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LOCALLY SIERPINSKI JULIA SETS OF WEIERSTRASS ELLIPTIC ℘ FUNCTIONS

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We define a locally Sierpinski Julia set to be a Julia set of an elliptic function which is a Sierpinski curve in each fundamental domain for the lattice. In order to construct examples, we give sufficient conditions on a lattice for which the corresponding Weierstrass elliptic ℘ function is locally connected and quadratic-like, and we use these results to prove the existence of locally Sierpinski Julia sets for certain elliptic functions. We give examples satisfying these conditions. We show this results in naturally occurring Sierpinski curves in the plane, sphere and torus as well.

Keywords: Complex dynamics; meromorphic functions; Julia sets; Sierpinski curves.

1. Introduction

Although studies have been done on the dynamical and measure theoretical properties of iterated elliptic functions and other meromorphic maps (see e.g. [Bergweiler, 1993; Hawkins & Koss, 2002, 2004; Kotus & Urbanski, 2003]), very few studies have been done on the topological properties of Julia sets of elliptic functions [Hawkins & Koss, 2005]. Elliptic functions are doubly periodic meromorphic functions that arise naturally in solutions to various classical mechanical problems [McKean & Moll, 1999]. In this paper we discuss topological properties of the Julia sets of some elliptic functions, namely the classical Weierstrass elliptic ℘ functions. In particular, we give conditions on the period lattice under which the Julia set for ℘ is locally connected, and sufficient conditions for the map to be quadratic-like. These results cannot be generalized directly from the case of rational maps, so new techniques are required; our focus is on constructing Sierpinski Julia sets in the meromorphic setting.

A Sierpinski curve, also called a Sierpinski carpet, is a planar set that contains a homeomorphic copy of any compact, connected one topological dimensional planar set. It was introduced by Sierpinski in 1916 and was described by G. T. Whyburn [Whyburn, 1958] to be the following set (see Fig. 1):

> The curve is obtained very simply as the residual set remaining when one begins with a square and applies the operation of dividing it into nine equal squares and omitting the interior of the center one, then repeats the operation on each of the surviving eight

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squares, then repeats again on the surviving 64 squares, and so on indefinitely.

A characterization of a Sierpinski curve, defined to be a set homeomorphic to the set described above, was given in [Whyburn, 1958] where it was shown that a set satisfying the following definition is homeomorphic to the classical Sierpinski carpet and is therefore a universal planar set.

Definition 1.1. An $\mathcal{S}$-curve, is a subset S of the plane such that

(1) S is compact;
(2) S is connected; (the first two properties are that S should be a planar continuum.)
(3) S is nowhere dense; (an $\mathcal{S}$-curve has topological dimension one);
(4) S is locally connected;
(5) $S = \mathbb{C} \setminus \cup_\alpha U_\alpha$, where each U_α is a simply connected open set and ∂U_α's are pairwise disjoint simple closed curves.

Whenever S is a subset of a two-dimensional manifold M (not necessarily the plane) satisfying Properties (1)–(5), then we call S a *Sierpinski curve*.

Remark 1.1. The following from [Whyburn, 1958] is used in later examples in Sec. 6. Let γ be any simple closed curve lying in an $\mathcal{S}$-curve S, and let I denote the interior of γ. If no complementary domain boundary ∂U lying in $I \cup \gamma$ intersects γ, then $T = S \cap (\gamma \cup I)$ is again an $\mathcal{S}$-curve.

The properties listed in Definition 1.1 are immediately reminiscent of properties of Julia sets arising in complex dynamics of rational maps, so this is a natural place in which to search for such curves, and indeed Milnor and Tan Lei in 1993 were the first to give an example [Milnor, 2000]. Soon thereafter many parametrized families of examples were produced by Devaney, the second author, and others in [Blanchard *et al.*, 2005; Devaney & Look, 2005; Devaney *et al.*, 2005] and other publications. This work is nicely summarized in [Devaney, 2004].

Recent studies on the Weierstrass elliptic $\wp$ function, [Hawkins & Koss, 2002, 2004, 2005; Kotus & Urbanski, 2003] suggest that Sierpinski curves occur as Julia sets in this setting as well; unlike the rational setting, instead of getting a single Sierpinski carpet for a Julia set in this setting we obtain a "Sierpinski carpet tile" that then is used to tile the entire plane. This is due to the periodicity in the Julia resulting from the periodicity of the elliptic function. We refer to this phenomenon as *wall-to-wall Sierpinski carpeting* and say the Julia set is *locally Sierpinski*. The Julia set of an elliptic function is not compact in the plane due to the double periodicity with respect to a lattice; so Property 1 in Definition 1.1 is never satisfied. Therefore we extend the definition to make precise the notion of wall-to-wall Sierpinski carpeting; (all of the terms used in Definition 1.2 are defined in Sec. 2 of this paper).

Definition 1.2. A *locally Sierpinski Julia set* of an elliptic function f is a Julia set $J(f)$ of a meromorphic function f with the property that there exists a period parallelogram Q for the period lattice Λ such that $J(f) \cap Q$ is a Sierpinski curve.

However, on $\mathbb{C}_\infty = \mathbb{C} \cup \infty$, the Riemann sphere, we have a compact Julia set and we will show that for some elliptic functions we have Sierpinski curve Julia sets on the sphere. Furthermore, we can view the Julia sets on the compact torus $\mathbb{C}/\Gamma$ where Γ is the lattice of periods of the functions; so we obtain in a natural way Sierpinski curves on the torus. The purpose of this paper is to prove the existence of locally Sierpinski Julia sets for certain lattices and for many Weierstrass elliptic $\wp$ functions that are not pairwise conformally conjugate. In order to do this, and of independent interest, we give sufficient conditions (on the lattice) for which an elliptic $\wp$ function is quadratic-like and for which the Julia set is locally connected.

In Fig. 2 we show a Sierpinski curve Julia set for the rational map of the form $R(z) = z^3 + \lambda/z^3$

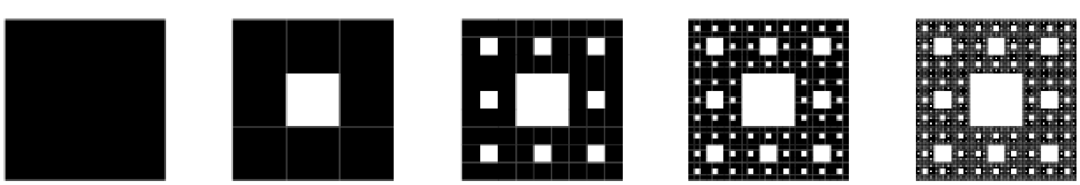

Fig. 1. The "original" Sierpinski curve.

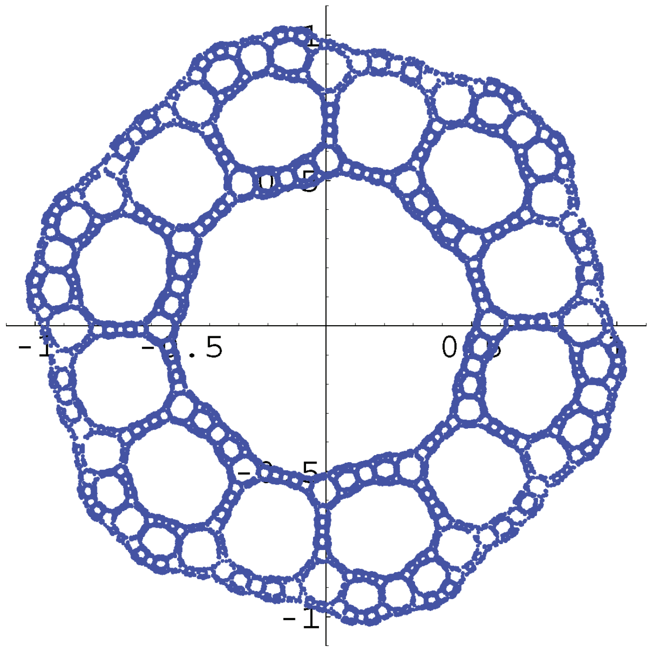

Fig. 2. A Sierpinski curve Julia set for a rational map.

