cf. [2],[10],[1])**.** A *trace* on a directed graph E is a function $g : E^0 \to [0, \infty)$ with two properties:

(i) For any regular vertex $v \in E^0$,

$$g(v) = \sum_{e \in E^1, r(e) = v} g(s(e)).$$

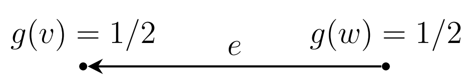
(ii) For any infinite receiver v $\in E^0$ and any finite collection of edges $e_1, \dots, e_n \in r^{-1}(v)$, we have

$$g(v) \geq \sum_{i=1}^{n} g(s(e_i)).$$

Definition 2.6. A *normalized graph trace* on a directed graph E is a graph trace on E with the additional property that $\sum_{v \in E^0} g(v) = 1$. We will often refer to normalized graph traces simply as graph traces. The collection of normalized graph traces on E (which may be empty) is denoted by $T(E)$.

As all graph traces considered henceforth will be normalized graph traces, it will be a convention to refer to these normalized graph traces simply as graph traces.

Example 2.7. Below is the unique normalized graph trace g of the directed graph G shown in Example 2.2, as can verified by the reader.



Lemma 1. If g and g' are graph traces and $t \in [0, 1]$, then $g'' = tg + (1-t)g'$ is a graph trace, where $g''(v) = tg(v) + (1-t)g'(v)$.

Proof. Let g_0, g_1 be graph traces on a directed graph E, and let $t \in [0, 1]$.
Let $g_t(v) = (1-t)g_0(v) + tg_1(v)$.
So, for $v_1, \dots, v_n \in E^0$, $\sum_{k=1}^{n} g_0(v_k) = 1$ and $\sum_{k=1}^{n} g_1(v_k) = 1$.

Then,

$$\begin{aligned}\sum_{k=1}^{n} g_t(v_k) &= \sum_{k=1}^{n}(1-t)g_0(v) + tg_1(v)\\ &= \sum_{k=1}^{n}(1-t)g_0(v) + \sum_{k=1}^{n} tg_1(v)\\ &= (1-t)\sum_{k=1}^{n} g_0(v) + t\sum_{k=1}^{n} g_1(v)\\ &= (1-t)+t\\ &= 1\end{aligned}$$

Let $v \in E^0$ and $e_1, \dots, e_n$ be edges with $r(e) = v$. Then,

$$g_0(v) \geq \sum_{k=1}^{n} g_0(s(e_k))$$

and

$$g_1(v) \geq \sum_{k=1}^{n} g_1(s(e_k))$$

. As $t \in [0,1]$, $t \geq 0$ and $1-t \geq 0$. So,

$$(1-t)g_0(v) \geq (1-t)\sum_{k=1}^{n} g_0(s(e_k))$$

and

$$tg_1(v) \geq t\sum_{k=1}^{n} g_1(s(e_k))$$

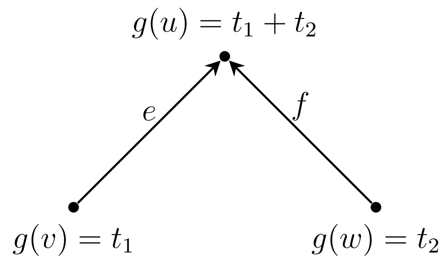
.
Then,

$$\begin{aligned}g_t(v) &= (1-t)g_0(v) + tg_1(v)\\ &\geq (1-t)\sum_{k=1}^{n} g_0(s(e_k)) + t\sum_{k=1}^{n} g_1(s(e_k))\\ &= \sum_{k=1}^{n}(1-t)g_0(v) + tg_1\\ &= \sum_{k=1}^{n} g_t(v)\end{aligned}$$

As $g_t(v) \geq \sum_{k=1}^{n} g(s(e_k))$ for all edges with $r(e) = v$, and for $v_1, \dots, v_n \in E^0$, $\sum_{k=1}^{n} g_t(v_k) = 1$, $g_t(v)$ is a graph trace of E by Definition 2.5. □

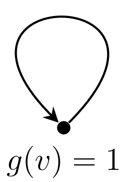
Definition 2.8. An *extreme* graph trace is a graph trace that cannot be written as a convex combination of other graph traces. That is, if g is an extreme graph trace and $g = tg' + (1-t)g''$ for $t \in (0,1)$, then $g' = g'' = g$.

Example 2.9. Some directed graphs have multiple graph traces.



As the sum of graph trace values on the vertices must be 1, we know that $2(t_1+t_2) = 1$ and that $t_1 + t_2 = \frac{1}{2}$, setting a relationship between the graph's two sources which remains true for all graph traces. Because of this relationship, specifying the value of the trace at one source is enough to determine the graph trace on all vertices. Note that the value at u will always be $\frac{1}{2}$ regardless of the value of the traces at the sources. Furthermore, this graph has two extreme graph traces, one where $t_1 = \frac{1}{2}$, and one where $t_2 = \frac{1}{2}$. All other graph traces (of which there are infinitely many) are convex combinations of these two extreme graph traces, which can be found by $g = (2g(v))g_v + (2g(w))g_w = (2(g(v)g_v + (1-2g(v))g_w = sg_v + (1-s)g_w$. One can imagine the set of graph traces of this graph as a line segment with the extreme traces at the ends and all other graph traces ranging between the two extreme traces.

Example 2.10. Graph traces are not limited to graphs with no cycles.

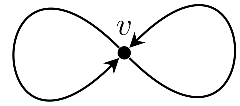


It will now be our convention to label vertices simply with the numeric value associated with the graph trace at that vertex rather than with an equation. If it is necessary to label vertices otherwise, these name labels will be in the form of lowercase letters and should be distinguishable from graph traces.

Example 2.11. Graph traces are not limited to finite graphs. Consider the following example of an infinite graph with graph trace as shown.

However, not all directed graphs have graph traces. Following are examples of directed graphs without traces.

Example 2.12. Graphs with loops do not always have graph traces. Consider the following graph:

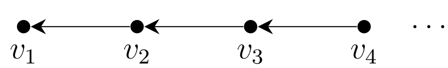


As v is a regular vertex, to have a graph trace, we must have

$$g(v) = \sum_{e \in E^1, r(e)=v} g(s(e)).$$

Note that v receives two edges, both with source v. Then, $g(v) = 2g(v)$. This is only true when g(v) = 0. However, this means that $\sum_{v \in E^0} g(v) \neq 1$ and g(v) is not a graph trace.

Example 2.13. Not every infinite graph holds a trace.



As each vertex is a regular vertex, we know that:

$$g(v_1) = c = \sum_{e \in E^1, r(e)=v} g(s(e)) = g(v_2) = g(v_3) = g(v_4) \cdots$$

So, each vertex must have the same graph trace value. However, as there are infinitely many vertices, $\sum_{v \in E^0} g(v) \neq 1$ and $g(v)$ is not a graph trace.

Definition 2.14. Let E be a finite graph with no cycles and let v, $w \in E^0$. Then, define the number of finite paths from v as $n(v) = |\{\lambda \in E^* : s(\lambda) = v\}|$. Also define the number of paths between v and w as $n(v, w) = |\{\lambda \in E^* : s(\lambda) = v, r(\lambda) = w\}|$.

Theorem 2.15 (Johnson)**.** *Let E be a finite graph with no cycles. Then there is a bijection between the extreme traces on E and the sources of E, given by*

$$S_E \ni v \mapsto g_v \in T(E),$$

where $g_v(w) = \frac{n(v,w)}{n(v)}$.

REMARK 2.16. The set of graph traces forms a convex compact subset of $C_0(E^0, [0,1])$ hence by the Krein-Millman Theorem, $T(E)$ is the closed convex hull of its extreme points, the extreme traces. Knowing the extreme traces of a graph defines all the graph traces of the graph, as all other graph traces are convex combinations of extreme graph traces. Furthermore, when the graph is finite, by Carathéodory's theorem we know that every trace can be written as a convex combination of $|E^0| + 1$ or fewer extreme traces.

Theorem 2.17. *[cf. Kadison-Ringrose, Vol. 1, Lemma 3.4.6] Let g be a graph trace on a directed graph E, where E does not have any infinite receivers. Then the following are equivalent:*

(i) If h is another graph trace on E and there exists some $t \in (0,1)$ such that $g(v) \geq th(v)\ \forall v \in E^0$, then $h = g$.

(ii) g is an extreme graph trace.

