Sierpinski Carpets as Julia sets for Imaginary 3-Circle Inversions

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The dynamics of the map

$$G_\rho(z) = \frac{-\rho^2 z^2}{z^3 - 1}$$

is investigated for $\rho \in \mathbb{R}$. This map is derived from multiple geometric circle inversions. Conditions are given for Cantor set and Sierpinski curve Julia sets for this family.

0. Introduction

The antiholomorphic function

$$z \mapsto \frac{r^2}{\overline{z - a}} + a$$

sends the point $z \in \mathbb{C}$ to its inversion image about the circle of radius $r \in \mathbb{R}^+$ centered at the point $a \in \mathbb{C}$. Dynamically, this map is not interesting since iterating twice yields the identity mapping. Using three circles we invert a point z about each circle and form a new map by sending z to the arithmetic mean of the three inversion images. We refer to this as inverting z about the three circles and refer to the three circles as the *generating circles*. We define *symmetric 3-circle inversion* as the particular three circle inversion where all generating circles have the same radius, r, and each is centered on a cube root of unity. It is easy to verify this yields the degree-three antiholomorphic rational map

$$F_r(z) = \frac{r^2 \overline{z}^2}{\overline{z}^3 - 1}$$

with $r \in \mathbb{R}^+$.

In [4] it was shown that there exists a single bifurcation value, r^*, for this family. If $r < r^*$ the Julia set for F_r is a Cantor set while $r > r^*$ yields Apollonian-like Julia sets. An Apollonian-like Julia set is a Julia set with key features in common with the Apollonian gasket (see Figure 1). Namely, the Fatou set consists of finitely many immediate basins of attraction and all of their pre-images with the property that each Fatou component boundary is a simple closed curve intersecting infinitely many other component boundaries. Further, for two Fatou components A and B, $\partial A \cap \partial B$ is either empty or consists of a single point.

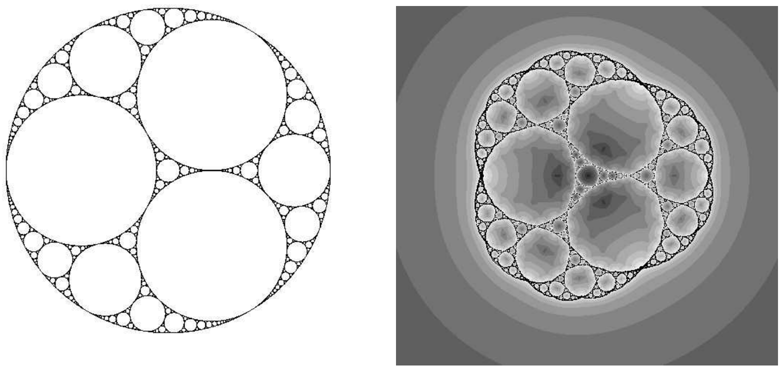

Figure 1. The Apollonian gasket and an Apollonian-like Julia set.

This work extends the notion of symmetric 3-circle inversion to include circles with purely imaginary radii, ρi with $\rho \in \mathbb{R}^+$. Specifically, we examine the map

$$G_\rho(z) = \frac{-\rho^2 z^2}{z^3 - 1}.$$

The dynamics of $G_\rho(z)$ is quite different from the dynamics of $F_r(z)$. For example, there are infinitely many ρ-values for which the Julia set of G_ρ is a Sierpinski curve, a Sierpinski curve being any planar curve homeomorphic to the Sierpinski carpet fractal (see Figure 2). In [6] it was shown that any planar set that is compact, connected, locally connected, nowhere dense and has the property that its complementary regions are bounded by disjoint simple closed curves is a Sierpinski curve. Sierpinski curves are interesting topologically as there exists a homeomorphic copy of any planar, one-dimensional continuum within the Sierpinski curve. Hence, the Sierpinski curve can be thought of as a universal planar curve.

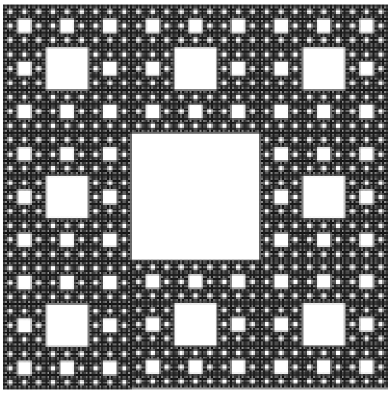

Figure 2. The Sierpinski carpet fractal.