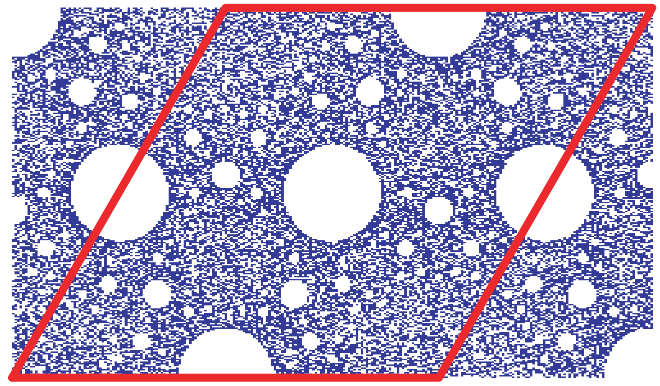

Fig. 4. A single Sierpinski carpet tile.

studied by the second author in [Devaney *et al.*, 2005], while in Fig. 3 we show a Julia set comprised of wall-to-wall Sierpinski carpeting. In Fig. 4 we show one Sierpinski curve carpet tile. In each of the figures, the Sierpinski Julia set is represented by the blue points.

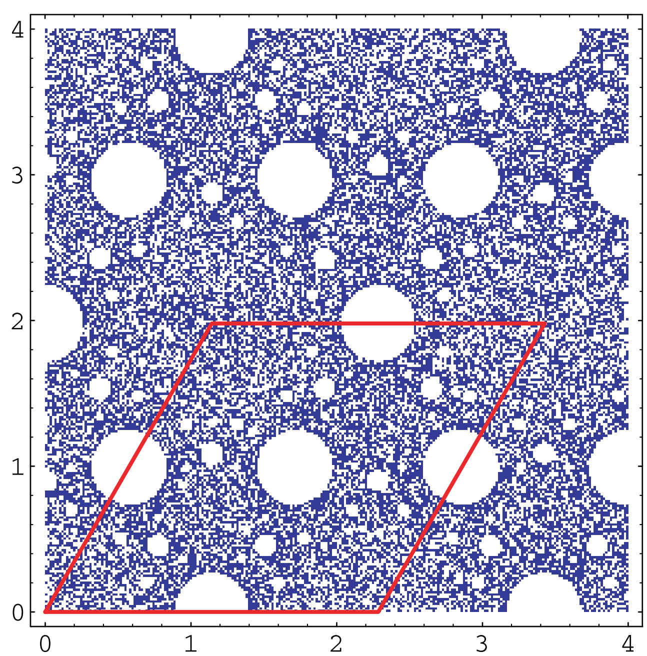

Fig. 3. A locally Sierpinski Julia set for an elliptic map with period lattice shown.

The paper is organized as follows. In Sec. 2 we outline the basic dynamics of elliptic functions. Section 3 presents some results on hyperbolic elliptic functions and connected Julia sets. The main results of the paper are given in Secs. 4–6. We first give sufficient conditions under which a Weierstrass elliptic $\wp$ function is locally connected and provide examples of lattices giving rise to locally connected Julia sets. We then give criteria for an elliptic function to be quadratic-like, and we use these results in Sec. 6 to prove the existence of locally Sierpinski Julia sets for elliptic functions.

2. Preliminaries on the Dynamics of Weierstrass Elliptic $\wp$ Function

Let $\lambda_1, \lambda_2 \in \mathbb{C}\backslash\{0\}$ such that $\lambda_2/\lambda_1 \notin \mathbb{R}$, and define $\Lambda = [\lambda_1, \lambda_2] \equiv \{m\lambda_1 + n\lambda_2 : m, n \in \mathbb{Z}\}$; the generators are not unique. We view Λ as a group acting on $\mathbb{C}$ by translation, each $\omega \in \Lambda$ inducing the transformation of $\mathbb{C}$:

$$T_\omega : z \mapsto z + \omega.$$

Definition 2.1. A closed, connected subset Q of $\mathbb{C}$ is defined to be a *fundamental region* for Λ if

(1) for each $z \in \mathbb{C}$, Q contains at least one point in the same Λ-orbit as z;
(2) no two points in the interior of Q are in the same Λ-orbit.

If Q is any fundamental region for Λ, then for any $s \in \mathbb{C}$, the set

$$Q + s = \{z + s : z \in Q\}$$

is also a fundamental region. If Q is a parallelogram we call it a *period parallelogram* for Λ.

The "appearance" of a lattice $\Lambda = [\lambda_1, \lambda_2]$ is determined by the ratio $\tau = \lambda_2/\lambda_1$. (We generally choose the generators so that $\mathrm{Im}(\tau) > 0$.) If $\Lambda = [\lambda_1, \lambda_2]$, and $k \neq 0$ is any complex number, then $k\Lambda$ is the lattice defined by taking $k\lambda$ for each $\lambda \in \Lambda$; $k\Lambda$ is said to be *similar* to Λ. Similarity is an equivalence relation between lattices, and an equivalence class of lattices is called a *shape*. A lattice is *real* if $\Lambda = \overline{\Lambda}$.

Definition 2.2

(1) $\Lambda = [\lambda_1, \lambda_2]$ is *real rectangular* if there exist generators such that λ_1 is real and λ_2 is purely imaginary. Any lattice similar to a real rectangular lattice is *rectangular*.
(2) $\Lambda = [\lambda_1, \lambda_2]$ is *real rhombic* if there exist generators such that $\lambda_2 = \overline{\lambda_1}$. Any similar lattice is *rhombic*.
(3) A lattice Λ is *square* if $i\Lambda = \Lambda$. (Equivalently, Λ is square if it is similar to a lattice generated by $[\lambda, \lambda i]$, for some $\lambda > 0$.)
(4) A lattice Λ is *triangular* if $\Lambda = e^{2\pi i/3}\Lambda$ in which case a period parallelogram can be made from two equilateral triangles.

In each of cases (1)–(3) the period parallelogram with vertices $0, \lambda_1, \lambda_2$, and $\lambda_3 := \lambda_1 + \lambda_2$ can be chosen to be a rectangle, rhombus, or square respectively.

Definition 2.3. An *elliptic function* is a meromorphic function in $\mathbb{C}$ which is periodic with respect to a lattice Λ.

For any $z \in \mathbb{C}$ and any lattice Λ, the *Weierstrass elliptic function* is defined by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

Replacing every z by $-z$ in the definition we see that $\wp_\Lambda$ is an even function. The map $\wp_\Lambda$ is meromorphic, periodic with respect to Λ, and has order 2.

The derivative of the Weierstrass elliptic function is also an elliptic function which is periodic with respect to Λ defined by

$$\wp'_\Lambda(z) = -2 \sum_{w \in \Lambda} \frac{1}{(z-w)^3}.$$

The Weierstrass elliptic function and its derivative are related by the differential equation

$$\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2\wp_\Lambda(z) - g_3, \tag{1}$$

where $g_2(\Lambda) = 60 \sum_{w \in \Lambda \setminus \{0\}} w^{-4}$ and $g_3(\Lambda) = 140 \sum_{w \in \Lambda \setminus \{0\}} w^{-6}$.

The numbers $g_2(\Lambda)$ and $g_3(\Lambda)$ are invariants of the lattice Λ in the following sense: if $g_2(\Lambda) = g_2(\Lambda')$ and $g_3(\Lambda) = g_3(\Lambda')$, then $\Lambda = \Lambda'$. Furthermore given any g_2 and g_3 such that $g_2^3 - 27g_3^2 \neq 0$ there exists a lattice Λ having $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ as its invariants [DuVal, 1973].

Theorem 2.1 [DuVal, 1973]. *For $\Lambda_\tau = [1, \tau]$, the functions $g_i(\tau) = g_i(\Lambda_\tau), i = 2, 3$, are analytic functions of τ in the open upper half plane* $\mathrm{Im}(\tau) > 0$.

We have the following homogeneity in the invariants g_2 and g_3 [DuVal, 1973].

Lemma 2.2. *For lattices Λ and $\Lambda', \Lambda' = k\Lambda \Leftrightarrow$*

$$g_2(\Lambda') = k^{-4}g_2(\Lambda) \quad \textit{and} \quad g_3(\Lambda') = k^{-6}g_3(\Lambda).$$

Theorem 2.3 [Jones & Singerman, 1997]. *The following are equivalent*:

(1) $\wp_\Lambda(\overline{z}) = \overline{\wp_\Lambda(z)}$;
(2) Λ *is a real lattice*;
(3) $g_2, g_3 \in \mathbb{R}$.

