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A CRITERION FOR SIERPINSKI CURVE JULIA SETS

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Abstract. This paper presents a criterion for a Julia set of a rational map of the form $F_\lambda(z) = z^2 + \lambda/z^2$ to be a Sierpinski curve.

1. Introduction

In this paper we discuss the one-parameter family of rational maps given by $F_\lambda(z) = z^2 + \lambda/z^2$ where $\lambda \neq 0$ is a complex parameter. Our goal is to give a criterion for the Julia set of such a map to be a Sierpinski curve. A Sierpinski curve is a rather interesting topological space that is homeomorphic to the well known Sierpinski carpet fractal. The interesting topology arises from the fact that a Sierpinski curve contains a homeomorphic copy of any one-dimensional plane continuum. Hence, any such set is a universal planar continuum.

When λ is small, this family of maps may be regarded as a singular perturbation of the map $z \mapsto z^2$. The Julia set of z^2 is well understood: it is the unit circle in $\overline{\mathbb{C}}$, and the restriction of the map to the Julia set is just the angle doubling map on the circle. For $\lambda \neq 0$, the Julia set changes dramatically. In [1], it is shown that, in every neighborhood of $\lambda = 0$ in the parameter plane, there are infinitely many disjoint open sets of parameters for which the Julia set is a Sierpinski curve. This result should be contrasted with the situation that occurs for the related family $G_\lambda(z) = z^n + \lambda/z^m$ with $1/n + 1/m < 1$. Curt McMullen [8] has shown that, provided

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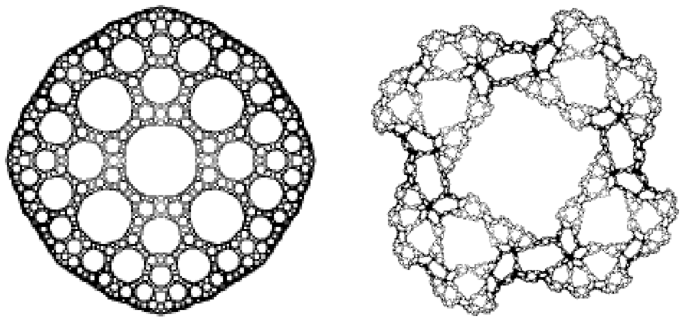
λ is sufficiently small, the Julia set of G_λ is always a Cantor set of circles. A dynamical criterion for this is given in [4]. On the other hand, Jane Hawkins [7] has shown that very different phenomena arise in the family $H_\lambda(z) = z + \lambda/z$.

Our goal in this paper is to investigate the dynamics of the family F_λ for all λ-values, not just those close to the origin. Our main result is a criterion for the Julia set of F_λ to be a Sierpinski curve.

Theorem 1.1. *Suppose that the critical orbit of F_λ tends to ∞ but the critical points of F_λ do not lie in the immediate basin of ∞. Then the Julia set of F_λ is a Sierpinski curve. In particular, any two Julia sets corresponding to an eventually escaping critical orbit are homeomorphic.*

In Figure 1, we display two Sierpinski curve Julia sets drawn from the family F_λ.



$\lambda = -1/16$ $\lambda = 0.132 + 0.097i$

FIGURE 1. The Sierpinski curve Julia sets for two values of λ.

We say *critical orbit* in this theorem because, despite the fact that this family consists of rational maps of degree 4, all of the free critical points for F_λ eventually land on the same orbit. This is reminiscent of the situation for the family of quadratic polynomials $Q_c(z) = z^2 + c$, where the orbit of the sole critical point 0 plays a significant role in determining the dynamics. As is well known, the Julia set of a quadratic polynomial is either a connected set or a Cantor set, and it is the behavior of the critical orbit that determines which case we are in. For if $Q_c^n(0) \to \infty$, then the

Julia set of this map is a Cantor set, whereas if the orbit of 0 is bounded, the Julia set is a connected set. This determines whether c lies outside or inside the Mandelbrot set [6]. For the family F_λ, we shall prove that there is a similar fundamental dichotomy, but there is a subtle but extremely important difference.

Theorem 1.2. *If the entire critical orbit of F_λ lies in the immediate basin of attraction of ∞, then the Julia set of F_λ is a Cantor set. On the other hand, if the entire critical orbit does not lie in the immediate basin, then the Julia set is connected.*

The subtle difference here lies in our assumption that the entire critical orbit lies in the *immediate basin* of ∞. For quadratic polynomials, if the critical orbit escapes to ∞, then its entire orbit must lie in the immediate basin of ∞. However, for F_λ, it is possible that the critical orbit escapes to ∞ but that the entire orbit does not lie in the immediate basin. That is, the critical points may lie in one of the (disjoint) preimages of the immediate basin, or, said another way, the critical orbit may jump around before entering the immediate basin of B. This is the case in which we find Sierpinski curve Julia sets.