Proof. $(i) \to (ii)$: Let g be a graph trace on a directed graph E that satisfies condition (i). Suppose that there are graph traces g', g'' on E such that $g = tg' + (1-t)g''$ for some $t \in (0,1)$. From this we see that $g \geq tg'$, and then setting $h = g'$ in condition (i) gives $g = g'$ (and hence also $g'' = g$), thus g is extreme.

$(ii) \to (i)$: Let g be an extreme graph trace on the directed graph E. Let h be a graph trace on E, and $t \in (0,1)$ such that $g(v) \geq th(v)\ \forall v \in E^0$. Define $f = \frac{g-th}{1-t}$. As $g > th$ and $1 > t$, we know that the value of f will always be non-negative. As

$$\sum_{v \in E^0} g(v) = 1$$

and

$$\sum_{v \in E^0} th(v) = t$$

we know

$$\frac{\sum_{v \in E^0} g(v) - th(v)}{1-t} = 1$$

, so $\sum_{v \in E^0} f(v) = 1$. As the Cuntz-Krieger relations are linear, and f is a linear combination of graph traces which satisfy these conditions, we know f is a graph trace. This would then mean that g is a convex combination of other graph traces, specifically $g = th + (1-t)f$. However, g is extreme, so by Definition 2.8, $g = h = f$. So, when g is extreme, (i) holds true. □

Theorem 2.18. *If E is a graph with* finitely many vertices*, then (i) and (ii) of Theorem 2.17 are equivalent to*

(iii) If h is a graph trace on E such that $h(v) = 0$ whenever $g(v) = 0$, then $g = h$.

Proof. $(iii) \to (i)$: Let g, h be graph traces on the finite directed graph E, such that $h(v) = 0$ whenever $g(v) = 0$, then $g = h$. Let c $(0, 1)$ and $c \neq \frac{g(v)}{h(v)}$ for all v where $g(v)/h(v)$ is non-zero. Let $t = \min\{\frac{g(v)}{h(v)}, c\}$ for all v such that $h(v), g(v) \neq 0$. As E has finitely many vertices and all $\frac{g(v)}{h(v)}$ values are positive, t will be positive.
Consider the inequality, $g(v) \geq th(v)$. There are two cases to examine. First, when $g(v) = 0$, by definition $h(v) = 0$ and the inequality holds. Second, if $g(v) \neq 0$, then either $h(v) = 0$ and the inequality holds, or $h(v) \neq 0$. Then, $\frac{g(v)}{h(v)} \geq t$, by definition of t and $g(v) \geq th(v)$, showing that the inequality holds.
Therefore, when h is a graph trace on E such that $h(v) = 0$ whenever $g(v) = 0$, then $g = h$, we know that there exists some $t \in (0, 1)$ such that $g(v) \geq th(v)$ $\forall v \in E^0$, then $h = g$. $\square$

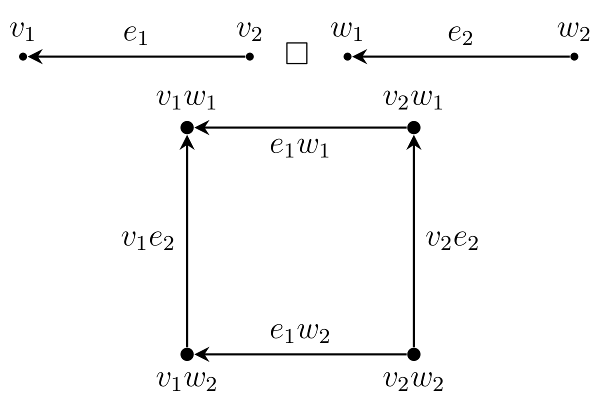
Given two directed graphs, E and F, one can combine them to form a larger graph, which we will refer to as their *product graph*. The graphs, E and F used to create product graphs will be referred to as *factor graphs*. In this section we will discuss several ways of combining directed graphs into product graphs. We will refer to these combination techniques as *graph products* and will show several relations between aspects of the factor graphs and their product graphs.

In this section it will be helpful to identify when two graphs are "the same": if E and F are directed graphs, then an isomorphism from E to F consists of an edge bijection $\phi : E^1 \to F^1$ and a vertex bijection $\phi_0 : E^0 \to F^0$ such that $r_F(\phi(e)) = \phi_0(r_E(e))$ and $s_F(\phi(e)) = \phi_0(s_E(e))$

Definition 2.19 (Johnston)**.** The *box (Cartesian) product* of E with F is the graph $E\square F = (E^0 \times F^0, (E^1 \times F^0) \cup (E^0 \times F^1), r_\square, s_\square)$, where $r_\square, s_\square$ are defined as follows: For all $e \in E^1$, $f \in F^1$, $u \in E^0$, $v \in F^0$:

$$\begin{aligned} r_\square(e, v) &= (r_E(e), v) \qquad & r_\square(u, f) &= (u, r_F(f)) \\ s_\square(e, v) &= (s_E(e), v) \qquad & s_\square(u, f) &= (u, s_F(f)) \end{aligned}$$

Example 2.20. Given the directed graph E shown below, we can compute the box product of E with itself, $E\square E$.



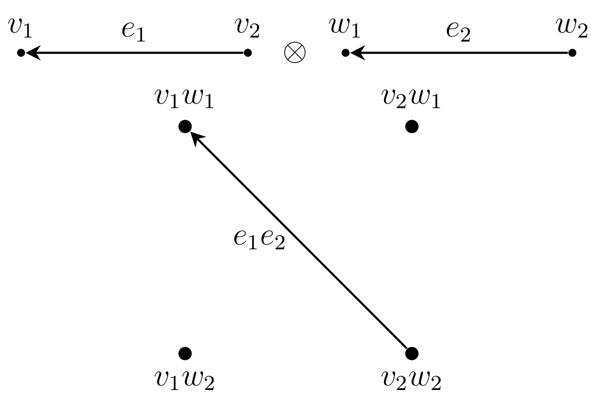
Proposition 2.21. *Let E and F be finite, directed graphs without cycles with m and n extreme traces, respectively. Then $E \square F$ has mn extreme traces.*

Proof. Let E and F be finite, directed graphs without cycles. Let E have m extreme traces and F have n extreme traces. As shown by Johnson in [2], extreme traces correspond to sources, so $|S_E| = m$ and $|S_F| = n$. Futhermore, a vertex vw in $E \square F$ is a source if and only if $v \in S_E$ and $w \in S_F$. Therefore, $|S_{E \square F}| = mn$. As extreme traces correspond to sources, $E \square F$ has mn extreme traces. $\square$

Definition 2.22 (Johnston)**.** The *tensor (categorical) product* of E with F is the graph $E \otimes F = (E^0 \times F^0, E^1 \times F^1, r_\otimes, s_\otimes)$, such that for all $(e, f) \in E^1 \times F^1$ we define:

$$r_\otimes(e, f) = (r_E(e), r_F(f)) \text{ and } s_\otimes(e, f) = (s_E(e), s_F(f)).$$

Example 2.23. Given the directed graph E shown below, we can compute the tensor product of E with itself, $E \otimes E$.

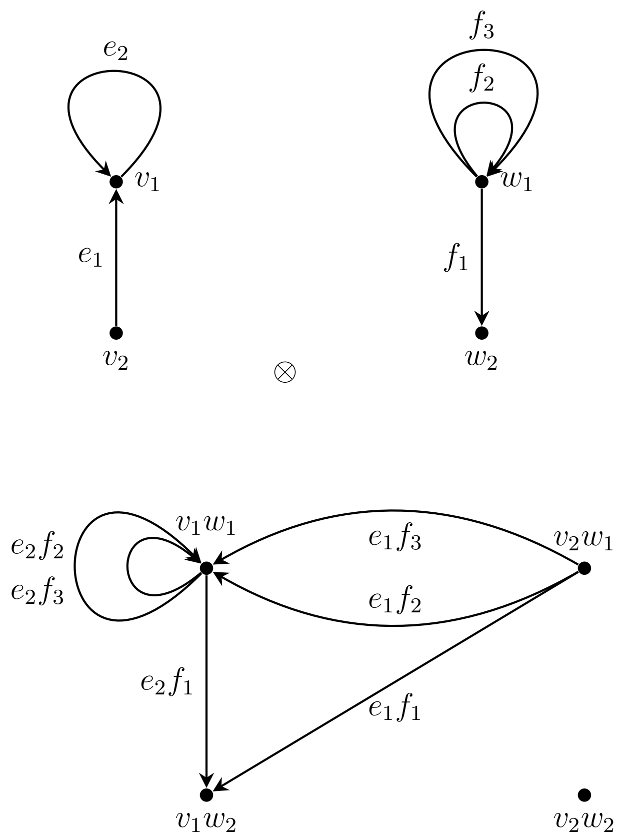


Lemma 2. Let E be a directed graph, let P denote the graph with one vertex and no edges, and let C denote the graph with one vertex and one edge.