Another distinction between the dynamics of F_r and G_ρ arises when we consider the dynamics of the maps restricted to their Julia sets. Let r_1 and r_2 in $\mathbb{R}^+$ be parameter values yielding Apollonian-like Julia sets for F_r. It was shown in [4] that the dynamics of F_{r_1} and F_{r_2}, restricted to their respective Julia sets, are topologically conjugate. However, this is not the case for G_ρ. For G_ρ there are infinitely many values of $\rho \in \mathbb{R}$ that yield Sierpinski curve Julia sets (hence, have homeomorphic Julia sets) whose maps are not topologically conjugate when restricted to their respective Julia sets.

The map G_ρ, when conjugated by $\phi(z) = 1/z$, is dynamically similar to the map $E_c(z) = z^2 + c/z$ for $c \in \mathbb{R}$. The map E_c is one of the maps studied in [2], where the map was presented as a singular perturbation of $z \mapsto z^2$. The work that follows shows that there is also a purely geometric interpretation, in terms of 3-circle inversion, for the map E_c.

The outline of this paper is as follows. In the next section we discuss some preliminaries from the field and establish basic properties of symmetric 3-circle inversion. In Section 2 we prove that the Julia sets of G_ρ are Cantor sets provided ρ is sufficiently small. We discuss the theory of polynomial-like mappings as formulated by Douady and Hubbard in Section 3. This theory is used in Section 4 to give conditions on the parameter values sufficient for Sierpinski curve Julia sets.

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1. Background, Preliminaries and Notation

Let $F : \mathbb{C}_\infty \mapsto \mathbb{C}_\infty$ be a holomorphic map on the extended complex plane, $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. We use the notation $F^n(z)$ to represent $F \circ F \circ \cdots \circ F(z)$ where the composition occurs n times. A point z lies on an n-cycle if $F^n(z) = z$. If n is the smallest positive integer for which this is true we say z has period n (if $n = 1$ we say z is a fixed point.) An n-cycle is attracting if $|(F^n)'(z)| < 1$ for any point z on the n-cycle. Conversely, if $|(F^n)'(z)| > 1$ we say the cycle is repelling.

Of particular interest are the Julia and Fatou sets of F. The Julia set for a holomorphic function F consists of the points for which the iterates of F do not form a normal family in the sense of Montel (see [5]). The Julia set is denoted $\mathcal{J}(F)$. The Fatou set is equal to $\mathbb{C}_\infty - \mathcal{J}(F)$. It is well known that, for rational maps with degree ≥ 2, the Julia set is equal to the closure of the set of repelling periodic points.

For $r \in \mathbb{R}^+$ we define inversion about the circle of radius r centered at the origin by the map $z \mapsto r^2/\overline{z}$. For the more general case where $r \in \mathbb{C}$ we use $z \mapsto \overline{r}^2/\overline{z}$. This map equals the standard definition of circle inversion when $r \in \mathbb{R}$. Further, the map agrees with geometric intuition by adding a clockwise rotation equal to twice the argument of r into the geometric interpretation of circle inversion when r is complex. Using this interpretation we define symmetric 3-circle inversion for any radius $r \in \mathbb{C}$,

$$z \mapsto \frac{\overline{r}^2\overline{z}^2}{\overline{z}^3 - 1}.$$

We are interested in the particular case $r = i\rho$ with $\rho \in \mathbb{R}$. Hence, we investigate

$$F_{\rho i}(z) = \frac{-\rho^2\overline{z}^2}{\overline{z}^3 - 1}.$$

For $\rho \in \mathbb{R}$ we have $F_{\rho i}(\overline{z}) = \overline{F_{\rho i}(z)}$. Therefore, the domains of normality for $F_{\rho i}$ and $\overline{F_{\rho i}}$ are the same implying that $\mathcal{J}(F_{\rho i}) = \mathcal{J}\left(\overline{F_{\rho i}}\right)$. Note that

$$\overline{F_{\rho i}(z)} = \frac{-\rho^2 z^2}{z^3 - 1}.$$

As $\overline{F_{\rho i}(z)}$ is holomorphic we work with this map. To simplify notation we let $G_\rho = \overline{F_{\rho i}}$. When we have need to speak of the symmetric 3-circle inversion with real radius r we use the notation F_r. For the remainder of this work both r and ρ are in $\mathbb{R}^+$. All work regarding $\rho \in \mathbb{R}^+$ carries trivially to $\rho \in \mathbb{R}^-$ as $G_\rho(z) = G_{-\rho}(z)$.