There are other significant differences between the Julia sets of the family of rational maps and those of the quadratic polynomials. For example, in the case of connected quadratic Julia sets, it is often the case that the boundary of the basin at ∞ has infinitely many pinch points. That is, the complement of the closure of the immediate basin of ∞ consists of infinitely many disjoint open sets. For example, if Q_c admits an attracting periodic point of period $n \geq 2$, then the complement of the closure of the immediate basin of ∞ always consists of infinitely many disjoint components made up of the various basins of attraction and their preimages. These are the Fatou components for the map.

For F_λ, a very different situation occurs. Let B denote the immediate basin of ∞ for F_λ. Then we shall prove the following theorem.

Theorem 1.3. *Suppose $J(F_\lambda)$ is connected. Then $\overline{\mathbb{C}} - \overline{B}$ is an open, connected, simply connected set.*

For "nice" simply connected open sets, the boundary of such sets is a simple closed curve, but as is well known, this need not be the case. For example, the topologists' sine curve and other, non-locally connected sets may bound a simply connected open set in the plane. In our case, however, we often have simple closed curves bounding the basin of ∞. We shall also show:

Theorem 1.4. *The boundary of the immediate basin of ∞ is a simple closed curve in each of the following cases:*

(1) $|\lambda| < 1/16$*;*
(2) *the critical orbits lie on the boundary of the basin of ∞ but are preperiodic (the Misiurewicz case);*
(3) *the critical points do not accumulate on the boundary of the basin of ∞, as in the special case where they eventually tend to ∞ and we have a Sierpinski curve Julia set.*

2. Preliminaries

In this section, we describe some of the basic properties of the family $F_\lambda(z) = z^2 + \lambda/z^2$ where, as always, we assume that $\lambda \neq 0$. Observe that $F_\lambda(-z) = F_\lambda(z)$ and $F_\lambda(iz) = -F_\lambda(z)$ so that $F_\lambda^2(iz) = F_\lambda^2(z)$ for all $z \in \overline{\mathbb{C}}$. Also note that 0 is the only pole for each function in this family. The points $(-\lambda)^{1/4}$ are prepoles for F_λ since they are mapped directly to 0. The four critical points for F_λ occur at $\lambda^{1/4}$. Note that $F_\lambda(\lambda^{1/4}) = \pm 2\lambda^{1/2}$ and $F_\lambda^2(\lambda^{1/4}) = 1/4 + 4\lambda$, so each of the four critical points lies on the same forward orbit after two iterations. We call the union of these orbits the *critical orbit* of F_λ.

Let $J = J(F_\lambda)$ denote the *Julia set* of F_λ. J is the set of points at which the family of iterates of F_λ fails to be a normal family in the sense of Montel. Equivalently, $J(F_\lambda)$ is the closure of the set of repelling periodic points of F_λ (see [9] for the basic properties of Julia sets).

The point at ∞ is a superattracting fixed point for F_λ. Let B be the immediate basin of attraction of ∞ and denote by ∂B the boundary of B. The map F_λ has degree 2 at ∞ and so F_λ is conjugate to $z \mapsto z^2$ on B, at least in a neighborhood of ∞. The basin B is a (forward) invariant set for F_λ in the sense that, if $z \in B$, then $F_\lambda^n(z) \in B$ for all $n \geq 0$. The same is true for ∂B.

We denote by $K = K(F_\lambda)$ the set of points whose orbit under F_λ is bounded. K is the *filled Julia set* of F_λ. K is given by $\overline{\mathbb{C}} - \cup F^{-n}(B)$. Both J and K are completely invariant subsets in the sense that if $z \in J$ (resp. K), then $F_\lambda^n(z) \in J$ (resp. K) for all $n \in \mathbb{Z}$. It is known that $J(F_\lambda)$ is the boundary of $K(F_\lambda)$ (see [9]).

Proposition 2.1 (Fourfold Symmetry)**.** *The sets B, ∂B, J, and K are all invariant under $z \mapsto iz$.*

Proof: We prove this for B; the other cases are similar. Let $U = \{z \in B \,|\, iz \in B\}$. U is an open subset of B. If $U \neq B$, there exists $z_0 \in \partial U \cap B$, where ∂U denotes the boundary of U. Hence, $z_0 \in B$, but $iz_0 \in \partial B$. It follows that $F_\lambda^n(iz_0) \in \partial B$ for all n. But since $F_\lambda^2(z_0) = F^2(iz_0)$, it follows that $z_0 \notin B$ as well. This contradiction establishes the result. □

There is a second symmetry present in this family. Consider the map $H(z) = \sqrt{\lambda}/z$. Note that we have two such maps depending upon which square root of λ we choose. H is an involution and we have $F_\lambda(H(z)) = F_\lambda(z)$. As a consequence, H preserves both J and K. The involution H also preserves the circle of radius $\lambda^{1/4}$ and interchanges the interior and exterior of this circle. Hence, both J and K are symmetric about this circle with respect to the action of H.