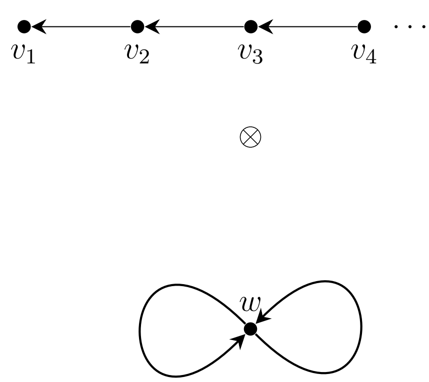
(i) $E \square P \cong E$

(ii) $E \otimes C \cong E$

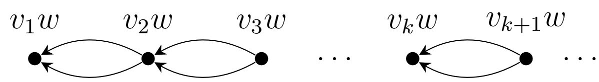
Example 2.24. As isolated vertices can be created in product graphs using the tensor product, it is possible to have a product graph that holds a trace whose factor graphs do not. In the following example, neither of the two factor graphs hold a graph trace. However, due to the isolated vertex, the product graph has a unique graph trace $g(v_2w_2) = 1$, $g(v_1w_1) = 0, g(v_1w_2) = 0, g(v_2w_1) = 0$.



Example 2.25. Creating isolated vertices is not the only way to achieve tensor product graphs with traces whose factor graphs do not have traces. Consider the following graphs that do not hold traces.



The tensor product of these two graphs, as shown below, does have a graph trace. This graph trace can be seen above in Example 2.11.



Proposition 2.26. *Let E and F be directed graphs with graph traces g_E and g_F respectively. Then $g = g_E \otimes g_F$ given by $g(vw) = g_E(v)g_F(w)$ defines a graph trace on $E \otimes F$*

Proof. Let E and F be directed graphs with graph traces g_E and g_F respectively. Consider $g = g_E \otimes g_F$ given by $g(vw) = g_E(v)g_F(w)$ on $E \otimes F$. As $g_E(v) \geq 0$ and $g_F(w) \geq 0$, we know that $g_E(v)g_F(w) \geq 0$. As g_E and g_F are graph traces, we know $\sum_{v\in E^0} g_E(v) = 1$ and $\sum_{w\in F^0} g_F(v) = 1$. So,

$$\begin{aligned} \sum_{vw\in E^0\times F^0} g(v) &= \sum_{v\in E^0} g_E(v) \sum_{w\in F^0} g_F(w) \\ &= 1. \end{aligned}$$

Let $vw \in E^0 \times F^0$ be a regular vertex. Then, $v \in E^0$ and $w \in F^0$ are regular vertices. Then, $g_E(v) = \sum_{e:r(e)=v} g(s(e))$ and $g_F(w) = \sum_{f:r(f)=w} g(s(f))$. So,

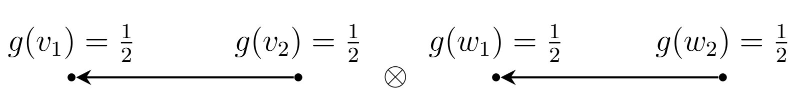
$$\begin{aligned} g(vw) &= \sum_{e:r(e)=v} g_E(s(e)) \sum_{f:r(f)=w} g_F(s(f)) \\ &= \sum_{ef:r(ef)=vw} g(s(ef)). \end{aligned}$$

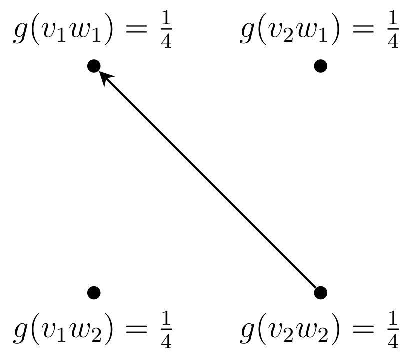
Let vw be an infinite reciever. As $g_E(v) \geq \sum_{e:r(e)=v} g_E(s(e))$ and $g_F(w) \geq \sum_{f:r(f)=w} g_F(s(f))$ (note: this is true for regular and non-regular vertices), we know that

$$\begin{aligned} g(vw) &\geq \sum_{e:r(e)=v} g_E(s(e)) \sum_{f:r(f)=w} g_F(s(f)) \\ &\geq \sum_{ef:r(ef)=vw} g(s(ef)). \end{aligned}$$

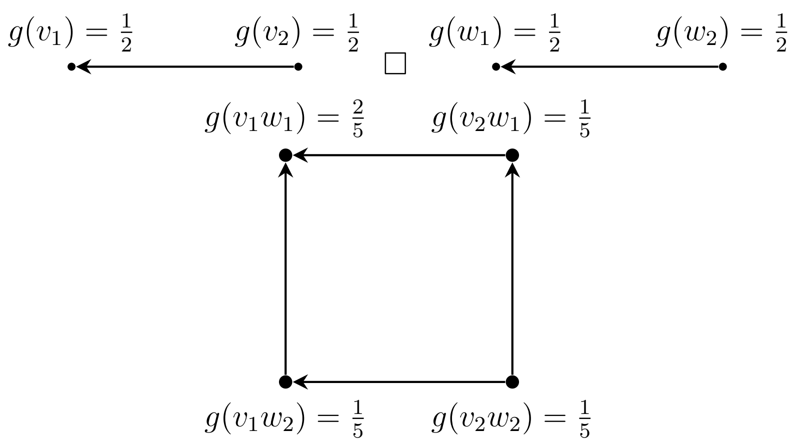
Therefore, by Definition 2.5, $g = g_E \otimes g_F$ is a graph trace on $E \otimes F$. □

Example 2.27. Consider the directed graph E, with unique trace, shown below and the tensor product, $E \otimes E$ with product graph trace. Note, however, that this is not the only graph trace on $E \otimes E$.





Example 2.28. There is no corresponding product trace operation for the box product. Consider the directed graph E shown below and the box product $E\square E$ with unique graph traces.



3 Traces on Locally Convex k-graphs

Higher-rank graphs are multi-dimensional generalizations of directed graphs which tend to be more complicated than directed graphs. They were introduced in citeKumjian-Pask as a way to generalize both graph C^*-algebras and Robertson-Steger algebras. Much of the theory of graph C^*-algebras carries over to the setting of higher-rank graphs, but some results become more complicated.

In order to define a higher-rank graph, it will be useful to have an understanding of the semigroup $\mathbb{N}^k$. (Note: we are using the convention that $0 \in \mathbb{N}$.)

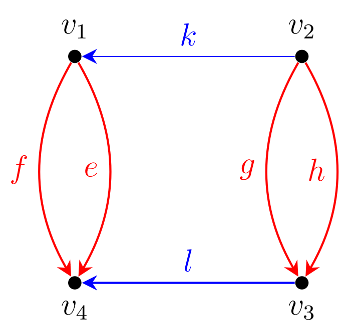
Definition 3.1. Let k be a positive integer. Then $\mathbb{N}^k = \{n = (n_1, \ldots, n_k) : n_i \in \mathbb{N} \text{ for all } i = 1, \ldots, k\}$. Then $\mathbb{N}^k$ is a semigroup with the addition inherited from vector addition in $\mathbb{R}^k$. If $m, n \in \mathbb{N}^k$, then we say that $m \leq n$ if $m_i \leq n_i$ for *all* $i = 1, \ldots, k$. (In particular, $\leq$ is a partial order and not a total order on $\mathbb{N}^k$.) Note that the identity element in $\mathbb{N}^k$ is the zero vector $0 = (0, \ldots, 0)$. For $i = 1, \ldots, k$, let e_i denote the ith standard basis vector, i.e. $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, and so on.

Remark 3.2. Any semigroup S with identity can be viewed as a category with one object, where the elements of S correspond to the morphisms in the category and the object corresponds to the identity element.