For any lattice Λ, the Weierstrass elliptic function and its derivative satisfy the following properties: for $k \in \mathbb{C}\setminus\{0\}$,

$$\begin{aligned} \wp_{k\Lambda}(ku) &= \frac{1}{k^2}\wp_\Lambda(u), \quad \text{(homogeneity of } \wp_\Lambda), \\ \wp'_{k\Lambda}(ku) &= \frac{1}{k^3}\wp'_\Lambda(u), \quad \text{(homogeneity of } \wp'_\Lambda). \end{aligned} \tag{2}$$

The critical points and values of the Weierstrass elliptic function on an arbitrary lattice $\Lambda = [\lambda_1, \lambda_2]$ are as follows. For $j = 1, 2$, we have $\wp_\Lambda(\lambda_j - z) = \wp_\Lambda(z)$ for all z. Taking derivatives of both sides we obtain $-\wp'_\Lambda(\lambda_j - z) = \wp'_\Lambda(z)$, so at $z = \lambda_1/2, \lambda_2/2$ or $\lambda_3/2$, we see that $\wp'_\Lambda(z) = 0$. We use the notation

$$e_1 = \wp_\Lambda\left(\frac{\lambda_1}{2}\right), \quad e_2 = \wp_\Lambda\left(\frac{\lambda_2}{2}\right), \quad e_3 = \wp_\Lambda\left(\frac{\lambda_3}{2}\right)$$

to denote the critical values. Since e_1, e_2, e_3 are the (distinct) zeros of Eq. (1), we also write

$$\wp'_\Lambda(z)^2 = 4(\wp_\Lambda(z) - e_1)(\wp_\Lambda(z) - e_2)(\wp_\Lambda(z) - e_3). \tag{3}$$

Equating like terms in Eqs. (1) and (3), we obtain

$$\begin{gathered} e_1 + e_2 + e_3 = 0, \\ e_1e_3 + e_2e_3 + e_1e_2 = \frac{-g_2}{4}, \quad e_1e_2e_3 = \frac{g_3}{4}. \end{gathered} \tag{4}$$

In general, similar lattices do not result in conformally conjugate elliptic functions [Hawkins & Koss, 2004]. We can start with a fixed shape lattice, say $\Lambda = [1, \tau]$ and produce many types of dynamics as was shown in [Hawkins & Koss, 2002, 2004, 2005]. Within a given simlarity class (shape), Eq. (2), is used to produce infinitely many lattices with fixed critical points (i.e. with real superattracting fixed points). These examples will be shown in many cases to yield locally Sierpinski Julia sets. The following proposition generalizes a result from [Hawkins & Koss, 2004] for real lattices.

Proposition 2.1. *Let $\Lambda = [1, \tau]$ be a lattice such that the critical value $\wp_\Lambda(1/2) = \epsilon \neq 0$. If m is any odd integer and $k = \sqrt[3]{2\epsilon/m}$ (taking any root) then the lattice $\Gamma = k\Lambda$ has a superattracting fixed point at $mk/2$.*

Proof. Equation (2) for $\wp'_{k\Lambda}$ implies that $k/2$ is a critical point for $\wp_\Gamma$. Since m is odd, periodicity implies that $\wp_\Gamma(mk/2) = \wp_\Gamma(k/2)$. Further, the homogeneity property implies that

$$\wp_\Gamma\left(\frac{mk}{2}\right) = \wp_\Gamma\left(\frac{k}{2}\right) = \wp_{k\Lambda}\left(\frac{k}{2}\right) = \frac{1}{k^2}\wp_\Lambda\left(\frac{1}{2}\right) = \frac{\epsilon}{k^2} = \frac{mk}{2}. \quad \blacksquare$$

Corollary 2.1. *Every similarity class contains a lattice Λ for which $\wp_\Lambda$ has a superattracting fixed point.*

Proof. We apply Proposition 2.1, and we need only to check that $\wp_\Lambda(1/2) = \epsilon \neq 0$ for each (nonreal) choice of τ. A critical value $e_j = 0$ if and only if $g_3 = 0$, by Eq. (4). This holds if and only if the lattice is square, in which case $\tau = i$. However in this case we know that $e_3 = 0$, and $e_1 = \wp_\Lambda(1/2) \neq 0$. $\blacksquare$

Figure 5 shows the graph of $\wp_\Lambda$ on $\mathbb{R}$ for a real square lattice which has a superattracting fixed point.

2.1. *Fatou and Julia sets for elliptic functions*

We review the basic dynamical definitions and properties for meromorphic functions which appear in [Baker *et al.*, 1991a; Bergweiler, 1993; Devaney & Keen, 1988; Devaney, 2004]. Let $f : \mathbb{C} \to \mathbb{C}_\infty$ be a meromorphic function. The *Fatou set* $F(f)$ is the set of points $z \in \mathbb{C}_\infty$ such that $\{f^n : n \in \mathbb{N}\}$ is defined

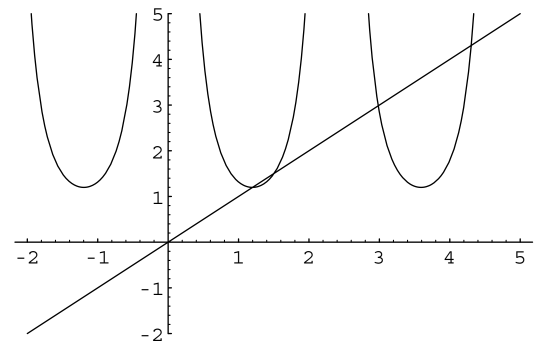

Fig. 5. Graph of $\wp_\Lambda|_\mathbb{R}$ with a superattracting fixed point.

and normal in some neighborhood of z. The *Julia set* is the complement of the Fatou set on the sphere, $J(f) = \mathbb{C}_\infty \backslash F(f)$. Notice that $\mathbb{C}_\infty \backslash \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}$ is the largest open set where all iterates are defined. Since $f(\mathbb{C}_\infty \backslash \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}) \subset \mathbb{C}_\infty \backslash \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}$, Montel's theorem implies that

$$J(f) = \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}.$$

Let Crit(f) denote the set of critical points of f, i.e.

$$\mathrm{Crit}(f) = \{z : f'(z) = 0\}.$$

If z_0 is a critical point then $f(z_0)$ is a *critical value*. For each lattice, $\wp_\Lambda$ has three critical values and no asymptotic values. The *singular set* Sing(f) of f is the set of critical and finite asymptotic values of f and their limit points. A function is called *Class S* if f has only finitely many critical and asymptotic values; for each lattice Λ, every elliptic function with period lattice Λ is of Class S. The *postcritical set* of $\wp_\Lambda$ is:

$$P(\wp_\Lambda) = \overline{\bigcup_{n\geq 0} \wp_\Lambda^n(e_1 \cup e_2 \cup e_3)}.$$

For a meromorphic function f, a point z_0 is *periodic* of period p if there exists a $p \geq 1$ such that $f^p(z_0) = z_0$. We also call the set $\{z_0, f(z_0), \ldots, f^{p-1}(z_0)\}$ a *p-cycle*. The *multiplier* of a point z_0 of period p is the derivative $(f^p)'(z_0)$. A periodic point z_0 is called *attracting*, *repelling*, or *neutral* if $|(f^p)'(z_0)|$ is less than, greater than, or equal to 1 respectively. If $|(f^p)'(z_0)| = 0$ then z_0 is called a *superattracting* periodic point. As in the case of rational maps, the Julia set is the closure of the repelling periodic points [Baker *et al.*, 1991a].

Suppose U is a connected component of the Fatou set. We say that U is *preperiodic* if there exists $n > m \geq 0$ such that $f^n(U) = f^m(U)$, and the minimum of $n - m = p$ for all such n, m is the *period* of the cycle.

Since every elliptic function is of Class S the basic dynamics are similar to those of rational maps with the exception of the poles. The first result holds for all Class S functions as was shown in [Bergweiler, 1993] (Theorem 12) and [Rippon & Stallard, 1999].

Theorem 2.4. *For any lattice Λ, the Fatou set of an elliptic function f_Λ with period lattice Λ has no wandering domains and no Baker domains.*

In particular, Sullivan's No Wandering Domains Theorem holds in this setting so all Fatou components of f_Λ are preperiodic. Because there are only finitely many critical values, we have a bound on the number of attracting periodic points that can occur.

The next result was proved in [Hawkins & Koss, 2004]; it is only known for the Weierstrass elliptic function.

Theorem 2.5. *For any lattice Λ, $\wp_\Lambda$ has no cycle of Herman rings.*

We summarize this discussion with the following result.