3. The Fundamental Dichotomy

We briefly recall a well known result for the family of quadratic polynomials $Q_c(z) = z^2 + c$. Each map Q_c has a single critical point at 0 and so, like F_λ, Q_c has a single critical orbit. The fate of this orbit leads to the well known fundamental dichotomy for quadratic polynomials:

(1) If $Q_c^n(0) \to \infty$, then $J(Q_c)$ is a Cantor set;
(2) but if $Q_c^n(0) \not\to \infty$, then $J(Q_c)$ is a connected set.

The set of parameter values c for which the Julia set of Q_c is connected forms the well known Mandelbrot set. Our goal in this section is to prove a similar result in the case of F_λ.

Before stating this result, note that, unlike the quadratic case, there are two distinct ways that the critical orbit of F_λ may tend to ∞. One possibility is that one (and hence all) of the critical points lies in the immediate basin B. The other possibility is that these critical points lie in one of the preimages of B that is disjoint from B. For quadratic polynomials this second possibility does not occur.

Our goal in this section is to prove the fundamental dichotomy for the family F_λ.

Theorem 3.1. *If one and hence all of the critical points of F_λ lie in B, then $J(F_\lambda)$ is a Cantor set; if the critical points of F_λ do not lie in B, then both $J(F_\lambda)$ and $K(F_\lambda)$ are compact, connected sets.*

Proof: Suppose first that no critical point lies in B. Then we may extend the conjugacy between F_λ and z^2 to all of B, and so B is a simply connected open set in $\overline{\mathbb{C}}$. Let $U_0 = \overline{\mathbb{C}} - B$. U_0 is compact and connected with boundary ∂B. Let $U_1 = U_0 - F_\lambda^{-1}(B)$. $F_\lambda^{-1}(B) - B$ is a simply connected open set containing 0 which is mapped two-to-one onto B. Hence, $F_\lambda^{-1}(B) - B$ lies in U_0 and is disjoint from ∂B since orbits in ∂B remain bounded. Therefore, U_1 is compact and connected. Inductively, U_k is given by $U_{k-1} - F_\lambda^{-k}(B)$. Since $F_\lambda^{-k}(B)$ is a collection of disjoint, simply connected, open sets which do not intersect the boundary of U_{k-1}, it follows that U_k is also compact and connected. Then $K(F_\lambda) = \cap U_k$ is compact and connected. Since J is the boundary of K, J too is compact and connected.

The proof that $J(F_\lambda)$ is a Cantor set when all critical points lie in B is standard. See, for example, [9]. □

We emphasize again that the critical points for F_λ may eventually escape but not lie in B. In this case, we still have a connected Julia set. In fact, we shall show in section 5 that $J(F_\lambda)$ is a Sierpinski curve in this case.

We denote the set of parameter values for which $J(F_\lambda)$ is connected by $\mathcal{M}$; $\mathcal{M}$ is called the *connectedness locus* for this family. This set is the analogue of the Mandelbrot set for quadratic polynomials.

In the case where no critical points lie in B, we denote $F_\lambda^{-1}(B) - B$ by T. Since $F_\lambda \,|\, B$ is only two to one, T is nonempty. Since 0 is a pole of order 2, it follows that T is an open set about 0 on which F_λ is two to one. We call T the *trap door*, since any orbit that enters T immediately "falls through" it and enters the basin at ∞. Just as in the fourfold symmetry proposition, T is invariant under $z \mapsto iz$. Also, the involution H interchanges B and T.

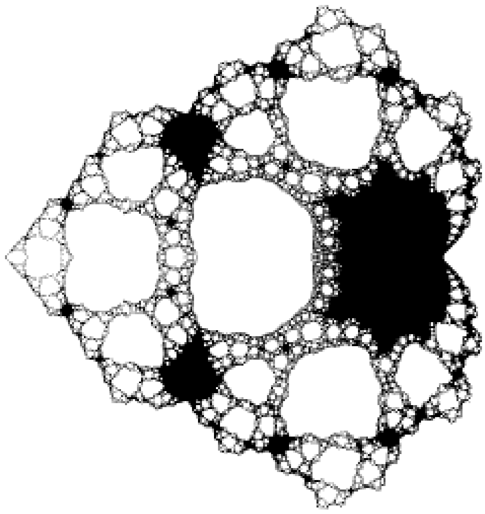


FIGURE 2. The parameter plane for the family $z^2 + \lambda/z^2$. White regions correspond to λ-values for which the critical orbit escapes to ∞.