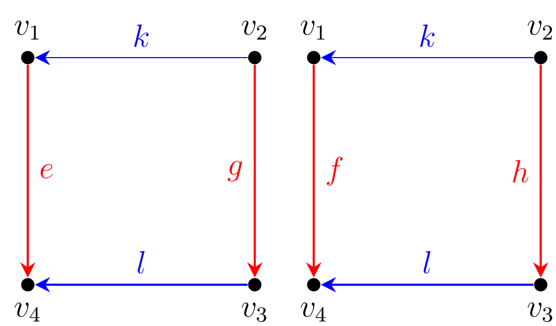
Definition 3.3. A k-graph (Λ, d) consists of a (countable small) category Λ and a *degree functor* $d: \Lambda \to \mathbb{N}^k$ (i.e. $d(\lambda_1\lambda_2) = d(\lambda_1) + d(\lambda_2)$) satisfying the *factorization property*: for any $\lambda \in \Lambda$, if $d(\lambda) = m + n$ for $m, n \in \mathbb{N}^k$, then there exist unique $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m$,$d(\nu) = n$. We will refer to elements of Λ as *paths* in Λ, and if $d(\lambda) = 0$ we will often refer to λ as a *vertex*.

Remark 3.4. The degree functor and factorization property of a higher-rank graph encodes the category structure as well: for $\lambda \in \Lambda$ say of degree $n \in \mathbb{N}^k$, we can write $n = n + 0$, and then the factorization rules give us a unique λ', λ'', of degree n and 0 respectively, such that $\lambda = \lambda'\lambda''$. We must have $\lambda = \lambda'$, and we define $s(\lambda) := \lambda''$. Similarly we can define $r(\lambda)$ to be the unique path of degree 0 such that $r(\lambda)\lambda = \lambda$.

Example 3.5 ([7])**.** We can visualize a k-graph by drawing its 1-skeleton, which is the colored directed graph $(\Lambda^0, \bigcup_{i=1}^k \Lambda^{e_i}, r, s, c)$ where the coloring c is defined by $c(\Lambda^{e_i}) = \{i\}$. The 1-skeleton alone does not determine a k-graph, as it does not take into consideration the factorization property of k-graphs. This factorization can be determined through the creation of factorization squares, as shown below.

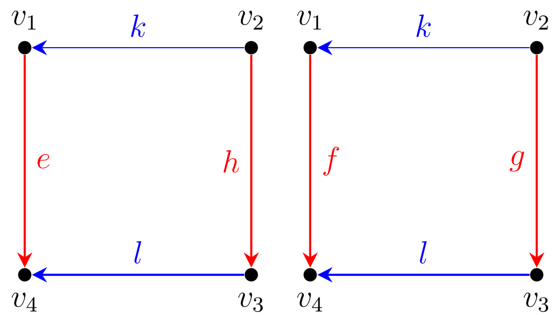


To determine the factorization rules for this graph, we need to know the factorizations of ek and fk. There are two possible sets of factorization rules, which are shown below:



If the first set of potential factorization rules, as shown above, we have $ek = lg$ and $fk = lh$, while in the second, we have $ek = lh$ and $fk = lg$, as shown below. A

1-skeleton and its associated factorization property, as shown by the collection of factorization squares, is enough to define a k-graph. Each set of factorization rules defines a different k-graph for the 1-skeleton.



Definition 3.6. Let (Λ, d) be a k-graph. Then for any $n \in \mathbb{N}^k$, we set $\Lambda^n = d^{-1}(n) = \{\lambda \in \Lambda : d(\lambda) = n\}$. If $\lambda \in \Lambda$, then $\lambda\Lambda$ denotes the set of all *extensions* of λ, that is, paths $\nu = \lambda\mu$ for some $\mu \in \Lambda$. Specializing to the case $\lambda = v \in \Lambda^0$, we have $v\Lambda^n = \{\mu \in \Lambda : d(\mu) = n, r(\mu) = v\}$.

We say that λ is a *maximal path of degree less than or equal to* n if $d(\lambda) \leq n$ and the only $\nu \in \lambda\Lambda$ with $d(\nu) \leq n$ is λ. The set of maximal paths of degree less than or equal to n is denoted by $\Lambda^{\leq n}$; the subset of these which have range vertex $v \in \Lambda^0$ is denoted by $v\Lambda^{\leq n}$.

We say that Λ is *row-finite* if for every $v \in \Lambda^0$ and every $n \in \mathbb{N}^k$, the set $v\Lambda^{\leq n}$ is finite.

We say that Λ is *locally convex* if whenever there exist paths $\lambda \in \Lambda^{e_i}$ and $\mu \in \Lambda^{e_j}$ with $i \neq j$ and $r(\lambda) = r(\mu)$, there must also exist paths $\nu \in s(\lambda)\Lambda^{e_j}$ and $\rho \in s(\mu)\Lambda^{e_i}$.

Remark 3.7. It is not too difficult to see that Λ being row-finite is equivalent to E_Λ (the 1-skeleton of Λ) being row-finite. If Λ has no sources, in the sense that $v\Lambda^n$ is non-empty for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, then Λ is locally convex (the converse is not true, as Example 3.5 shows).

Definition 3.8. Let Λ be a row-finite, locally convex k-graph, and let Λ^0 be its set of vertices. A function $g : \Lambda^0 \to [0, \infty)$ is called a *higher-rank graph trace* if

(i) for any vertex $v \in \Lambda^0$ and any degree $n \in \mathbb{N}^k$, we have

$$\sum_{\lambda \in v\Lambda^{\leq n}} g(s(\lambda)) = g(v);$$

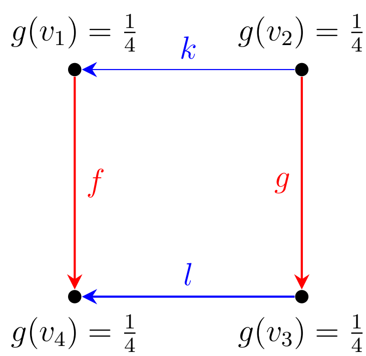
(ii)

$$\sum_{v \in \Lambda^0} g(v) = 1$$

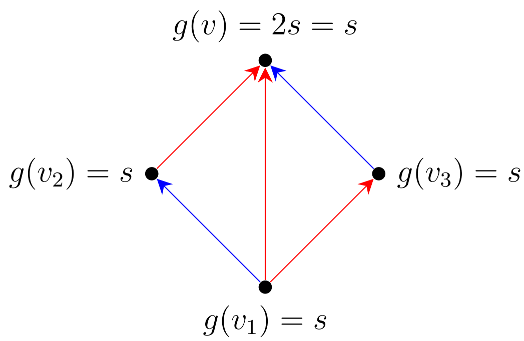
.

Remark 3.9. We will often refer to such higher-rank graph traces simply as graph traces when there is no chance of confusion. One can define graph traces on non-row-finite and non-locally convex k-graphs, but care must be taken in modifying the definition.

Example 3.10. Below is an example of a graph trace on a higher-rank graph. Note that the values are different than they would be on a directed graph with analogous form (Example 2.28).

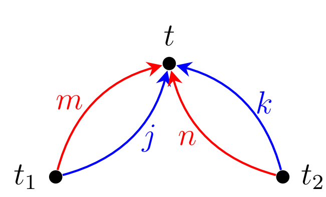


Example 3.11. Not every k-graph carries a graph trace.



Propagating the trace from a value at the graph's source, one can see that, by Definition 3.8, $g(v) = g(v_1) + g(v_2) = g(v_3)$. As $g(v_1) = g(v_2) = g(v_3)$, this gives the relation $2g(v_1) = g(v_1)$ which is only true if $g(v_1) = 0$. This would mean that the value of g is 0 on all vertices and $\sum_{v\in\Lambda^0} g(v) \neq 1$. Therefore, the above graph has no graph trace.

Example 3.12. Consider the 2-graph with three vertices, t, t_1, and t_2, with m lines of one degree and j lines of the other with source t_1 and range t, and with n lines of the first degree and k lines of the second degree with source t_2 and range t.



Traces on higher rank graphs can be understood by examining the relationships between the values of sources on the graph. Looking at $g(t)$, we know that $mg(t_1) + ng(t_2) = jg(t_1)+kg(t_2)$. This give the relation $(n-k)g(t_2) = (j-m)g(t_1)$. As the sum of graph trace values equals 1, we also know that $g(t_1) + g(t_2) + mg(t_1) + ng(t_2) = 1$. This gives the relation $(1 + m)g(t_1) + (1 + n)g(t_2) = 1$. A graph trace on this graph must satisfy both of these conditions. However, not all values of n, m, j, k satisfy these relations while maintaining appropriate values for the graph trace. For example, due to the first relation, when $n > k$, $j > m$ to sustain a graph trace. Similarly, when $n < k$, $j < m$, as the values of the graph trace must be positive. Furthermore, when $n = k$, the relationship between j and m has no restrictions. The same is true for n and k when $j = m$. If the values of n, m, j, k do not satisfy one of these conditions, then the graph has no trace.