It should not be surprising that the map G_ρ displays a great deal of symmetry.

Theorem 1.1 (3-fold symmetry) For any ω such that $\omega^3 = 1$ we have $G_\rho(\omega z) = \omega^2 G_\rho(z)$.

Proof: Assuming that $\omega^3 = 1$ we obtain

$$G_\rho(\omega z) = \frac{-\rho^2(\omega z)^2}{(\omega z)^3 - 1} = \frac{-\rho^2\omega^2 z^2}{\omega^3 z^3 - 1} = \omega^2\left(\frac{-\rho^2 z^2}{z^3 - 1}\right) = \omega^2 G_\rho(z).$$

□

A consequence of this theorem is that symmetric points behave symmetrically. If we know the orbit of a point z then we know certain information about the orbits of its symmetric images, ωz and $\omega^2 z$. Namely, if z goes to 0 then so do its symmetric images. If z is attracted to a non-zero attracting cycle then so are its symmetric images (although the cycles will not necessarily be the same cycles or even of the same period).

The maps G_ρ have poles at the cube roots of unity (the centers of the generating circles). For any ρ the origin is a super-attracting fixed point. We denote the immediate basin of attraction for the origin by $\mathcal{O}$. A rational map with degree $d \geq 2$ has $2d - 2$ critical points (counting multiplicity). Since G_ρ is degree 3 we have 4 critical points, one of which is the origin. The remaining three critical points for the map G_ρ are given by $\sqrt[3]{-2}$. Since the critical point 0 is fixed we distinguish the symmetric critical points $\sqrt[3]{-2}$ by referring to them as the *free critical points*. In general the dynamics of maps with a large number of critical points can be difficult to study. However, by Theorem 1.1 all of the free critical points for G_ρ behave symmetrically. We denote the real value of $\sqrt[3]{-2}$ by c_0. Hence, the free critical points are given by $c_j = \omega^j c_0$ where $\omega \neq 1$ is a cube root of unity and $j = 1, 2$.

2. Cantor Sets

In this section we prove that if ρ is sufficiently small then $\mathcal{J}(G_\rho)$ is a Cantor set. This is accomplished by showing that the free critical points for G_ρ are in $\mathcal{O}$ for sufficiently small values of ρ. For each c_i we define the *critical segment* S_{c_i} to be the line segment connecting c_i to the origin. Specifically, S_{c_i} consists of all points $z = ae^{i\theta_{c_i}}$ where $0 \le a \le |c_i|$ and $\theta_{c_i} = Arg(c_i) + 2k\pi$ with $k \in \mathbb{Z}$. We show through two lemmas the existence of ρ_c such that $S_{c_i} \in \mathcal{O}$ for $i = 0, 1, 2$ if $\rho < \rho_c$. Once this is demonstrated we prove our main result for this section:

Theorem 2.1 Let $\rho_c = \sqrt{3}/2^{2/3}$. If $\rho < \rho_c$ then $\mathcal{J}(G_\rho)$ is a Cantor set.

We begin with a lemma:

Lemma 2.1 For all ρ there exists $z_\rho^* \in \mathbb{R}^+$ such that

(1) $G_\rho(z_\rho^*) = z_\rho^*$,
(2) $G_\rho(z) < z$ for all $0 < z < z_\rho^*$,
(3) z_ρ^* is strictly increasing as ρ decreases, and
(4) $z_\rho^* \mapsto 1$ as $\rho \mapsto 0$.