4. THE CASE $|\lambda| < 1/16$

Here, we deal with the very special case where $|\lambda| < 1/16$. We first prove the following theorem.

Theorem 4.1. *Suppose that $|\lambda| < 1/16$. Then the boundary of B is a simple closed curve.*

Proof: Consider the *critical circle* S_λ given by $r = |\lambda|^{1/4}$. Note that S_λ contains all four critical points as well as the four prepoles. Write $\lambda = \rho \exp(i\psi)$ and $z = \rho^{1/4} \exp(i\theta) \in S_\lambda$. Then we compute

$$\begin{aligned} F_\lambda(z) &= \rho^{1/2}(e^{2i\theta} + e^{i(\psi - 2\theta)}) \\ &= \rho^{1/2}\left((\cos(2\theta) + \cos(\psi - 2\theta)) + i(\sin(2\theta) + \sin(\psi - 2\theta))\right). \end{aligned}$$

If we set $x = \cos(2\theta) + \cos(\psi - 2\theta))$ and $y = \sin(2\theta) + \sin(\psi - 2\theta)$, then a computation shows that

$$\frac{d}{d\theta}\left(\frac{y}{x}\right) = 0.$$

Hence, the image of S_λ under F_λ is a line interval passing through the origin. F_λ maps S_λ onto this line in four-to-one fashion, except at the two endpoints, which are the critical values $\pm 2\sqrt{\lambda}$. Note that these two critical values lie inside S_λ provided we have

$$2|\sqrt{\lambda}| < |\lambda|^{1/4},$$

which occurs when $|\lambda| < 1/16$. Hence, the condition $|\lambda| < 1/16$ guarantees that the image of S_λ lies strictly inside S_λ.

Now if V_λ is another circle surrounding the origin whose radius is slightly larger than $|\lambda|^{1/4}$, then the image of V_λ also lies inside S_λ and hence inside V_λ. Moreover, the image of V_λ is a simple closed curve since there are no critical points or prepoles on V_λ. The involution H maps V_λ to a second circle W_λ that lies strictly inside the critical circle and we have $F_\lambda(V_\lambda) = F_\lambda(W_\lambda)$. The annular region between V_λ and W_λ is mapped in four-to-one fashion onto the disk surrounding the origin and bounded by $F_\lambda(V_\lambda)$. In particular, the image of this annulus is disjoint from the annulus provided that V_λ is sufficiently close to the critical circle.

We claim that the preimage of V_λ consists of a pair of disjoint simple closed curves, one lying inside the critical circle and one lying outside V_λ. This follows from the fact that F_λ maps the exterior of V_λ in two-to-one fashion onto the exterior of the curve $F_\lambda(V_\lambda)$. The interior of the circle W_λ is mapped in similar fashion onto the exterior of $F_\lambda(V_\lambda)$. Let U_λ denote the preimage of V_λ lying outside V_λ, and let A_λ denote the annular region bounded by V_λ and U_λ. Note that A_λ is mapped in two-to-one fashion onto the annulus bounded by V_λ and $F_\lambda(V_\lambda)$.

We now use quasiconformal surgery to modify F_λ to a new map G_λ which agrees with F_λ on the exterior of A_λ but which is conjugate to $z \mapsto z^2$ in the interior of U_λ with a fixed point at the origin. To obtain G_λ, we first replace F_λ in the disk bounded by V_λ by a map which is a quasiconformal deformation of $z \mapsto z^2$ on $|z| < 1/2$. Then we extend G_λ to A_λ so that the new map is quasiconformally conjugate to z^2 on and inside A_λ and agrees with F_λ on the outer boundary U_λ of A_λ. The new map G_λ is continuous and has degree

2 with two superattracting fixed points, one at 0 and one at ∞. Hence G_λ is everywhere conjugate to z^2. Therefore the boundary of the basin of attraction of ∞ for G_λ is a simple closed curve. Since G_λ agrees with F_λ in the exterior of A_λ, the same is true for F_λ. This proves that ∂B is a simple closed curve when $|\lambda| < 1/16$. □

We now use this result to prove the following theorem.