The condition that a graph be locally convex is very strong, as the following lemma shows. (Note: for $n \in \mathbb{N}^k$, $|n| := \sum_{i=1}^{k} n_i \in \mathbb{N}$ is referred to as the *weight* of n.)

Lemma 3. Let Λ be a locally convex k-graph. Suppose that $\lambda \in \Lambda$ and e_i is a standard basis vector in $\mathbb{N}^k$ such that both

- $s(\lambda)$ receives no paths of degree e_i, and
- $r(\lambda)$ receives a path of degree e_i.

Then $d(\lambda) \geq e_i$.

Proof. The proof is by induction on $|d(\lambda)|$. If $|d(\lambda)| = 1$ then by local convexity and the fact that $s(\lambda)$ receives no paths of degree e_i, $d(\lambda) = e_i$ so our base case holds. Assume that for if $\lambda \in \Lambda^*$, $|d(\lambda)| = n$, and the conditions given in the statement of the Lemma hold, then $d(\lambda) \geq e_i$. Let $\lambda \in \Lambda^*$ be such that $|d(\lambda)| = n + 1$ and such that the conditions given in the statement of the lemma hold.

By factorization property, we can factor λ into as $\lambda = \lambda'\lambda''$ of degrees e_j and $d(\lambda) - e_j$ respectively, for some basis vector $e_j \in \mathbb{N}^k$, where $r(\lambda'') = s(\lambda_0)$. Note that $|d(\lambda'')| = |d(\lambda)| - 1$.

Case 1: $e_j = e_i$. In this case, $d(\lambda) \geq e_i$, so we are finished

Case 2: $e_j \neq e_i$. By local convexity there must be a path μ_0 such that $r(\mu_0) = s(\lambda_0)$ and $d(\mu_0) = e_i$. Therefore $r(\lambda'')$ receives a path of degree e_i and $s(\lambda'') = s(\lambda)$, which does not receive a path of degree e_i by hypothesis. Thus $d(\lambda'') \geq e_i$ by induction, and $d(\lambda) = d(\lambda'') + e_j$, so we have established the Lemma.

□

Definition 3.13. Let Λ be a finite graph with no cycles with v, $w \in \Lambda^0$. Then, define the number of finite paths from v as $n(v) = |\{\lambda \in \Lambda : s(\lambda) = v\}|$. Also define the number of paths between v and w as $n(v, w) = |\{\lambda \in \Lambda : s(\lambda) = v, r(\lambda) = w\}|$.

Proposition 3.14. *Let Λ be a finite locally convex k-graph, and suppose that $v \in S_\Lambda$ is a source. For $w \in \Lambda^0$, let $g_v(w) = \frac{n(v,w)}{n(v)}$. Then g_v is a higher-rank graph trace on Λ.*

Proof. It suffices to show that for every $w \in \Lambda^0$ and every basis vector $e_i \in \mathbb{N}^k$, we have $g_v(w) = \sum_{\lambda \in w\Lambda^{\leq e_i}} g_v(s(\lambda))$. This amounts to showing that

$$|n(v,w)| = \sum_{\lambda \in \Lambda^{\leq e_i}} |n(v, s(\lambda))| \tag{1}$$

There are two cases to consider, depending on whether or not the vertex w receives any edges of degree e_i.

Case I: w receives no edges of degree e_i. Then $w\Lambda^{\leq e_i} = \{w\}$ and the sum on the right-hand side of the above equation is trivially equal to $|n(v,w)|$.

Case II: $w\Lambda^{e_i} = \{f_1, \ldots, f_m\}$. Define a bijection between $w\Lambda v$ and $\cup_{j=1}^m \{f_j\} \times s(f_j)\Lambda v$ as follows: for any path $\lambda \in w\Lambda v$, use Lemma 3 to see that $d(\lambda) \geq e_i$. Then use the factorization property to get $\lambda = f_j \lambda'$ for some $j \in \{1, \ldots, m\}$ and $\lambda' \in s(f_j)\Lambda v$. Then define $\phi(\lambda) = (f_j, \lambda')$; as $\lambda = f_j\lambda'$, the map ϕ is injective. Since $f_j\mu \in w\Lambda v$ for any j and $\mu \in s(f_j)\Lambda v$, we have that ϕ is surjective as well. This establishes the desired equation, so that g_v is a trace. $\square$

Lemma 4. Let Λ be a finite locally-convex k-graph, and let g be a trace on Λ. Then the following are equivalent:

- g is extreme;
- if h is another graph trace on Λ, and there exists $t \in (0,1)$ such that $g(v) \geq th(v)$ for all $v \in \Lambda^0$, then $g = h$;
- if h is another graph trace on Λ such that $h(v) = 0$ whenever $g(v) = 0$, then $g = h$.
- if h is another graph trace such that the set of $v \in S_\Lambda$ such that $h(v) = 0$ is the same as the set of $v \in S_\Lambda$ such that $g(v) = 0$, then $g = h$.

Lemma 5. Let Λ be a finite, locally convex k-graph with no cycles and suppose that $g \in T(\Lambda)$ is a graph trace on Λ such that there is a unique source $v \in S_\Lambda$ with $g(v) \neq 0$. Then g is extreme and $g = g_v$.

Proof. Suppose that h is another trace such that $h(v) = 0$ whenever $g(v) = 0$. We show that $h = g$. Immediately we see that the only source where h has nonzero value is v. It is a general fact that, in a finite k-graph, $h(w) = \sum_{v \in S_\Lambda} n(v,w)h(v)$ (take n to be the maximum degree of any path in the definition of a graph trace). Thus we have $h(w) = n(v,w)h(v)$ for all $w \in \Lambda^0$. Summing over all $w \in \Lambda^0$ and applying the normalization condition, we obtain $h(v) = \frac{1}{n(v)}$. This shows that $h(w) = g_v(w)$ for all w, whence $h = g$, and g is extreme. $\square$

Proposition 3.15 (cf. [2]). *If Λ is a finite locally convex k-graph with no cycles, then there is a one-to-one correspondence between sources and extreme traces defined by $S_\Lambda \ni v \mapsto g_v \in T(\Lambda)$ where $g_v(w) = \frac{n(v,w)}{n(v)}$.*

Proof. Since each g_v is only nonzero on a single source, namely v, it is clear that the map $S_\Lambda \to \operatorname{ext} T(\Lambda)$ given above is injective. It remains to show that it is surjective, i.e. that every extreme trace $g \in \operatorname{ext} T(\Lambda)$ must have the form g_v for some $v \in S_\Lambda$. If g is extreme, consider a source $v \in S_\Lambda$ such that $g(v) \neq 0$. Then one can check that g_v vanishes on every vertex that g vanishes on, hence $g = g_v$ by Lemma 4. Thus the map $S_\Lambda \ni v \mapsto g_v \in \operatorname{ext} T(\Lambda)$ is indeed a bijection.

□

4 Products of higher-rank graphs

Definition 4.1. Let Λ be a k-graph and let Π be an ℓ-graph. The product of Λ and Π, denoted $\Lambda \times \Pi$, is simply the Cartesian product of Λ and Π, equipped with the following structure:

(i) $d(\lambda, \pi) = (d(\lambda), d(\pi))$, where the left hand side of this equation is interpreted as a vector in $\mathbb{N}^{k+\ell}$.

(ii) $r(\lambda, \pi) = (r(\lambda), r(\pi))$ and likewise for the source map.

(iii) $(\lambda, \pi)(\lambda', \pi') = (\lambda\lambda', \pi\pi')$ whenever both compositions in the factor graphs are defined.

Note that this with degree map, the vertex set of $\Lambda \times \Pi$ is just $\Lambda^0 \times \Pi^0$.

Lemma 6. With the structure given, $\Lambda \times \Pi$ is a $(k+\ell)$-graph. (That is, it satisfies the factorization rules.)

Proof. See [3] for a proof. □

Remark 4.2. Let Λ be a k-graph and let Π be an ℓ-graph, with 1-skeletons E_Λ and E_Π respectively. The 1-skeleton of $\Lambda \times \Pi$ is the same as the box product of E_Λ and E_Π, as can be verified fairly straightforwardly. Therefore, $\Lambda \times \Pi$ is row-finite if and only if both Λ and Π are row-finite.