Proof: Note that $G_\rho(z) = z$ for $z \in \mathbb{R}^+$ iff

$$\frac{-\rho^2 z^2}{z^3 - 1} = z$$

iff $z^3 + \rho^2 z - 1 = 0$. Let $P_\rho(z) = z^3 + \rho^2 z - 1$. We note that $P_\rho(0) = -1$ and $P_\rho(1) = \rho^2$. Hence, the existence of z_ρ^* is guaranteed by the continuity of $P_\rho(z)$ and the intermediate value theorem. Since $G_\rho(0) = 0$ and $G_\rho(z)$ is concave up and strictly increasing on $(0, 1)$, z_ρ^* is the only fixed point in $(0, 1)$. This implies $G_\rho(z) < z$ for $0 < z < z_\rho^*$. As z_ρ^* satisfies $z^3 + \rho^2 z - 1 = 0$, implicit differentiation yields

$$\frac{dz}{d\rho} = \frac{-2\rho z}{3z^2 + \rho^2}.$$

Clearly $dz/d\rho$ is negative for all $z > 0$. Hence, z_ρ^* is strictly decreasing as ρ increases, implying that z_ρ^* is strictly increasing as ρ decreases. That z_ρ^* approaches 1 as ρ decreases to 0 is clear from examining $z^3 + \rho^2 z - 1 = 0$ as ρ decreases to 0.

□

It is possible to solve for z_ρ^*, obtaining

$$z_\rho^* = \frac{2(3)^{1/3}\rho^2 - 2^{1/3}\left(-9 + \sqrt{81 + 12\rho^6}\right)^{2/3}}{6^{2/3}\left(-9 + \sqrt{81 + 12\rho^6}\right)^{1/3}}.$$

A simple calculation reveals that z_ρ^* is a repelling fixed point for $G_\rho(z)$. One consequence of Lemma 2.1 is that the interval $\left[0, z_\rho^*\right) \subset \mathbb{R}^+$ is in $\mathcal{O}$. We are now in a position to show:

Lemma 2.2 If $\rho < \rho_c = \sqrt{3}/2^{2/3}$ then the critical points of G_ρ are in $\mathcal{O}$.

Proof: Due to symmetry, we can show that any one of our free critical points is in $\mathcal{O}$ and obtain the desired result. For this proof we use the critical point $c_0 \in \mathbb{R}^-$. This critical point is mapped into $\mathbb{R}^+$, the critical value being $G_\rho(c_0) = \left((2)^{2/3}/3\right)\rho^2 = v_\rho$. G_ρ maps S_{c_0} in one-to-one fashion onto the interval $[0, v_\rho]$. Note that v_ρ strictly decreases to 0 and z_ρ^* strictly increases to 1 as ρ decreases to 0. Hence, there exists a value of ρ for which $v_\rho = z_\rho^*$. This value is $\rho_c = \sqrt{3}/2^{2/3}$. If $\rho < \rho_c$ then $v_0 < z_\rho^*$ and we see that S_{c_0} is mapped into $[0, z_\rho^*) \subset \mathcal{O}$. As S_{c_0} is a curve connecting 0 to c_0 all of which is mapped to the immediate basin of 0 it must be that $S_{c_0} \subset \mathcal{O}$, and hence $c_0 \in \mathcal{O}$.

□

The main result now follows.

Proof of Theorem 2.1: If $\rho < \rho_c$ then $G_\rho(z)$ is a degree 3 rational map whose critical points are all in the immediate basin of attraction for the super-attracting fixed point at the origin. By Theorem 9.8.1 in [1], $\mathcal{J}(G_\rho)$ is a Cantor set.

3. Polynomial-like Maps

We begin this section with a brief discussion of polynomial-like maps as described by Douady and Hubbard (see [3]). Let U and V be open topological disks and h a holomorphic map such that $h : U \mapsto V$ is a proper map of degree 2. If the closure of U is contained in V then $h : U \mapsto V$ is a polynomial-like map of degree 2. In this case, h restricted to U is conjugate to a polynomial $z \mapsto z^2 + c$ restricted to a disk with radius $r > 1$ centered at the origin.

For any $\epsilon \in (0, 1)$, if

$$\rho > \rho_\epsilon = \sqrt{\frac{2 - \epsilon(\epsilon^2 - 3\epsilon + 3)}{1 - \epsilon}}$$

then G_ρ maps the circle of radius $1 - \epsilon$ strictly outside itself. We can use this result to show that G_ρ is a polynomial-like mapping from a pre-image of the disk of radius $1 - \epsilon$ to the disk of radius $1 - \epsilon$.