Theorem 4.2. *Suppose that $|\lambda| < 1/16$ and that the critical points of F_λ tend to ∞ but do not lie in the the immediate basin B of ∞. Then $J(F_\lambda) = K(F_\lambda)$ is a Sierpinski curve.*

Proof: It is known [12] that any planar set that is compact, connected, locally connected, and nowhere dense, and that has the property that any two complementary domains are bounded by simple closed curves that are disjoint is homeomorphic to the Sierpinski carpet and is therefore a Sierpinski curve. In our case, the fact that both J and K are compact and connected was shown in the previous section. Since all of the critical orbits tend to ∞, it follows that $J = K$ and hence, using standard properties of the Julia set, J is nowhere dense. Also, since no critical points accumulate on J, it is known [9] that J is locally connected.

It therefore suffices to show that the complementary domains are all bounded by disjoint simple closed curves. By the previous result, ∂B is bounded by a simple closed curve lying strictly outside the critical circle. Using the involution H, the boundary of the trap door is given by $H(\partial B)$, and so this region is bounded by a simple closed curve lying inside the critical circle and therefore disjoint from ∂B.

Now consider the preimage of the trap door. This preimage is an open set. It cannot consist of a single component, for if this were the case, this component would necessarily surround the origin (by fourfold symmetry) and thereby disconnect the Julia set. Hence, each of the components of the preimage of T is an open set that is mapped in either one-to-one or two-to-one fashion onto T depending upon whether or not a critical point lies in the preimage. (In fact, the critical points cannot lie in the first preimage of T, but we do not need this fact here.)

It follows that each component of the preimage of T is a simply connected open set whose boundary is a simple closed curve that is mapped onto ∂T. The boundaries of these components are disjoint from ∂B, since this curve is invariant under F_λ and hence cannot be mapped to ∂T. They are also disjoint from ∂T since the boundary of the trap door is mapped to ∂B, whereas the boundary of the components are mapped to ∂T, and we know that $\partial T \cap \partial B = \emptyset$. Finally, the boundary of each component is disjoint from any other such boundary for a point in the intersection would necessarily be a critical point. If this were the case, then the critical orbit would eventually map to ∂B, contradicting our assumption that the critical orbit tends to ∞. Hence, the first preimages of T are all bounded by simple closed curves that are disjoint from each other as well as the boundaries of B and T. Continuing in this fashion, we see that the preimages $F_\lambda^{-n}(T)$ are similarly bounded by simple closed curves that are disjoint from all earlier preimages of ∂B. This gives the result. □

5. The Boundary of $\mathbf{B}$

In this section, we consider any λ-value for which the Julia set of F_λ is connected, not just those that satisfy $|\lambda| < 1/16$. Our aim is to show that the open set $\overline{\mathbb{C}} - \overline{B}$ is a connected and simply connected set. This implies that the interior of the set containing the origin and bounded by the boundary of B has just one connected component. Moreover, we show below that if z lies in the intersection of the boundaries of both B and T, then z must be a critical point of F_λ. Hence, there are at most four points in the intersection of these two boundaries.

Proposition 5.1. *The open set $\overline{\mathbb{C}} - \overline{B}$ is connected and simply connected whenever $B \cap T$ is empty.*

Proof: Let W_0 denote the open connected component of $\overline{\mathbb{C}} - \overline{B}$ that contains 0. Note that W_0 contains all of T since the boundary of B does not meet T. Hence, the closure of W_0 also contains ∂T.

Lemma 5.2. *W_0 is symmetric under $z \mapsto iz$ and hence has fourfold symmetry.*

Proof: Let X denote the set of points z in W_0 for which iz also lies in W_0. Note that X is an open subset of W_0. Note also that $X \supset T$ since T possesses fourfold symmetry and lies in W_0. Hence, X is nonempty. Now suppose that $X \neq W_0$. Then there must be a point $z_1 \in \partial X \cap W_0$. So $z_1 \in W_0$, but $iz_1 \notin W_0$. Therefore,

iz_1 lies in ∂W_0, which is contained in ∂B. Since $iz_1 \in \partial B$ and it was earlier shown that ∂B has fourfold symmetry, we know that $z_1 \in \partial B$, contradicting our assumption that $z_1 \in W_0$. This proves the lemma. □

Lemma 5.3. *All four preimages of any point in W_0 lie in W_0.*

Proof: Since $H(B) = T$ and $T \subset W_0$, we have $H(\partial B) \subset \overline{W}_0$. Therefore, $H(\partial W_0) \subset \overline{W}_0$ and so H maps $\overline{\mathbb{C}} - \overline{W}_0$ into W_0.