Theorem 2.6. *For any lattice Λ, at most three different types of forward invariant Fatou cycles can occur for $\wp_\Lambda$, and each periodic Fatou component contains one of these*:

(1) *a linearizing neighborhood of an attracting periodic point*;
(2) *a Böttcher neighborhood of a superattracting periodic point*;
(3) *an attracting Leau petal for a periodic parabolic point. The periodic point is in $J(\wp_\Lambda)$*;
(4) *a periodic Siegel disk containing an irrationally neutral periodic point.*

2.1.1. *Symmetries of the Julia and Fatou sets*

Lemma 2.7. *If Λ is any lattice and f_Λ is an elliptic function with period lattice Λ, then*

(1) $J(f_\Lambda) + \Lambda = J(f_\Lambda)$, *and*
(2) $F(f_\Lambda) + \Lambda = F(f_\Lambda)$.

The algebraic and analytic symmetry of the Weierstrass elliptic function manifests itself in a large amount of symmetry in each Julia set arising from an elliptic function. The proof of Theorem 2.8 is given in [Hawkins & Koss, 2002].

Theorem 2.8. *If $\Lambda = [\lambda_1, \lambda_2]$ is any lattice then*

(1) $J(\wp_\Lambda) + \Lambda = J(\wp_\Lambda)$ *and* $F(\wp_\Lambda) + \Lambda = F(\wp_\Lambda)$.
(2) $(-1)J(\wp_\Lambda) = J(\wp_\Lambda)$ *and* $(-1)F(\wp_\Lambda) = F(\wp_\Lambda)$.
(3) $\overline{J(\wp_\Lambda)} = J(\wp_{\overline{\Lambda}})$ *and* $\overline{F(\wp_\Lambda)} = F(\wp_{\overline{\Lambda}})$.
(4) *If Λ is square, then* $e^{\pi i/2}J(\wp_\Lambda) = J(\wp_\Lambda)$ *and* $e^{\pi i/2}F(\wp_\Lambda) = F(\wp_\Lambda)$.
(5) *If Λ is triangular, then* $e^{2\pi i/3}J(\wp_\Lambda) = J(\wp_\Lambda)$ *and* $e^{2\pi i/3}F(\wp_\Lambda) = F(\wp_\Lambda)$. *Moreover,* $e^{2\pi i/3}\wp_\Lambda(z) = \wp_\Lambda(e^{2\pi i/3}z)$ *for all* $z \in \mathbb{C}\backslash\Lambda$.

In addition to a basic Julia set pattern repeating on each fundamental period, we also see symmetry within the period parallelogram.

Proposition 2.2. *For the lattice $\Lambda = [\lambda_1, \lambda_2]$, $J(\wp_\Lambda)$ and $F(\wp_\Lambda)$ are symmetric with respect to any critical point $\lambda_1/2 + \Lambda$, $\lambda_2/2 + \Lambda$, and $(\lambda_1 + \lambda_2)/2 + \Lambda$. That is, if c is any critical point of $\wp_\Lambda$, then $c + z \in J(\wp_\Lambda)$ if and only if $c - z \in J(\wp_\Lambda)$.*

In particular, if F_o is any component of $F(\wp_\Lambda)$ that contains a critical point c, then F_o is symmetric with respect to c.

3. Connected Julia Sets and Hyperbolic Weierstrass $\wp_\Lambda$ Functions

We give sufficient conditions under which $J(\wp_\Lambda)$ is connected if $\wp_\Lambda$ is the Weierstrass elliptic $\wp$ function with period lattice Λ. The proof of the next result appears in [Hawkins & Koss, 2005]. It uses the fact that although there are infinitely many critical points for $\wp_\Lambda$, there are exactly three critical values and $\wp_\Lambda$ is locally two-to-one in each fundamental region.

Theorem 3.1. *If Λ is a lattice such that each critical value of $\wp_\Lambda$ that lies in the Fatou set is the only critical value in that component, then $J(\wp_\Lambda)$ is connected. In particular, if each Fatou component contains either* 0 *or* 1 *critical value, then $J(\wp_\Lambda)$ is connected.*

We now turn to the definition of a hyperbolic elliptic function; the concept of hyperbolicity for a meromorphic function is similar to that for a rational map, but the equivalent characterizations in the rational settings are not always the same for meromorphic functions (e.g. see [Rippon & Stallard, 1999]).

Definition 3.1. We say that an elliptic function is *hyperbolic* if $J(f)$ is disjoint from $P(f)$.

We adapt the following more general result from [Rippon & Stallard, 1999] (Theorem C) to our setting. We define the set

$$A_n(f) = \{z \in \mathbb{C} : f^n \text{ is not analytic at } z\}.$$

Since f is elliptic,

$$A_n = A_n(f) = \bigcup_{j=1}^{n} f^{-j}(\{\infty\}).$$

We therefore denote the set of prepoles for f by

$$\mathcal{A} = \bigcup_{n\geq 1} A_n.$$

We say that ω is an *order k prepole* if $\wp_\Lambda^k(\omega) = \infty$; so a pole is an order 1 prepole.

Theorem 3.2 [Rippon & Stallard, 1999]. *If an elliptic function f is hyperbolic then there exist $K > 1$ and $c > 0$ such that*

$$|(f^n)'(z)| > cK^n \frac{|f^n(z)|+1}{|z|+1},$$

for each $z \in J(f)\backslash A_n, n \in \mathbb{N}$.

As an immediate corollary we get a lower bound on the derivative of f^n when its image is large.

Corollary 3.1. *If an elliptic function f is hyperbolic then there exist $K > 1$ and $c > 0$, and numbers $0 < r < R$ such that if $|z| < r$ and $z \in J(f)\backslash A_n$, $n \in \mathbb{N}$, and $|f^n(z)| > R$, then*

$$|(f^n)'(z)| > cK^n \frac{R+1}{r+1}.$$

Remark 3.1. For a fixed lattice Λ, and any elliptic function $f = f_\Lambda$, we have that $f(z+\lambda) = f(z)$ and $f'(z+\lambda) = f'(z)$ for every $\lambda \in \Lambda$. From this it follows that $(f^n)'(z+\lambda) = (f^n)'(z)$ for all $n \in \mathbb{N}$. Therefore we strengthen Corollary 3.1 and obtain a generalization of a standard result for hyperbolic rational maps (Theorem 3.3).

Corollary 3.2. *For a hyperbolic elliptic function f with period lattice Λ, there exist constants $K > 1$, $\alpha = \alpha(\Lambda) > 0, R > 0$, and a neighborhood U of $J(f)$ such that for any $z \in U\backslash A_n$, if $|f^n(z)| > R$, then*

$$|(f^n)'(z)| > \alpha K^n(R+1).$$

Proof. We choose $\alpha = c/(r+1)$, where r is chosen so that there exists a fundamental region for Λ completely contained in $\{z : |z| < r\}$ and c and R are chosen as in Corollary 3.1 (or slightly larger if necessary). ■

The proof of the next result appears in [Hawkins & Koss, 2005].

Theorem 3.3. *An elliptic function f is hyperbolic if and only if there exist $K > 1$ and $C > 0$, $C = C(\Lambda)$, such that*

$$|(f^n)'(z)| > CK^n, \tag{5}$$

for each $z \in J(f)\backslash A_n, n \in \mathbb{N}$.

Corollary 3.3. *If f is hyperbolic, then there exist $K > 1, C > 0$, and a neighborhood U of $J(f)$, such that*

$$|(f^n)'(z)| > CK^n \quad \text{for all } z \in U\backslash A_n,\ n \in \mathbb{N}.$$

3.1. *Hyperbolic elliptic functions with triangular period lattices*

We begin with some properties of $\wp_\Lambda$ for Λ a triangular period lattice.

Proposition 3.1 [Hawkins & Koss, 2004, 2005]. *Assume Λ is triangular; then the following hold.*

(1) *$g_2 = 0$; in this case e_1, e_2, e_3 all have the same modulus and are the cube roots of $g_3/4$. Furthermore, e_i is real for some $i = 1, 2, 3$ if and only if g_3 is real, if and only if Λ is a real lattice.*
(2) *$g_3 > 0$ if and only if some $e_j > 0$, in which case there exists $\lambda > 0$ such that $\Lambda = [\lambda e^{\pi i/3}, \lambda e^{-\pi i/3}]$.*
(3) *The postcritical set $P(\wp_\Lambda)$ is contained in three forward invariant sets: one set $\alpha = \overline{\bigcup_{n\geq 0} \wp_\Lambda^n(e_1)}$, and the sets $e^{2\pi/3}\alpha$ and $e^{4\pi/3}\alpha$. (These sets are not necessarily disjoint.)*
(4) *$J(\wp_\Lambda)$ is connected.*

Corollary 3.4. *Assume Λ is triangular and there is an attracting periodic point. Then the following hold:*

(1) *If p_0 denotes the attracting periodic point, then $p_1 = e^{2\pi/3}p_0$ and $p_2 = e^{4\pi/3}p_0$ are attracting periodic points as well.*
(2) *$\wp_\Lambda$ is hyperbolic.*

The main result of this section then is the following.