Lemma 3.1 The circle of radius α centered at the origin is mapped strictly outside of itself by G_ρ if $H_\rho(\alpha) = \alpha^3 - \rho^2\alpha + 1 < 0$.

Proof: Assume that $H_\rho(\alpha) < 0$. Then $\rho^2\alpha > \alpha^3 + 1$. Since $\alpha^3 + 1 \geq \left|\alpha^3 e^{3i\theta} - 1\right|$ we have $\rho^2\alpha > \alpha^3 + 1 \geq \left|\alpha^3 e^{3i\theta} - 1\right|$. This implies, since $\alpha > 0$, that $\rho^2\alpha^2 > \alpha\left|\alpha^3 e^{3i\theta} - 1\right|$ further implying that

$$\left|\frac{\rho^2\alpha^2 e^{2i\theta}}{\alpha^3 e^{3i\theta} - 1}\right| > \alpha.$$

Since

$$\left|\frac{\rho^2\alpha^2 e^{2i\theta}}{\alpha^3 e^{3i\theta} - 1}\right| = \left|G_\rho\left(\alpha e^{i\theta}\right)\right|$$

we have $\left|G_\rho\left(\alpha e^{i\theta}\right)\right| > \alpha$ concluding the proof of the lemma.

□

Since $H_\rho(1 - \epsilon) = (1 - \epsilon)^3 - \rho^2(1 - \epsilon) + 1 = \rho^2(\epsilon - 1) + 2 - \epsilon(\epsilon^2 - 3\epsilon + 3)$ we know $H_\rho(1 - \epsilon) < 0$ whenever

$$\rho > \sqrt{\frac{2 - \epsilon(\epsilon^2 - 3\epsilon + 3)}{1 - \epsilon}} = \rho_\epsilon.$$

Therefore, for $0 < \epsilon < 1$, the circle of radius $1 - \epsilon$ is mapped strictly outside of itself by G_ρ when $\rho > \rho_\epsilon$. Although these results hold for all $\epsilon \in (0, 1)$ we note that ρ_ϵ tends to ∞ as ϵ tends to 1.

Theorem 3.1 For each $\rho > \rho_\epsilon$ there exists an invariant simple closed curve, β_ρ, surrounding the origin on which G_ρ is conjugate to the map $z \mapsto z^2$. All orbits inside β_ρ tend to 0.

Proof: Let U be the set of all z such that $|z| < 1 - \epsilon$. Since the circle of radius $1 - \epsilon$ is mapped strictly outside itself we can define $U' = G_\rho^{-1}(U) \cap U$. The free critical points are given by $\sqrt[3]{-2}$ and hence are not in U or U'. Further, the poles of G_ρ are the cubes roots of unity and are also not in U or U'. Hence, $G_\rho(z)$ is a proper holomorphic map of degree 2 from U' onto U. As the boundary of U is mapped strictly outside of U it cannot be the case that any point in the closure of U' is also in the boundary of U. Hence, $\overline{U'} \subset U$, where $\overline{U'}$ is the closure of U'. Therefore, $G_\rho : U' \mapsto U$ is a polynomial-like map of degree 2. As a polynomial-like map on U' its filled Julia set is

$$\mathcal{K}_{G_\rho} = \{z \in U' | G_\rho^m(z) \in U' \text{ for all } m \in \mathbb{Z}^+\}.$$

On a neighborhood of $\mathcal{K}_{G_\rho}$, G_ρ is quasiconformally conjugate to a degree 2 polynomial P. Since 0 is a super-attracting fixed point for $G_\rho|\mathcal{K}_{G_\rho}$ we know that $P(z) = z^2$. The invariant curve β_ρ is the image of the Julia set of P, the unit circle, under the quasiconformal conjugacy conjugating z^2 to $G_\rho(z)$.

□

Corollary 3.2 The curve β_ρ is the boundary of $\mathcal{O}$, the immediate basin of attraction for 0.

Theorem 3.2 also holds for F_r when $r > \rho_\epsilon$. As discussed in [4] the Julia sets for F_r with $r > \rho_\epsilon$ are topological sets resembling Apollonian gaskets. For G_ρ the case is quite different: we can choose ρ such that $\mathcal{J}(G_\rho)$ is a Sierpinski curve Julia set. Before continuing, it is worth noting that Theorem 3.2 only applies to the boundary of $\mathcal{O}$, basins for other attracting cycles may or may not have simple closed curves for boundaries.