Now H maps prepoles to prepoles. If one of the prepoles lies in $\overline{\mathbb{C}} - \overline{W}_0$, then its image under H lies in W_0. This cannot occur since, by the previous lemma, W_0 has fourfold symmetry. Hence, each prepole lies in $\overline{W}_0$. In fact, each prepole must lie in W_0 since ∂W_0 is mapped to ∂B.

It follows that all four preimages of 0 lie in W_0. Therefore, the entire set $F_\lambda^{-1}(W_0)$ is contained in W_0 for, otherwise, there would be points in $\partial W_0 \subset \partial B$ that are mapped into W_0. This cannot happen since ∂B is invariant. □

Remark 5.4. By the lemma, $F_\lambda : F_\lambda^{-1}(W_0) \to W_0$ is a proper map of degree four. By the Riemann-Hurwitz Theorem, either $F_\lambda^{-1}(W_0)$ is an annulus containing all four critical points, or else $F_\lambda^{-1}(W_0)$ is a union of four disjoint disks, each of which is mapped homeomorphically onto W_0. This latter case occurs when the critical points all lie in ∂B, as we discuss below.

We now complete the proof of Proposition 5.1 that $\overline{\mathbb{C}} - \overline{B}$ is connected and simply connected. It suffices to show that W_0 is the only component of $\overline{\mathbb{C}} - \overline{B}$.

Assume that there is an additional component of $\overline{\mathbb{C}} - \overline{B}$ that is disjoint from W_0. Call this component W_1. Note that $-W_1$ is also a component of $\overline{\mathbb{C}} - \overline{B}$ and that $\pm W_1$ **are**?? disjoint, since otherwise this component would surround W_0. We have that $F_\lambda(W_1)$ does not meet W_0 since all preimages of points in W_0 lie in W_0. Also, as above, we have that $H(\pm W_1)$ lies in W_0.

We claim that there are no critical points in W_1. For, if $c_\lambda \in W_1$, then we must have $-c_\lambda \in -W_1$ and so F_λ maps both $\pm W_1$ onto an open set Q in two-to-one fashion. Now Q lies in $\overline{\mathbb{C}} - \overline{B}$ and hence, Q must be some connected component of this set, say $Q = W_k$. Then we have $k \neq 0$ and all four preimages of any point in W_k lie in $\pm W_1$. But, since $F_\lambda(H(z)) = F_\lambda(z)$, there must also be preimages of these points in $H(\pm W_1) \subset W_0$, as we saw above. Thus, we have more than four preimages for these points, so this cannot happen. We conclude that there can be no critical points in W_1.

Thus, we have that any additional component of $\overline{\mathbb{C}} - \overline{B}$ cannot contain either a critical point or a prepole of F_λ. Now we know that the set of components $\cup W_j$ excluding W_0 is mapped onto itself by F_λ. But then either one of these domains must be periodic under F_λ or else we have no periodic domains in $\cup W_j$. The former is impossible, since such a periodic domain would necessarily have a critical point belonging to it, while the latter is impossible by the Sullivan No Wandering Domains Theorem. See [9].

We conclude that there are no other W_j to start with in $\overline{\mathbb{C}} - \overline{B}$, and so $\overline{\mathbb{C}} - \overline{B} = W_0$, an open, connected, simply connected set as claimed. □

As a remark, the fact that there is only one component to the complement of $\overline{B}$ does not preclude the existence of quadratic-like filled Julia sets with infinitely many pinch points. These often reside as subsets of W_0 as depicted in Figure 3.

Corollary 5.5. *Suppose $z_0 \in \partial B \cap \partial T$. Then z_0 is a critical point of F_λ.*

Proof: Suppose that z_0 is not a critical point of F_λ. Then $F_\lambda(z_0) = w_0$ has four distinct preimages: $\pm z_0$ and $\pm z_1$ with $z_0 \neq \pm z_1$. Let $\pm U_i$ be open neighborhoods of $\pm z_i$ and suppose that the $\pm U_i$ are disjoint and that $F_\lambda(\pm U_i) = W$ where W is an open neighborhood of w_0.

Since $z_0 \in \partial B$, we may find a pair of external rays α_0 and β_0 that land at distinct points in $U_0 \cap \partial B$. Let $\gamma(\alpha_0, \beta_0)$ denote the union of the external rays contained between α_0 and β_0 (where we assume that the angle between these two rays is smaller than $\pi/2$). We may choose α_0 and β_0 so that the closure of $\gamma(\alpha_0, \beta_0)$ contains z_0, i.e., that these external rays land on either "side" of z_0. The set $-\gamma(\alpha_0, \beta_0)$ lies in B and has similar properties near $-z_0$.