Theorem 3.4. *If Λ is a triangular lattice and $\wp_\Lambda$ has an attracting cycle, then $J(\wp_\Lambda)$ is hyperbolic and connected.*

Proof. The result follows from Proposition 3.1 and Corollary 3.4. ■

In order to determine which triangular lattices give rise to Fatou sets such that the boundary of each simply connected component is a simple closed curve, we need to apply the theory of polynomial-like mappings (see Figs. 3 and 4). However, first we turn to the notion of local connectivity.

4. Local Connectivity of Julia Sets for Weierstrass $\wp$ Functions

Local connectivity of a Julia set is a difficult property to verify for an arbitrary rational map (even for a quadratic polynomial, see e.g. [Milnor, 2000c]). Furthermore results in the rational setting cannot always pass to the meromorphic setting. Therefore we begin with a careful exposition of sufficient criteria for local connectivity for elliptic functions. We follow the treatment of Milnor for rational maps [Milnor, 2000a] (Chaps. 17 and 19), adding detail where the meromorphic setting requires new techniques.

Definition 4.1. A Hausdorff space J is *locally connected* if every point $z \in J$ has arbitrarily small connected neighborhoods.

Noting that the neighborhoods in Definition 4.1 need not be open, it is straightforward to verify the following result (Lemma 17.13 from [Milnor, 2000a]).

Lemma 4.1. *If J is a compact metric space, then J is locally connected if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that any two points of distance $< \delta$ are contained in a connected subset of J of diameter $< \epsilon$.*

We endow the Riemann sphere $\mathbb{C}_\infty$ with the spherical metric, which we denote by d. Given an elliptic function f and a component W of the Fatou set of f, we define the distance between two points $w, z \in W$, $\text{dist}_W(w, z)$ to be the infimum of the d lengths of paths between w and z, using only paths that lie in W. Then the diameter of W is given by: $\text{diam}_d(W) = \sup_{w,z\in W} \text{dist}_W(w, z)$. Since W lies completely in a bounded region in $\mathbb{C}$ in our setting, by $\text{diam}_{\text{Euclidean}}(W)$ we mean the analogous diameter defined using the Euclidean metric. The next result uses Lemma 4.1 and appears in [Milnor, 2000a].

Proposition 4.1. *Given a lattice Λ and an elliptic function f with period lattice Λ, if every component of $F(f)$ has a locally connected boundary, and if for any $\epsilon > 0$ there are at most finitely many components with diameter $> \epsilon$, then $J(f)$ is locally connected.*

We define the elliptic function f to have *small Fatou components* if $F(f)$ is such that for any $\epsilon > 0$ there are at most finitely many components with diameter $> \epsilon$. Examples have been given of lattices and maps $\wp_\Lambda$ with infinitely many Fatou components all having the same fixed size; namely, the toral band examples from [Hawkins & Koss, 2005]. Some of these examples are hyperbolic, so the direct application of results from rational mappings ends at this point. The local connectivity of the Julia sets when toral bands are present, and even the connectivity, is yet to be determined.

However, it is known that there are many lattices which give rise to elliptic functions f with the property that each Fatou component is completely contained in a single period parallelogram. We focus on these examples here and show that they have small Fatou components. Recall that for any lattice, $J(f)$ consists of a set in a period parallelogram which is then translated by the elements of the lattice throughout the plane; we compactify $J(f)$ by including the point at ∞ (see Theorem 2.7).

The next result is from Milnor ([Milnor, 2000a], Lemma 19.3), proved for rational maps; however the proof uses only the local form of the mapping so it can be extended to the elliptic setting. The extra hypothesis included in the meromorphic setting is to avoid having ∞ on the boundary of the Fatou component. We reprove this result for a class of examples using other methods in the next section.

Proposition 4.2. *If f is a hyperbolic elliptic map and W is a simply connected component of $F(f)$ which is completely contained in one fundamental region, then the boundary ∂W is locally connected.*

We cite another result from [Milnor, 2000a] which applies in our setting. This result is obtained using different methods for our examples in the next section as well (see e.g. Theorem 5.2).

Lemma 4.2. *If $\wp_\Lambda$ is hyperbolic, and each component of $F(\wp_\Lambda)$ lies in a fundamental region, then for any component $W \subset F(\wp_\Lambda), \operatorname{diam}_d(W) < \infty$.*

Given any Fatou component W, we have $W \subset \mathbb{C}$ and W occurs with countably many translates of itself under the period lattice Λ by Lemma 2.7. It is well known that if a set W of fixed diameter in the plane is translated by $\lambda \in \mathbb{C}$, the diameters of the translated sets shrink *in the spherical metric d* as $|\lambda|$ gets large. Given a component $W \in F(\wp_\Lambda)$ then, each translate $W + \lambda$, $\lambda \in \Lambda$ is mapped under $\wp_\Lambda$ to $\wp_\Lambda(W)$. In the Euclidean metric, the diameters of $W + \lambda$ are the same, so in the spherical metric the diameters are arbitrarily small near ∞. This leads to the following lemma.

Corollary 4.1. *If $\wp_\Lambda$ is hyperbolic, and each component of $F(\wp_\Lambda)$ is contained in a fundamental region, then for any component $W \subset F(\wp_\Lambda)$, and any $\epsilon > 0$, there exists an $R > 1$ such that $\operatorname{diam}_d(W + \lambda) < \epsilon$ for all $\lambda \in \Lambda$ such that $|\lambda| > R$.*

Lemma 4.3. *Assume we have an elliptic function f and $F(f)$ is such that each Fatou component is completely contained in a single fundamental region. Then f has small Fatou components if and only if there exists a period parallelogram Q containing 0 (in its closure) such that given any $\epsilon > 0$, there exist only finitely many components of $F(f) \cap Q$ with Euclidean diameter $> \epsilon$.*

Proof. ($\Rightarrow$) This direction is clear since in $\mathbb{C}$, the spherical and Euclidean metrics are equivalent.

($\Leftarrow$) Given $1 > \epsilon > 0$, using the spherical metric we consider the closed ball $\overline{B_r(\infty)}$ for some $r > 0$. All but finitely many fundamental regions of $\wp_\Lambda$ are contained in $\overline{B_r(\infty)}$; let B denote the union of fundamental regions completely contained in $\overline{B_r(\infty)}$. By our assumption on Q and by applying Corollary 4.1, we can choose r small enough so that every Fatou component in B has diameter $< \epsilon/2$. Clearly Q is not completely contained in B since $0 \notin B$; by assumption we have only finitely many Fatou components of Euclidean diameter $> \epsilon/2$ in Q, and also for the finitely many translates of Q under Λ not in B (this is because Euclidean distance is invariant under translation so the diameter of a translated set will not change). Since for any planar set A

$$\operatorname{diam}_d(A) \leq 2 \operatorname{diam}_{\text{Euclidean}}(A),$$

the result follows. ∎

Proposition 4.3. *If Λ is triangular and $\wp_\Lambda$ has an attracting cycle, then $\wp_\Lambda$ has small Fatou components.*

Proof. By Theorem 3.4 we have that $\wp_\Lambda$ is hyperbolic. Since it was established in [Hawkins & Koss, 2005] that each Fatou component of $\wp_\Lambda$ is completely contained in one fundamental region under these hypotheses, we can apply Lemma 4.3. If $\Lambda = [\lambda_1, \lambda_2]$, we define and fix the period parallelogram

$$Q = \{z : z = s\lambda_1 + t\lambda_2, t, s \in [0, 1]\}.$$

Our hypothesis that each Fatou component is completely contained in a fundamental region does not imply that all Fatou components intersecting Q lie completely in Q. By symmetry each Fatou component intersecting ∂Q has two mirror image halves, each lying on one side of the boundary. Noting this, in what follows we estimate the diameter of the entire Fatou component intersecting Q, since our assumption implies that the component cannot intersect any fundamental region other than the nearest neighbors of Q. We say W is "a component in Q" when W is a Fatou component intersecting Q.

We find an open neighborhood U of $J(\wp_\Lambda)$ on which $\wp_\Lambda$ is expanding as in Corollary 3.3, and extend the cover to all of Q by adding components of $F(\wp_\Lambda)$, labeling them W_j. (Use the entire component for components W_j intersecting the boundary of Q.)