4. Sierpinski Curves Julia Sets

When $\rho > \rho_\epsilon$, $\partial\mathcal{O}$ is a simple closed curve and $\mathcal{O}$ is simply connected. Since the pre-image of 0 consists of 0 and ∞ there exists a set $\mathcal{B}$, containing ∞, such that $G_\rho(\mathcal{B}) = \mathcal{O}$ and $G_\rho^{-1}(\mathcal{O}) = \{\mathcal{O}, \mathcal{B}\}$.

Lemma 4.1 Assume $\rho > \rho_\epsilon$. If $w \in \partial\mathcal{O} \cap \partial\mathcal{B}$ then w is a critical point of $G_\rho(z)$.

Proof: Assume that w is in $\partial\mathcal{O}\cap\partial\mathcal{B}$. Since w is in $\partial\mathcal{O}$ there exists a ray, γ, in $\mathcal{O}$ connecting 0 to w. Then $G_\rho(\gamma)$ is in $\mathcal{O}$. Hence, in $\mathcal{B}$ there exists a pre-image of $G_\rho(\gamma)$, call it ζ, that connects ∞ to w. Hence we have two curves, ζ and γ, with ζ connecting w to ∞ and γ connected 0 to w. Both of these curves have the same image, namely a ray from 0 to $G_\rho(w)$, implying that w is a critical point.

□

Theorem 4.1 Assume $\rho > \rho_\epsilon$. If c_0 is in the basin of attraction for 0 but not in the immediate basin of attraction for 0 then the Julia set, $\mathcal{J}(G_\rho)$, is a Sierpinski curve.

Proof: We must show that $\mathcal{J}(G_\rho)$ meets the 5 necessary and sufficient conditions for Sierpinski curves. Standard arguments (see [5] for example) from complex dynamics show that the Julia set for a rational map with degree ≥ 2 (that is not the entire Riemann sphere) is closed and nowhere dense. Since ∞ is a pre-image of 0 there exists an open set surrounding ∞ which is mapped into $\mathcal{O}$. Hence, $\mathcal{J}(G_\rho)$ is bounded. Since $\mathcal{J}(G_\rho)$ is closed and bounded in $\mathbb{C}$ it is compact.

By the *3-fold symmetry* theorem all of the free critical points are in the basin of attraction for 0, with none being in the immediate basin. Since all critical points (free and non-free) are in the basin of attraction for 0 there cannot exist any other stable domain. In other words, the Fatou set consists only of $\mathcal{O}$ and all of its pre-images. Since we have assumed that all of our critical points are in the basin of attraction of 0 they are bounded away from the Julia set. $\mathcal{O}$ is simply connected with a simple closed curve boundary. Therefore, since the critical points are bounded away from $\mathcal{J}(G_\rho)$, the pre-images of the immediate basin are all simply connected with simple closed curve boundaries. Hence, the Julia set is the Riemann sphere with countably many simply connected open sets removed, implying that $\mathcal{J}(G_\rho)$ is connected. Since $G_\rho(z)$ is dynamically hyperbolic, $\mathcal{J}(G_\rho)$ being connected implies $\mathcal{J}(G_\rho)$ is locally connected (see Theorem 19.2 in [5].)

All that remains is showing the boundaries of complementary domains are pairwise disjoint simple closed curves. The boundaries of the complementary domains of $\mathcal{J}(G_\rho)$ are precisely the boundaries of the pre-images of the immediate basin of attraction for 0. We have already established that these boundaries are simple closed curves. To establish these boundaries are pairwise disjoint we assume this is not the case. Assume there exist Fatou components $S \subset G_\rho^{-d}(\mathcal{O})$ and $T \subset G_\rho^{-m}(\mathcal{O})$ such that $S \neq T$ and $\partial S \cap \partial T \neq \emptyset$. If $d \neq m$ we can map these sets forward obtaining $\partial\mathcal{O}\cap\partial\mathcal{B}$, which we have established only occurs if the intersection point is a critical point. This clearly cannot happen since, by assumption, all critical points are in the basin of attraction for 0. If $d = m$ there exists a smallest $k \leq m$ such that $G_\rho^k(S) = G_\rho^k(T)$. Hence, there exists a point in $\partial G_\rho^{k-1}(S) \cap \partial G_\rho^{k-1}(T)$. Since $G_\rho^{k-1}(S) \neq G_\rho^{k-1}(T)$ and $G_\rho^k(S) = G_\rho^k(T)$, this intersection point must be a critical point. Therefore, this case cannot occur either.