Now z_0 lies in $\partial T \cap \partial B$. Hence, $\pm z_1$ also lies in $\partial T \cap \partial B$ since $H(\pm z_0) = \pm z_1$ and H maps ∂T to ∂B and ∂B to ∂T. Thus, we may find two other external rays α_1 and β_1 in B such that α_1 and β_1 terminate in U_1 on either side of z_1 and, moreover, have the property that $\gamma(\alpha_1, \beta_1)$ is disjoint from $\pm\gamma(\alpha_0, \beta_0)$. As above, $-\gamma(\alpha_1, \beta_1)$ lies in B and terminates in $-U_1$.

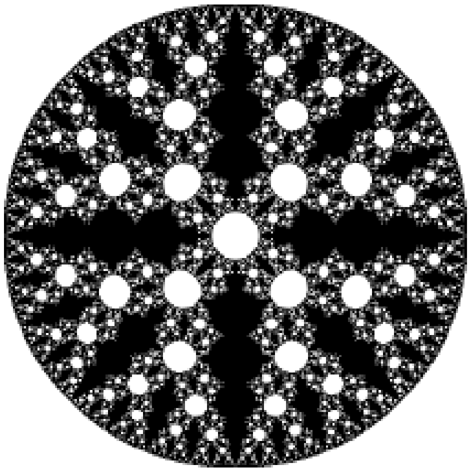


FIGURE 3. The Julia set of F_λ when $\lambda = 0.01$. For this λ-value, F_λ admits an attracting cycle of period 2. Note the black regions lying inside W_0 that resemble the Julia set of $z^2 - 1$; these are the basins of attraction of the two-cycle for F_λ.

Since $\pm\gamma(\alpha_0, \beta_0)$ and $\pm\gamma(\alpha_1, \beta_1)$ are disjoint, it follows that the images of these sets are distinct. (Since F_λ is even, we know that $F_\lambda(\gamma(\alpha_i, \beta_i)) = F_\lambda(-\gamma(\alpha_i, \beta_i))$ and hence if the images of these four sets were not distinct, then there would exist points in B with four preimages in B and this cannot happen since F_λ is $2-1$ on B.) Also, these images accumulate near w_0. Thus, we have two disjoint intervals of rays in B that accumulate on ∂B, and both contain w_0 in their closure. It follows that there must be a subset R of ∂B that is disjoint from the boundary of W_0 and R is separated from W_0 by these sets of rays. (See Figure 4.)

The set R cannot bound an open component of $\overline{\mathbb{C}} - \overline{B}$, as we saw above. Hence R, which is separate from W_0, must have empty interior. But there must be preimages of this region on the boundary of W_0, and so there must be points in W_0 that map arbitrarily close to these points. This is impossible.

We conclude that w_0 could not have had four preimages and so z_0 must have been a critical point. □

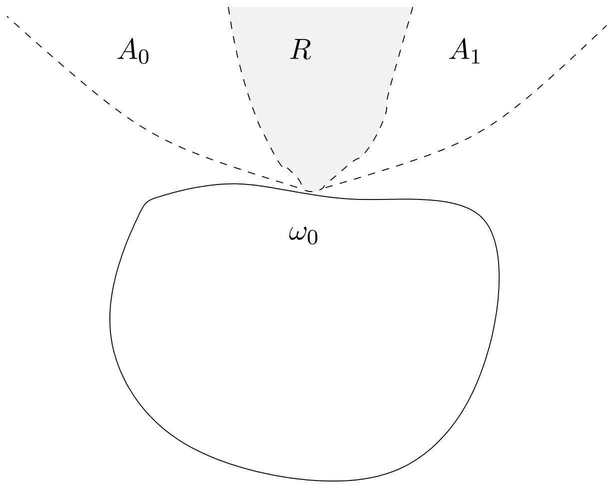


FIGURE 4. The region R where $F_\lambda(\pm\gamma(\alpha_i, \beta_i)) = A_i$

6. Proof of the Sierpinski Curve Criterion

In this section, we investigate the general case where the critical orbit escapes through the trap door into B. Here, we complete the proof that, when this occurs, the Julia set is a Sierpinski curve.

In this case, F_λ is a rational map of degree $d \geq 2$ whose postcritical closure is disjoint from its Julia set. Hence, F_λ is dynamically hyperbolic. Since B is a simply connected Fatou component for F_λ, we have that ∂B is locally connected and that the Julia set of F_λ is connected and locally connected [9]. Since B is simply connected, we know by the Riemann mapping theorem that there is a conformal isomorphism $\psi : \mathbb{D} \to B$ where $\mathbb{D}$ is the open unit disk. The following result is well known. See [9].