By compactness of Q, we can find a finite subcover. Each component W of $F(\wp_\Lambda)$ intersecting Q is of one of two types: either it is one of the W_j's, and there are only finitely many of them (so we do not need a small diameter), or W is completely contained in the expanding neighborhood U. We assume then that $W \subset U$.

Some finite iterate of W lands on an attracting cycle of components and therefore leaves U completely. We say W is a level k component if k is the smallest non-negative integer such that $\wp_\Lambda^k(W)$ contains an attracting cycle. For a hyperbolic triangular lattice, we have either one or three attracting orbits; when there are three, they are rotations of each other, so have the same modulus (as described in Corollary 3.4). We clarify the definition of level k components of $F(\wp_\Lambda)$ in what follows. We define the fundamental regions which are translates of Q by: $Q_{m,n} = Q + m\lambda_1 + n\lambda_2$ for each $m, n \in \mathbb{Z}$. We consider any component $W \subset F(\wp_\Lambda)$.

- A component in Q is a level 0 component if it contains a point in the attracting orbit. Note that Q might not contain any level 0 components.
- A level 1 component in Q maps under $\wp_\Lambda$ in one iterate to an attracting orbit component in any fundamental region. There are finitely many of these, and every fundamental region $Q_{m,n}$ contains level 1 components.
- A level 2 component W in Q is defined by saying $\wp_\Lambda(W)$ is a level 1 component in any $Q_{m,n}$. Since every $Q_{m,n}$ contains finitely many level 1 components, and each of these has exactly two components in its preimage in Q, there are infinitely many level 2 components. Their translations in $Q_{m,n}$ are level 2 components.
- If $\wp_\Lambda(W)$ is a level $k-1$ component in any $Q_{m,n}$ we call W a level k component in Q. Since every $Q_{m,n}$ contains infinitely many level $k-1$ components for $k \geq 2$, and each of these has exactly two components to its preimage in Q, there are infinitely many level k components.

We let M be the maximum of the Euclidean diameters of W, taken over the W components in Q of level 1. Note that M is also the maximum diameter taken over all level 1 components in $\mathbb{C}$ by invariance of the Euclidean metric under translation. Suppose $K > 1$ is the expansion constant, then by Corollary 3.3,

$$|(\wp_\Lambda^n)'(z)| > CK^n \quad \text{for all } z \in U \backslash A_n,\ \ n \in \mathbb{N},$$

so

$$\mathrm{diam}_{\mathrm{Euclidean}}(W) \leq \frac{M}{CK^{k-1}}$$

for each component W of level $k \geq 1$ (and note that $W \cap (\bigcup_{n\geq 1} A_n) = \emptyset$). As $k \to \infty$, the diameter goes to 0, so given $\epsilon > 0$, there exists $k_o \in N$ such that we only have to consider components of level k for finitely many levels $k \leq k_o$.

For each W in Q of level 2, if $z \in W \subset U$, suppose $\wp_\Lambda(z) \in Q_{m,n}$ (and it follows that $\wp_\Lambda(z)$ is in a level 1 component). Let $\alpha_{m,n}$ denote the minimum Euclidean distance from $Q_{m,n}$ to the origin. Then by Corollary 3.2

$$\wp_\Lambda'(z) > \alpha K \cdot (\alpha_{m,n} + 1),$$

where α depends on the lattice.

Then although we have level 2 components $W \subset Q$ sent under $\wp_\Lambda$ to every $Q_{m,n}$, for each fixed (m, n) there are only finitely many such W, and

$$\mathrm{diam}_{\mathrm{Euclidean}}(W) \leq \frac{M}{\alpha K(\alpha_{m,n} + 1)}.$$

Clearly as $m, n \to \infty$, the diameter of level 2 components in Q goes to 0. Then for each fixed $\epsilon > 0$ we have only finitely many level 2 components in Q with diameter $> \epsilon$.

We use an induction argument on k to complete the argument. Given $\epsilon > 0$, suppose we have shown there are only finitely many level $k-1$ components in Q of diameter $> \epsilon$. (And hence only finitely many in any $Q_{m,n}$.)

Then we consider a level k component W in the expanding neighborhood U. By definition $\wp_\Lambda(W)$ is a level $k-1$ component in some $Q_{m,n}$ and so the diameter of W is bounded above by $(1/A)\ \mathrm{diam}_{\mathrm{Euclidean}}(\wp_\Lambda(W))$ if A is a lower bound for $\wp_\Lambda'(z), z \in W$. Applying the inequality

$$A > \alpha K(\alpha_{m,n} + 1)$$

from Corollary 3.2, we have that for all but finitely many values of m and n we will have diameters $< \epsilon$, and the result is shown. ■

Theorem 4.4. *If Λ is triangular and $P(\wp_\Lambda)$ has an attracting cycle, then $J(\wp_\Lambda)$ is locally connected.*

Proof. We apply Proposition 4.1, Lemma 4.3, and Proposition 4.3. ■

5. Quadratic-Like Elliptic Functions

In this section we show that for many lattices the Weierstrass elliptic $\wp$ function acts locally like a quadratic polynomial in the sense introduced by Douady and Hubbard [1985]. (This was conjectured and shown experimentally by the first author and Koss [2004].) While this section is of independent interest, it is also useful for determining local connectivity of the Julia set of a Weierstrass elliptic function. For the discussion which follows we need the following results regarding period parallelograms for elliptic functions.

Proposition 5.1. *If $\Lambda = [\lambda_1, \lambda_2]$ is any lattice and $u \in \mathbb{C}$, let $Q_u = \{u + s\lambda_1 + t\lambda_2 : 0 \leq s, t < 1\}$. Then $\wp_\Lambda : Q_u \to \mathbb{C}_\infty$ is onto and two-to-one except at the points that lie in the Λ-orbit of 0, $\lambda_1/2$, $\lambda_2/2$, and $(\lambda_1 + \lambda_2)/2$ (the critical points).*

Corollary 5.1. *For any lattice $\Lambda = [\lambda_1, \lambda_2]$, if u is a lattice point or half lattice point (i.e. of the form $u = m\lambda_1 + (2n+1)(\lambda_2/2)$ or $u = (2m+1)(\lambda_1/2) + n\lambda_2$, for any $m, n \in \mathbb{Z}$), then $\overline{Q}_u$ is a fundamental period parallelogram containing in its interior no poles and* *one critical point, and $\wp_\Lambda : Q_u \to \mathbb{C}_\infty$ is analytic, onto, and two-to-one counting multiplicity.*

Let int Q_u denote the interior of Q_u. Our goal is to export to our setting the following result from Mañé *et al.* [1983]. A simple closed curve γ is a *quasicircle* if it is the image of a circle under a quasiconformal homeomorphism (of the sphere); quasicircles are locally connected.

Theorem 5.1. *If p is a quadratic polynomial with an attracting fixed point, then $J(p)$ is a quasicircle.*

Definition 5.1. Let U and V be simply connected bounded open subsets of $\mathbb{C}$ such that $\overline{U} \subset V$ is compact. A map $f : U \to V$ is a *polynomial-like map* if f is a d-fold covering map for some integer $d \geq 2$ which we call the degree of f. A *quadratic-like map* has $d = 2$. A polynomial-like map of degree d has $d - 1$ critical points in U.

We denote a quadratic-like map by $(f : U, V)$. The *filled Julia set* K_f of $(f : U, V)$ is the set of points which do not escape U under f, i.e.

$$K_f = \{z \in U | f^n(z) \in U \text{ for all } n \geq 0\}.$$

Two polynomial-like maps f and g are *hybrid equivalent* if there is a quasiconformal conjugacy ϕ between f and g defined on a neighborhood of their filled Julia sets.

The following result was shown in [Douady & Hubbard, 1985] in the setting of rational maps, but since it is a local result, it applies equally well to elliptic functions.

Proposition 5.2. *For a polynomial-like map f, $K(f)$ is connected if and only if it contains every critical point of f. If $K(f)$ is connected, and f is of degree d then f is hybrid equivalent to a polynomial of degree d.*

We are now in a position to prove a result about certain elliptic functions.

Theorem 5.2. *Consider a lattice Λ such that $\wp_\Lambda$ has the following properties:*

- *$\wp_\Lambda$ has an attracting (or superattracting) fixed point q;*
- *the forward invariant Fatou component F_o containing q has exactly one critical value and one critical point in it;*
- *there is a period parallelogram Q_u, with u a lattice or half lattice point, such that*

$$\overline{F}_o \subset \text{int } Q_u.$$

- *there are no other critical values of $\wp_\Lambda$ contained in* int Q_u.