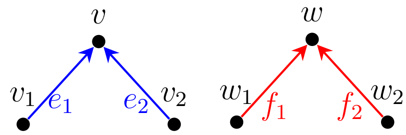
Lemma 7. Let Λ be a k-graph and let Π be an ℓ-graph. Then $\Lambda \times \Pi$ is locally convex if and only if both Λ and Π are locally convex.

Proof. (if): Suppose that e_i and e_j ($i \neq j$) are two basis vectors in $\mathbb{N}^{k+\ell} \cong \mathbb{N}^k \times \mathbb{N}^\ell$ (where as usual we identify the last ℓ entries in a vector as a vector in $\mathbb{N}^\ell$). Let $\nu = \lambda\pi, \nu' = \lambda'\pi' \in \Lambda \times \Pi$ satisfy $r(\nu) = r(\nu')$, $d(\nu) = e_i$, and $d(\nu') = e_j$.

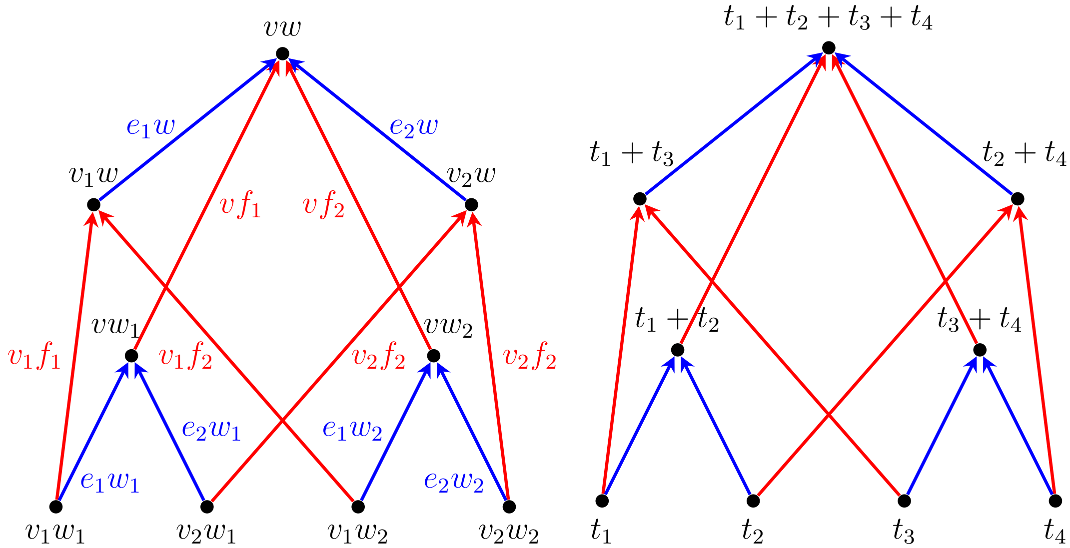
Case I: e_i and e_j both belong to $\mathbb{N}^k$ or both belong to $\mathbb{N}^\ell$. Assume without loss of generality the former; then $\pi = \pi' \in \Pi^0$ is a vertex (call it w), and we have λ, λ' two paths in Λ with $r(\lambda) = r(\lambda')$ and $d(\lambda) = e_i$, $d(\lambda') = e_j$. Applying local convexity of Λ gives $\mu, \mu' \in \Lambda$ with $r(\mu) = s(\lambda)$, $r(\mu') = s(\lambda')$, $d(\mu) = e_j$, and $d(\mu') = e_i$. Then the paths $\rho = \mu w$ and $\rho' = \mu' w$ in $\Lambda \times \Pi$ satisfy $r(\rho) = s(\nu), r(\rho') = s(\nu'), d(\rho) = d(\nu')$, and $d(\rho') = d(\nu)$.

Case II: e_i belongs to $\mathbb{N}^k$ and e_j belongs to $\mathbb{N}^\ell$, or vice versa. Assume without loss of generality the former; □

Example 4.3. Let E^* and F^* be the 1-graphs associated with the colored directed graphs below:

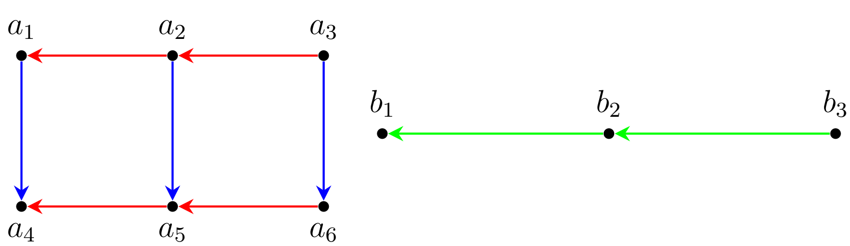


Then $E^* \times F^*$ is the 2-graph corresponding to the skeleton below.

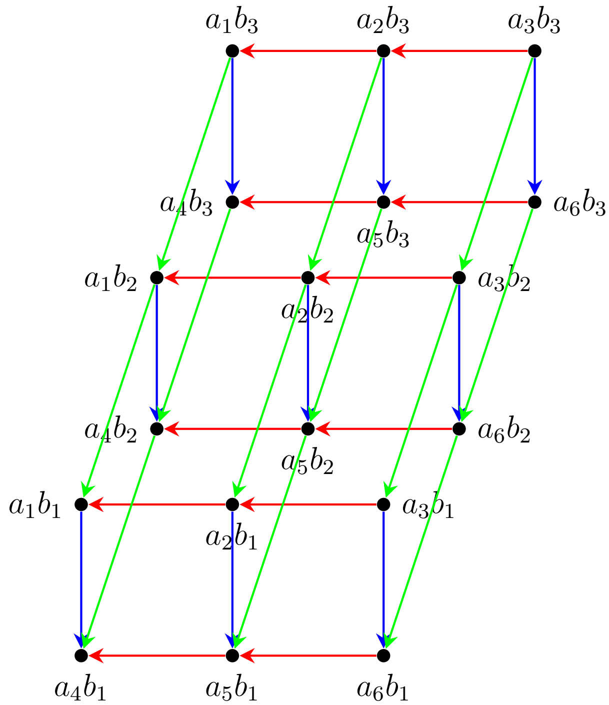


The graph traces on the product graph are represented by different combinations of t_1, t_2, t_3, t_4 as shown above, where each t_i corresponds to one of the four sources for $1 \leq i \leq 4$. Then we know that $4(t_1 + t_2 + t_3 + t_4) = 1$, so $t_1 + t_2 + t_3 + t_4 = 1/4$. This gives us a set of possible graph traces with four possible extreme traces. The graph trace on the product graph $E^* \times F^*$ can be concentrated at any one of the four sources, and be equal to zero at every other source. This is an example of a graph where the number of possible extreme traces is equal to the number of sources.

Example 4.4. We can also take the product of k-graphs when $k>1$. Consider the 2-graph and the 1-graph shown below.



We can take the product of these graphs, which will be a 3-graph represented by the following skeleton.



Lemma 8. Let Λ be a finite k-graph with no cycles, and let $g : \Lambda^0 \to [0, infty)$ such that $\sum_{v \in \Lambda^0} g(v) = 1$. If for every $v \in \Lambda^0$ and every standard basis vector $e_i \in \mathbb{N}^k$

$$g(v) = \sum_{\lambda \in v\Lambda^{\leq e_i}} g(s(\lambda)),$$

then g is a graph trace on Λ.