□

Examples can be generated by finding values of $\rho > \rho_\epsilon$ such that c_0 lands on 0 after $n \geq 3$ iterations. The construction of such ρ offers little in the way of insight and hence has been omitted.

It should be noted that the Apollonian-like Julia sets for F_r meet all but one of the topological classification conditions for the Sierpinski curve. Namely, while the complementary domains are bounded by simple closed curves, they fail to be disjoint. The Fatou component boundaries for G_ρ can not intersect because they are all pre-images of the boundary of $\mathcal{O}$, meaning any intersection point is a critical point. For F_r there is a super-attracting point at the origin along with three attracting fixed points. Two component boundaries can intersect here provided they are not pre-images of the same basin boundary.

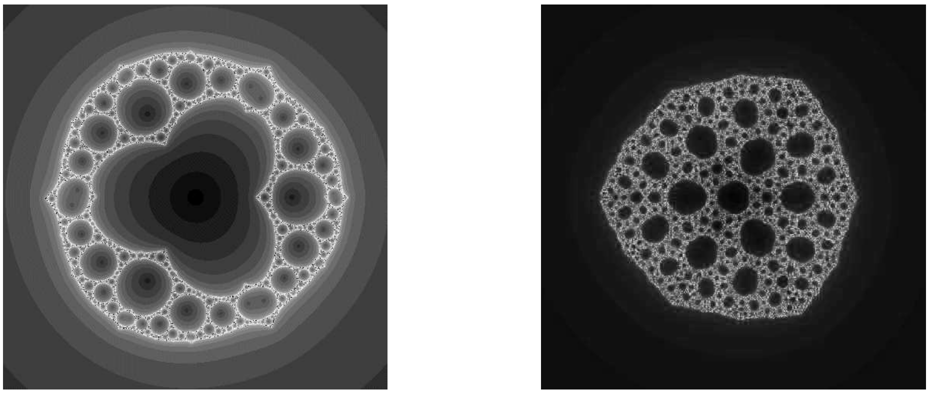

Figure 3. Sierpinski curve Julia sets for G_ρ.

Figure 3 shows Sierpinski curve Julia sets with c_0 landing in $\mathcal{O}$ in 5 and 11 iterations, respectively.

5. Conclusion

As discussed at the onset of this work, the map

$$F_r(z) = \frac{\overline{r}^2\overline{z}^2}{\overline{z}^3 - 1}$$

represents a form of 3-circle inversion with common circle radius $r \in \mathbb{C}$. For $r \in \mathbb{R}^+$ the Julia sets were limited to, essentially, one of two types with a single bifurcation value separating them. Further, if r_1 and r_2 are both on the same side of the bifurcation value then the maps F_{r_1} and F_{r_2} are dynamically conjugate on their Julia sets. As $F_r = F_{-r}$ for $r \in \mathbb{R}$ these results transfer trivially to $\mathbb{R}^-$. As we have seen, the structure for $r \in i\mathbb{R}^+$ is more interesting.

We note that, unlike the case for $r \in \mathbb{R}$, two parameter values both yielding Sierpinski curve Julia sets do not necessarily yield conjugate dynamics on their respective Julia sets. The distinction arises based on the number of iterations necessary for the critical point to land in $\mathcal{O}$. If G_{ρ_j} is conjugate to G_{ρ_k} on their Julia sets this conjugacy must take Fatou component boundaries to corresponding Fatou component boundaries and must take regions where G_{ρ_j} is two-to-one to regions where G_{ρ_k} is two-to-one. However, this causes a contradiction if $j \neq k$ because the j^{th} pre-images of $\mathcal{O}_{\rho_j}$ are mapped two-to-one onto their images under G_{ρ_j} but the same is not true for G_{ρ_k} (for G_{ρ_k} we need to take k pre-images). Although this work dealt with 3-circle inversion the theorems and proofs transfer trivially to m-circle inversion for any $m \geq 3$ that is odd.

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