We define $\mathcal{P} = \wp_\Lambda|_{Q_u}$. Then there exists an open set $\mathcal{O} \subset Q_u$ such that $(\mathcal{P} : \mathcal{O}, \mathcal{P}(\mathcal{O}))$ is a quadratic-like map and $\partial K(\mathcal{P})$ is a quasicircle.

Proof. Let $\Lambda = [\lambda_1, \lambda_2]$ be a lattice satisfying the hypotheses above. Our hypothesis on F_o implies that it is simply connected and contained in one fundamental period by [Hawkins & Koss, 2004]. Furthermore F_o contains the fixed point q of $\wp_\Lambda$. By hypothesis we can find a lattice or half lattice point u satisfying the hypothesis, and applying Corollary 5.1, there are poles of $\wp_\Lambda$ on the boundary of the parallelogram Q_u and none lie in the interior; so $\wp_\Lambda$ is analytic and contains exactly one critical value on int Q_u. We define the sets $V = \text{int } Q_u$ and $\mathcal{O} = \mathcal{P}^{-1}(V)$; i.e. $\mathcal{O}$ is the connected component of $\wp_\Lambda^{-1} V$ containing q. Clearly $\mathcal{P}(\mathcal{O}) = V$. Since $\mathcal{P}$ maps Q_u two-to-one onto $\mathbb{C}_\infty$, and all of the poles of $\mathcal{P}$ are on the boundary $\mathcal{O}$, we have:

(1) $\mathcal{O}$ is homeomorphic to a disk,
(2) $\wp_\Lambda$ is degree 2 on $\mathcal{O}$, and
(3) $\wp_\Lambda(\mathcal{O})$ contains $\mathcal{O}$ and is homeomorphic to a disk.

The first two statements are clear; we prove the third. Since V contains exactly one critical value, then $\wp_\Lambda^{-1}(V)$ is a single simply connected region in each fundamental period, bounded by a simple closed curve and containing a critical point on the interior of each parallelogram of the form $Q_{(u+m\lambda_1+n\lambda_2)}, m, n \in \mathbb{Z}$; the set $\mathcal{O}$ is just the component of $\wp_\Lambda^{-1}(V)$ contained in Q_u. Since by hypothesis F_o is forward invariant and $\overline{F}_o \subset V$, then $F_o \subset \mathcal{O}$ and therefore $\mathcal{P}$ maps $\mathcal{O}$ onto V by a ramified two-fold covering by Corollary 5.1.

Then Definition 5.1 and Proposition 5.2 imply that $(\mathcal{P} : \mathcal{O}, \mathcal{P}(\mathcal{O}))$ is quadratic-like. Theorem 5.2 gives that $\partial K(\mathcal{P})$ is a quasicircle. ∎

There are many lattices for which the associated function $\wp_\Lambda$ satisfies the hypotheses of Theorem 5.2. We prove the existence of some examples here. Other examples can be obtained by using Corollary 2.1.

5.1. *Examples of quadratic-like elliptic functions*

Theorem 5.3. *Suppose Λ is a real rectangular square lattice or a real triangular lattice with $g_3 > 0$, and suppose that $\wp_\Lambda$ has a real attracting fixed point p_o such that the forward invariant Fatou component F_{p_o} contains at most one critical value. Then for each forward invariant Fatou component F of $\wp_\Lambda$ there is an open set G and a period parallelogram Q of Λ such that*

$$\overline{F} \subset G \subset Q.$$

Proof. Assume first that $\Lambda = [\lambda, \lambda i]$ is a rectangular square for some $\lambda > 0$. By [Hawkins & Koss, 2002] (Proposition 6.7.2), the attracting fixed point $p_o > 0$ is the only nonrepelling cycle and we have that the immediate attracting basin $F_{p_o} = F$ contains the positive critical value $e_1 > 0$ which satisfies:

$$0 \leq n_o\lambda < e_1 \leq p_o < (n_o + 1)\lambda$$

for some non-negative integer n_o. Since $e_3 = 0$ is a pole, it is in the Julia set; by hypothesis $e_2 \notin F$. We claim therefore that F intersects exactly two quadrants in the plane: I and IV. If F were to intersect quadrant II or III, then by symmetry with respect to the origin and each axis, $e_2 = -e_1 \in F$, a contradiction. Furthermore F cannot intersect the imaginary axis because it is open and would therefore intersect quadrant II, and by symmetry, III. By the symmetry arising from the periodicity of $\wp_\Lambda$, it follows that F does not intersect any lines of the form $\Re(z) = m\lambda$, $m \in \mathbb{Z}$ (the vertical lines at lattice points).

Moreover we claim that F cannot intersect any line of the form $\Im(z) = (2m + 1)\lambda/2$, $m \in \mathbb{Z}$ (the horizontal half lattice lines). By periodicity of $\wp_\Lambda$ it is enough to consider points of the form $z = a + (\lambda/2)i$ for $0 \leq a \leq (\lambda/2)$. But it is well known (cf. [DuVal, 1973]) that for a rectangular square lattice, the horizontal line segment from $0 + (\lambda/2)i$ to $(\lambda/2) + (\lambda/2)i$ maps under $\wp_\Lambda$ onto the horizontal line segment from $-e_1$ to 0 along the negative real axis. In other words, its image does not lie in quadrants I and IV, so these lines form boundary lines for a region containing F.

In particular, we have just shown that F is completely contained in the period square Q with vertices (going clockwise around the square):

$$n_o\lambda + \left(\frac{\lambda}{2}\right)i, \quad (n_o + 1)\lambda + \left(\frac{\lambda}{2}\right)i,$$

$$(n_o + 1)\lambda - \left(\frac{\lambda}{2}\right)i, \quad n_o\lambda - \left(\frac{\lambda}{2}\right)i.$$

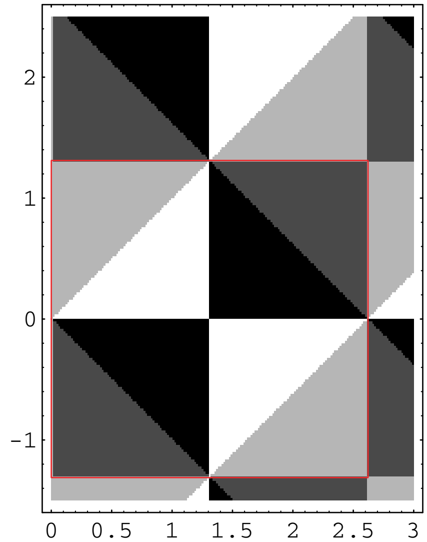

Fig. 6. Quadrant preimages for a real square lattice.

This is illustrated in Fig. 6 for $n_o = 0$. By [DuVal, 1973; Hawkins & Koss, 2002] (Lemma 4.4), we know that the lines with rectangular equations $y = x$ and all parallel lines of the form $y = x + m\lambda$, $m \in \mathbb{Z}$, all map to the non-negative imaginary axis, and lines $y = -x$, $y = -x + m\lambda$, $m \in \mathbb{Z}$, map to the nonpositive imaginary axis. The intersection of any two lines of this form are precisely the critical points of the form: $z = (m/2)(\lambda + i\lambda)$, $m \in \mathbb{Z}$; we know that these map to the origin or are poles, depending on whether m is even or odd.

We therefore consider the open square S bounded by the lines $y = x$, $y = -x$, $y = \lambda - x$, $y = -\lambda + x$, and its translation $S_{n_o} = S + n_o\lambda \subset Q$. Each square $S_m = S + m\lambda$, $m \in \mathbb{Z}$, maps two-to-one onto the open right half plane (quadrants I and IV) except at the one critical point in the center of S_m; by our construction, ∂S_m maps onto the imaginary axis and the point at ∞.

We claim that $\overline{F} \subset S_{n_o} \subset Q$. We have already shown that $F \subset S_{n_o} \subset Q$. If $\overline{F}$ contains a point in ∂S_{n_o}, then by continuity of $\wp_\Lambda$ and forward invariance of F, F contains points arbitrarily close to the line $\Re(z) = n_o\lambda$, which we have shown cannot happen. Since F is bounded, it must be a bounded distance from all poles (lattice points) and prepoles, which are the vertices of S_{n_o}. Therefore the result holds for square lattices using $G = S_{n_o}$.