Proof. For all sources w, we know

$$g(w) = \sum_{\lambda \in v\Lambda^{\leq n}} g(s(\lambda))$$

so we only have to consider vertices that receive an edge. Since for any v

$$g(v) = \sum_{\lambda \in v\Lambda^{\leq 0}} g(s(\lambda))$$

using our givens we have

$$g(v) = \sum_{\lambda \in v\Lambda^{\leq n}} g(s(\lambda))$$

for n=1. Assume that for a given k and any $v \in E^0$,

$$g(v) = \sum_{\lambda \in v\Lambda^{\leq k}} g(s(\lambda))$$

Then, for all m in the standard basis vectors of N^k,

$$\begin{aligned} g(v) &= \sum_{\lambda \in v\Lambda^{\leq k}} \sum_{\lambda_1 \in s(\lambda)\Lambda^{\leq m}} g(s(\lambda_1)) \\ &= \sum_{\lambda \in v\Lambda^{\leq k+1}} g(s(\lambda)). \end{aligned}$$

Thus by induction for all n we have

$$g(v) = \sum_{\lambda \in v\Lambda^{\leq n}} g(s(\lambda))$$

□

Proposition 4.5. *Let Λ be a k-graph and let Π be an ℓ-graph. Suppose that we are given graph traces g_1 on Λ and g_2 on Π. Define the map $g_1 g_2 : \Lambda^0 \times \Pi^0 \to [0, \infty)$ via $g_1 g_2(vw) = g_1(v) g_2(w)$. Then $g_1 g_2$ is a graph trace on $\Lambda \times \Pi$.*

Proof. for any regular vertex $v_i w_j$, any n in the set of standard basis vectors N^k,

$$\begin{aligned} \sum_{\lambda \in v_i w_j \Lambda^{=n}} g(s(\lambda)) &= \sum_{\lambda_1 \in v_i \Lambda_1^{\leq n}} g_1(s(\lambda_1) g_2(w_i) \\ &= g_2(w_i) \sum_{\lambda_1 \in v_i \Lambda_1^{\leq n}} g_1(s(\lambda_1) \\ &= g_1(v_i) g_2(w_j) \end{aligned}$$

Use a similar argument for the set of degrees from Π.

$$\begin{aligned} \sum_{vw \in (\Lambda \times \Pi)^0} g(vw) &= \sum_{v \in \Lambda^0, w \in \Pi^0} g_1(v) g_2(w) \\ &= \sum_{v \in \Lambda^0} g(v) \sum_{w \in \Pi^0} g(w) \\ &= 1 * 1 \\ &= 1 \end{aligned}$$

By Lemma 8 this is sufficient to prove that g is a graph trace. □

There is a construction for traces that moves in the opposite direction as well.

Proposition 4.6. *Let g be a graph trace on $\Lambda \times \Pi$. Then define $g_1 : \Lambda^0 \to [0, \infty)$ and $g_2 : \Pi^0 \to [0, \infty)$ by*

$$g_1(v) = \sum_{w \in \Pi^0} g(vw) \qquad g_2(w) = \sum_{v \in \Lambda^0} g_2(vw).$$

Then g_1 is a graph trace on Λ and g_2 is a graph trace on Π.

Proof. By symmetry it will suffice to prove that g_1 is a graph trace.

$$\begin{aligned}\sum_{v'\in\Lambda^0} g_1(v') &= \sum_{v'\in\Lambda^0}\sum_{w'\in\Pi^0} g(v'w') \\ &= \sum_{v'w'\in\Lambda^0\times\Pi^0} g(v'w') \\ &= 1\end{aligned}$$

Let Λ be a k-colored higher rank graph. Fix an arbitrary vertex $v\in\Lambda^0$ and a standard basis vector $e_i\in\mathbb{N}^k$. Then

$$\begin{aligned}g_1(v) &= \sum_{w\in\Pi^0} g(vw) \\ &= \sum_{w\in\Pi^0}\sum_{\lambda\in v\Lambda^{\leq e_i}} g(s(\lambda)w) \\ &= \sum_{\lambda\in v\Lambda^{\leq e_i}} g_1(s(\lambda)).\end{aligned}$$

Since the choice of v and e_i were arbitrary, this holds for all $v\in\Lambda^0$ and all $e_i\in\mathbb{N}^k$. Thus by Lemma 8 we have that g_1 is a graph trace on Λ. □

Lemma 9. If $g=g_\Lambda g_\Pi$, then $g_1=g_\Lambda$ and $g_2=g_\Pi$.

Proof. Fix a vertex $vw\in\Lambda\times\Pi$ with $v\in\Lambda$ and $w\in\Pi$, so that $g(vw)=g_\Lambda(v)g_\Pi(w)$. Then, because g_Π is a graph trace, we have that

$$\begin{aligned}g_1(v) &= \sum_{w'\in\Pi^0} g(vw') \\ &= \sum_{w'\in\Pi^0} g_\Lambda(v)g_\Pi(w') \\ &= g_\Lambda(v)\sum_{w'\in\Pi^0} g_\Pi(w') \\ &= g_\Lambda(v).\end{aligned}$$

Additionally, because g_Λ is a graph trace,

$$\begin{aligned}g_2(w) &= \sum_{v'\in\Lambda^0} g(v'w) \\ &= \sum_{v'\in\Lambda^0} g_\Lambda(v')g_\Pi(w) \\ &= g_\Pi(w)\sum_{v'\in\Lambda^0} g_\Lambda(v') \\ &= g_\Pi(w)\end{aligned}$$

These are what we wished to prove.

□

Lemma 10. Let g and g' be graph traces on Λ such that $g(v) = g'(v)$ whenever v is a source in Λ. Then $g = g'$.

Proof. We are given that for any source v on Λ, $g(v) = g'(v)$. Define the set of vertices S_n to be the set $\{v | \forall \lambda \in \Lambda^* \text{ with } r(\lambda) = v, |d(\lambda)| \leq n\}$ where if $d(\lambda) = c_1 e_1 + c_2 e_2 + \ldots + c_l e_l$, then $|d(\lambda)| = c_1 + c_2 + \ldots + c_l$. Note that by the givens we have that S_0, the set of all sources, satisfies $\forall v \in S_0, g(v) = g'(v)$. Assume that for any vertex $v \in S_n$, $g(v) = g'(v)$. Pick an arbitrary vertex $v_k \in S_{n+1}$.

Case 1: Suppose $v_k \in S_n$; then $g(v_k) = g'(v_k)$.

Case 2: Suppose $v_k \notin S_n$. Then $g(v_k) = \sum_{\lambda \in v_k \Lambda^{\leq m}} g(s(\lambda)) \forall m$ where $m \in \mathbb{N}^k$. Since $v_k \notin S_n$, it must receive at least one path of degree $n+1$. Note that $|n+1| > 0$ and thus no maximal paths will have degree 0. Then, because $\forall \lambda \in v_k \Lambda^{n+1}$ $s(\lambda) \in S_0$, we have

$$\begin{aligned} g(v_k) &= \sum_{\lambda \in v_k \Lambda^{\leq n+1}} g(s(\lambda)) \\ &= \sum_{\lambda \in v_k \Lambda^{\leq n+1}} g'(s(\lambda)) \\ &= g'(v). \end{aligned}$$

Since both cases result in $g(v_k) = g'(v_k)$ for an arbitrary $v_k \in S_{n+1}$, we have proven the desired result by induction.

$\square$

Definition 4.7. Let g be a trace on $\Lambda \times \Pi$. We say that g is a *product trace* if $g = g_1 g_2$, where g_1 is a trace on Λ and g_2 is a trace on Π. Additionally, we define $\text{proj}(g)$ to be the product trace $g_1 g_2$.

The preceding lemma shows that $g \mapsto \text{proj}(g)$ is an idempotent map onto the collection of product traces.

For the remainder of this section, g will refer to a trace on the product graph $\Lambda \times \Pi$. Furthermore, g_1 and g_2 will refer to factor traces/projections of g onto the factor graphs on Λ and Π, respectively.

Proposition 4.8 ([6, Lemma 1.1])**.** *If g is a trace on $\Lambda \times \Pi$ and g_1 is extreme, then $g = g_1 g_2$.*

Proof. Consider an arbitrary $w \in \Pi^0$. Then either $g_2(w) = 0$ or $g_2(w) \neq 0$.
Case 1: Suppose $g_2(w) = 0$. Then $\forall v \in \Lambda^0$ we must have $g(vw) = 0$ by the definition of g_2. Thus $g(vw) = g_1(v) g_2(w)$.
Case 2: Suppose $g_2(w) \neq 0$. Let $h : \Lambda \to [0, \infty)$ be defined by $h(v) = \frac{g(vw)}{g_2(w)}$.

$$\begin{aligned} \sum_{v \in \Lambda^0} h(v) &= \sum_{v \in \Lambda^0} \frac{g(vw)}{g_2(w)} \\ &= \frac{g_2(w)}{g_2(w)} \\ &= 1. \end{aligned}$$

Then, $\forall v \in \Lambda^0$, we also have the following:

$$\begin{aligned}
h(v) &= \frac{g(vw)}{g_2(w)} \\
&= \sum_{\mu \in vw(\Lambda\times\Pi)^{\leq(n,0)}} \frac{g(s(\mu))}{g_2(w)} \\
&= \sum_{\lambda \in v\Lambda^{\leq n}} h(s(\lambda)).
\end{aligned}$$