Theorem 6.1 (Caratheodory)**.** *A conformal isomorphism $\psi : \mathbb{D} \to U \subset \overline{\mathbb{C}}$ extends to a continuous map from the closed disk $\overline{\mathbb{D}}$ onto $\overline{U}$ if and only if the boundary ∂U is locally connected.*

This tells us that the Riemann map $\psi : \mathbb{D} \to B$ extends to a continuous map $\hat{\psi} : \overline{\mathbb{D}} \to \overline{B}$. In particular, we have a continuous map from $\partial\mathbb{D}$ to ∂B. Therefore, we know that all external rays R_t (with $t \in \mathbb{R}/\mathbb{Z}$) in B land on a single point in ∂B.

This allows us to prove the following theorem.

Theorem 6.2. *∂B is a simple closed curve.*

Combining this result with the techniques described in section 3 allows us to conclude that $J(F_\lambda)$ is a Sierpinski curve when the critical orbit escapes through the trap door.

Proof: ∂B is a simple closed curve if and only if exactly one external ray lands at each point in ∂B. Assume this is not the case. Suppose that there exists $p \in \partial B$ such that two external rays R_{t_1} and R_{t_2} land on p. Since these rays, together with the point p, form a Jordan curve and W_0 is connected and simply connected, we have that W_0 lies entirely within one of the two open components created by this Jordan curve. Without loss of generality, assume that W_0 is such that $W_0 \cap \gamma(t_1, t_2) = \emptyset$ (so W_0 is "outside" the sector $\gamma(t_1, t_2)$ between R_{t_1} and R_{t_2}).

We claim that there exist positive integers q and n such that the region

$$\gamma\left(\frac{q}{n}, \frac{q+1}{n}\right) \subset \gamma(t_1, t_2)$$

and neither $R_{q/n}$ nor $R_{(q+1)/n}$ lands on ∂W_0. If this is not possible, then all rays with angle s such that $R_s \subset \gamma(t_1, t_2)$ land at p, This gives a contradiction because the set of $s \in \mathbb{R}/\mathbb{Z}$ such that $\gamma(s) = p$ has measure zero [9]. Therefore, if we have two rays landing at p, then we can find q and n such that

$$\gamma\left(\frac{q}{n}, \frac{q+1}{n}\right) \subset \gamma(t_1, t_2)$$

and neither $R_{q/n}$ nor $R_{(q+1)/n}$ lands on ∂W_0. (See Figure 5.)

Assume that we have such q and n. As above, let $\gamma(q/n, (q+1)/n)$ denote the union of the external rays contained between q/n and $(q+1)/n$. After n iterations $\gamma(q/n, (q+1)/n)$ is mapped over all of ∂B. In particular, if R_θ is an external ray landing on ∂W_0, we know that there is a ray $R_\phi \in \gamma(q/n, (q+1)/n)$ such that $F_\lambda^n(R_\phi) = R_\theta$. Since $R_\phi \in \gamma(q/n, (q+1)/n)$, we know that the landing point $\gamma(\phi)$ is not on ∂W_0. Hence, there exists a neighborhood N_ϕ of $\gamma(\phi)$ such that $N_\phi \cap W_0$ is empty. However, since $F_\lambda^n(\gamma(\phi))$ is on the

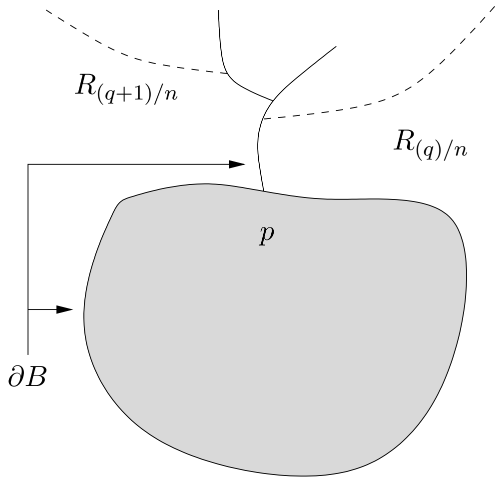


FIGURE 5. A possible landing pattern.

boundary of W_0, we know that $F_\lambda^n(N_\phi) \cap W_0$ is not empty. This is a contradiction since points not in W_0 never enter W_0. Hence, we can never have two rays landing at the same point on ∂B, implying that ∂B is a simple closed curve. □

Although the above was written for λ such that the critical points escape, the results also hold for the Misiurewicz case. In this case, F_λ is subhyperbolic and all of the theorems above hold with minor adjustments (i.e., all of the proofs depending on hyperbolicity still go through when hyperbolicity is replaced by subhyperbolicity). Therefore, we know that ∂B is a simple closed curve for the Misiurewicz case as well.

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