Thus h is a graph trace on Λ. We know that for any $v \in \Lambda^0$, if $g_1(v) = 0$ then $\sum_{w' \in \Pi^0} g(vw') = 0$. Since all values of g are non-negative, $g(vw) = 0$ which implies $h(v) = 0$. Then, $g_1(v) = 0$ implies h(v)=0. This combined with 2.17 tells us that $h = g_1$. Therefore we conclude that $\forall v \in \Lambda^0$, $g_1(v) = \frac{g(vw)}{g_2(w)}$. Then $g_1(v)g_2(w) = g(vw)$ for all w. □

Proposition 4.9. *If g_Λ and g_Π are extreme graph traces on Λ and Π respectively, then $g_\Lambda g_\Pi$ is an extreme trace on $\Lambda \times \Pi$.*

Proof. Express $g_\Lambda g_\Pi$ as $g' + (1-t)g''$ for some graph traces $g', g'' \in \Lambda \times \Pi$ and some $t \in (0,1)$. Then $\forall v \in \Lambda^0$,

$$\begin{aligned}
(g' + (1-t)g'')_1(v) &= \sum_{w\in\Pi^0} (g' + (1-t)g'')(vw) \\
&= \sum_{w\in\Pi^0} g'(vw) + \sum_{w\in\Pi^0} (1-t)g''(vw) \\
&= g_1'(v) + (1-t)g_1''(v)
\end{aligned}$$

So $g_\Lambda = g_1' + (1-t)g_1''$. Because g_Λ is extreme, we know $g_1' = g_1'' = g_\Lambda$. A similar argument holds to show that $g_2' = g_2'' = g_\Pi$. Since g_Λ is extreme, $g' + (1-t)g'' = g_\Lambda g_\Pi$. □

Lemma 11. The product trace $g_\Lambda g_\Pi$ is extreme if and only if g_Λ and g_Π are extreme.

Proof. If g_Λ and g_Π are extreme, then the above proposition shows that $g_\Lambda g_\Pi$ is extreme. For the other direction start off contrarily with the assumption that g_Λ is not extreme. Then $g_\Lambda = g' + (1-t)g''$ for some graph traces $g' \neq g''$ and some $t \in (0,1)$. Then $g_\Lambda g_\Pi = g' g_\Pi + (1-t)g'' g_\Pi$. Since $g' g_\Pi \neq g'' g_\Pi$ and $t \in (0,1)$, we have the contradiction that $g_\Lambda g_\Pi$ is not an extreme trace. Therefore, g_Λ must be an extreme trace. A symmetric argument may be used to prove that g_Π is an extreme trace as well. □

Proposition 4.10. *If E^* and F^* are 1-graphs, then every extreme trace on $E^* \times F^*$ is a product trace.*

Proof. Let g be an arbitrarily chosen extreme trace on $E^* \times F^*$. Then g must have nonzero trace on at least one source, which must be a product of sources on E^* and F^*. Call this source the product of some $v \in E^*$ with some $w \in F^*$ which we will call vw. As mentioned in Section 2 , all extreme traces on directed graphs have nonzero trace on exactly one source. Since 1-graphs are equivalent to directed graphs in terms of tracial properties, all extreme traces on both E^* and F^* have exactly one source with nonzero trace. Since the product of two extreme traces is always extreme, there is an extreme trace h on $E^* \times F^*$ with its only source with nonzero trace being vw. Since h is extreme, by 2.17 vw must be the only source that g has nonzero trace on. Also since g and h share the same set of sources and there are finite vertices on $E^* \times F^*$ and $g(v_i w_j) = 0$ whenever $h(v_i w_j) = 0$, $g = h$ by the extension of 2.17. Thus g is a product of extreme traces. □

Conjecture 4.11. If Λ is a k-graph and F^* is a 1-graph, then every extreme trace on $\Lambda \times F^*$ is a product of extreme traces

Conjecture 4.12. If g is an extreme graph trace on $\Lambda \times \Pi$, then $g = g_1 g_2$.

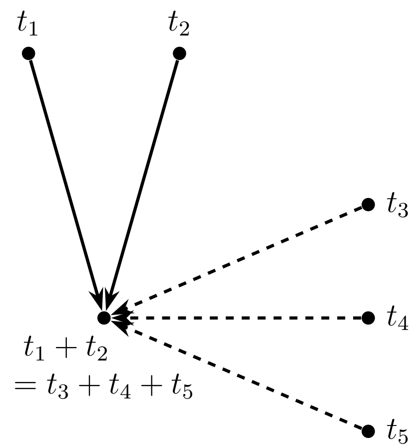
We have a number of statements; any of which would alone imply this result.

- if g is an extreme graph trace and $\forall v, v' \in S_\Lambda, \forall w, w' \in S_\Pi$, if $g(v'w) \neq 0$ and $g(vw') \neq 0$ then $g(vw) \neq 0$ (and, by switching the roles of primes and non primes, $g(v'w') \neq 0$).
- if g is an extreme graph trace then $g = g_1 g_2$.

5 Future directions

So far we have obtained several results pertaining to traces on locally convex k-graphs. It would be interesting to extend these to k-graphs which are not locally convex in a meaningful way. However, the technical difficulties arising from the failure of local convexity make this somewhat tricky. In this section we discuss the connection between traces on higher-rank graphs and tracial states on higher-rank graph C^*-algebras.

Example 5.1. One property of directed graphs that does not hold for non-locally convex higher-rank graphs is the correspondence of number of sources with number of extreme traces. These higher-rank graphs can have more extreme traces than sources, as shown in the graph below. This 2-graph has 5 sources but has 6 extreme traces because of the relation of traces on the sink (i.e., $t_1 + t_2 = t_3 + t_4 + t_5$). Additionally, this set of extreme traces does not form a simplex.



Because of the possible strange behavior of these higher-rank graphs, we will focus on higher-rank graphs that are product graphs, with tracial states that do form simplices.

If Λ is a locally convex, row-finite k-graph, then the C^*-algebra of Λ is defined (see [8]) to be the universal C^*-algebra generated by a family $\{s_\lambda : \lambda \in \Lambda\}$ of partial isometries, satisfying the following Cuntz-Krieger relations:

- $\{s_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections;
- $s_\lambda s_\mu = s_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;
- $s_\lambda^* s_\lambda = s_{s(\lambda)}$;
- $s_v = \sum_{\lambda \in v\Lambda^{\leq n}} s_\lambda s_\lambda^*$ for any $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

Thus if $\tau : C^*(\Lambda) \to \mathbb{C}$ is a tracial state (i.e. positive linear functional of norm 1 such that $\tau(xy) = \tau(yx)$ for all $x, y \in C^*(\Lambda)$), then we see that

$$\tau(s_v) = \sum_{\lambda \in v\Lambda^{\leq n}} \tau(s_{s(\lambda)}$$

for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. This is the motivation for the definition of a higher-rank graph trace in the locally convex case.

Local convexity of Λ is equivalent to the existence of a family $\{s_\lambda\}$ in which every s_λ is nonzero, as shown in [8].

If Λ is not locally convex, then it is still possibly to meaningfully assign to Λ a C^*-algebra generated by a family of nonzero partial isometries, but the Cuntz-Krieger relations have to be reformulated. Stating the new relations requires the introduction of some new graph-theoretic terminology, coming form [9].

Definition 5.2. Let Λ be a k-graph and let $\lambda, \mu \in \Lambda$. Then $\rho \in \Lambda$ is a *common extension* of λ and μ if there exist $\pi, \sigma \in \Lambda$ such that $\rho = \lambda\pi = \mu\sigma$. Define $d(\lambda) \vee d(\mu) \in \mathbb{N}^k$ by $(d(\lambda) \vee d(\mu))_i = \max\{d(\lambda)_i, d(\mu)_i\}$ for $i = 1, \ldots, k$. The set of *minimal common extensions* of λ and μ, denoted $MCE(\lambda, \mu)$ is the set of all common extensions ν of λ and μ such that $d(\nu) = d(\lambda) \vee d(\mu)$. By $\Lambda^{min}(\lambda, \mu)$ we denote the set of all ordered pairs $(\alpha, \beta) \in \Lambda \times \Lambda$ such that $\lambda\alpha = \mu\beta \in MCE(\lambda, \mu)$.

We say that Λ is *finitely aligned* if for any λ and μ in Λ, the set $MCE(\lambda, \mu)$ is finite (possibly empty). A set $E \subset v\Lambda$ is called *exhaustive* if for every $\lambda \in v\Lambda$, there is some $\tau \in E$ such that $MCE(\lambda, \tau) \neq \emptyset$.

REMARK 5.3. Every row-finite graph is finitely aligned,

With these definitions in place, we can define the C^*-algebra of a finitely aligned

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