We now turn to the case of a real triangular lattice with real attracting fixed point p_o. We argue as in the square lattice case; the immediate attracting basin F_{p_o} lies in quadrants I and IV. In the triangular case we have symmetry of the Fatou set about each axis, origin, and with respect to rotation through $2\pi/3$ radians. Furthermore our assumptions and Proposition 3.4 imply that there are three disjoint forward invariant Fatou components; from this we can conclude that $F = F_{p_o}$ cannot cross the imaginary axis. By Proposition 3.1 we can choose as generators of Λ the vector $\lambda e^{\pi i/3}$ and its conjugate; let Q be the period parallelogram containing $e_1 \in F$. There is a smaller interior region S in Q consisting of points sent into quadrants I and IV. Its description is more complicated than for the square case (cf. [DuVal, 1973] and Fig. 7), but its boundary is an analytic curve. The rest of the proof is the same as for the square case. Moreover, there are two other forward invariant components of $F(\wp_\Lambda)$, namely $e^{2\pi i/3}F$ and $e^{4\pi i/3}F$, and the result holds for these by symmetry. ■

In Figs. 6 and 7 we show the regions of a typical period parallelogram which get mapped under $\wp_\Lambda$ to quadrants I (the white points) and IV (the black points) for square and triangular lattices respectively. The points colored gray get mapped onto the remaining two quadrants (the left half plane).

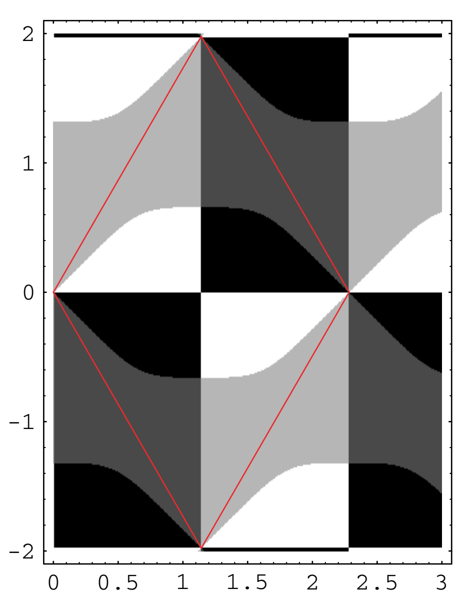

Fig. 7. Quadrant preimages for a real triangular lattice.

Theorem 5.4. *Suppose Λ is a real rectangular square lattice or a real triangular lattice with $g_3 > 0$, and suppose that $\wp_\Lambda$ has a real attracting fixed point p_o such that the forward invariant Fatou component F_{p_o} contains at most one critical value. Then $\wp_\Lambda$ restricted to the interior of an (appropriately chosen) period parallelogram containing p_o is quadratic-like.*

Proof. We apply Theorems 5.3 and 5.2. ■

Corollary 5.2. *If Λ satisfies the hypotheses of Theorem 5.4, then $F(\wp_\Lambda)$ consists of a countable union of simply connected open disks whose boundaries are quasicircles.*

Proof. This follows immediately from Theorem 5.2 and the fact that we do not have any other types of Fatou components present. ■

6. Examples of Locally Sierpinski Julia Sets

6.1. *Triangular lattices*

The space of Weierstrass elliptic $\wp$ functions with triangular period lattices was studied by the first author and Koss [2004]. It was shown there that there the hyperbolic parameters are quasiconformally stable, so the next result provides a large collection of examples.

Theorem 6.1. *If Λ is a real triangular lattice with $g_3 > 0$, and $\wp_\Lambda$ has an attracting fixed point, then $J(\wp_\Lambda)$ is a Sierpinski curve on $\mathbb{C}_\infty$ and $J(\wp_\Lambda)$ is locally Sierpinski.*

Proof. Suppose we have proved that $J(\wp_\Lambda)$ is a Sierpinski curve on $\mathbb{C}_\infty$; we now consider generators of Γ of the form $[\gamma, e^{2\pi i/3}\gamma]$, with $\gamma \in \mathbb{R}$, and use these to form the boundary of a period parallelogram Q. By Proposition 5.6 of [Hawkins & Koss, 2005] we have that the intersection of $J(\wp_\Lambda)$ with the boundary of (this particular) Q is a Cantor set. Setting $J_Q = J(\wp_\Lambda) \cap Q$, it follows from our verification of the properties in Definition 1.1 that the outer curve of J_Q is a simple closed curve homeomorphic to the graph of the Cantor function (instead of pieces of intervals between Cantor set points we have pieces of quasicircles). We then apply Remark 1.1 to the period parallelogram J_Q to obtain a locally Sierpinski curve.

We now turn to the proof that $J(\wp_\Lambda)$ is a Sierpinski curve on $\mathbb{C}_\infty$; we show the conditions of Definition 1.1 are satisfied. By Theorems 3.4 and 4.4 we have that $J(\wp_\Lambda)$ is connected and locally connected. Since the Julia set is closed and is not all of $\mathbb{C}_\infty$ we know that it is nowhere dense and compact on $\mathbb{C}_\infty$. It remains to check Property (5) of Definition 1.1.

By Corollary 3.4 there are three attracting fixed points; let us denote the attracting fixed points by p_1, p_2, and p_3, and each immediate basin of attraction by A_1, A_2, and A_3. Since p_j is an attracting fixed point and Λ is a triangular lattice, we know that no critical value tends to p_j apart from e_j. Therefore, the Fatou set of $\wp_\Lambda$ consists of A_j, $j = 1, 2, 3$ and all of the preimages.

We only need to check that the boundaries of pairwise disjoint complementary regions of $J(\wp_\Lambda)$ are disjoint simple closed curves. The boundaries of complementary regions of $J(\wp_\Lambda)$ are precisely the boundaries of the preimages of the A_j's. If ∂A_j is a simple closed curve, our hypothesis implies that its preimages will be simple closed curves as well.

By Theorems 5.3 and 5.4 we are guaranteed that Λ satisfies the hypotheses of Theorem 5.2. Therefore, by the theory of polynomial-like mappings we know that the boundary of A_j is a quasicircle and therefore a simple closed curve.

Finally we need to show that disjoint Fatou components have disjoint boundaries. Any Fatou component of $\wp_\Lambda$ is a component of $\wp_\Lambda^n(A_j)$ for some $n \le 0$ and some $j = 1, 2, 3$. Let W_1 and W_2 be two distinct Fatou components and assume that there is a point $q \in \partial W_1 \cap \partial W_2$. There are two cases:

Case 1. Both components terminate at the same fixed component A_j. Then there exists an integer $k > 0$ such that $\wp_\Lambda^k(W_1) = \wp_\Lambda^k(W_2)$ while $\wp_\Lambda^{k-1}(W_1) \neq \wp_\Lambda^{k-1}(W_2)$; i.e. $\wp_\Lambda$ takes $\wp_\Lambda^{k-1}(W_1)$ and $\wp_\Lambda^{k-1}(W_2)$, two disjoint open sets, onto the same set. Further, $\wp_\Lambda^{k-1}(q) \in \partial\wp_\Lambda^{k-1}(W_1) \cap \wp_\Lambda^{k-1}(W_2)$. Hence, $\wp_\Lambda^{k-1}(q)$ must be a critical point for $\wp_\Lambda$. But if a critical point is in $\partial\wp_\Lambda^{k-1}(W_1)$, then we have a critical value in $\partial\wp_\Lambda^{k}(W_1)$. This is not possible, however, since $\partial\wp_\Lambda^{k}(W_1) \subset J(\wp_\Lambda)$ and we know that none of our critical values are in the Julia set since $\wp_\Lambda$ is hyperbolic.

Case 2. W_1 and W_2 terminate at different fixed components, say A_1 and A_2 respectively. We note that we can relabel sets if necessary so that $e^{2\pi i/3}A_1 = A_2$. Then there exists an integer $k > 0$ such that $e^{2\pi i/3}\wp_\Lambda^k(W_1) = \wp_\Lambda^k(W_2)$ while $e^{2\pi i/3}\wp_\Lambda^{k-1}(W_1) \neq \wp_\Lambda^{k-1}(W_2)$. Then $q \in \partial W_1 \cap \partial W_2$ implies that $\wp_\Lambda^k(q) \in A_1 \cap A_2$. However it follows from the proof of Theorem 5.3 that $\overline{A}_1 \cap \overline{A}_2 = \emptyset$, so no point q exists.

Therefore, there is no point on the boundary of two pairwise disjoint complementary regions of $J(\wp_\Lambda)$. ■

A triangular lattice is characterized by $g_2 = 0$, so we can parametrize the dynamical behavior of the Weierstrass elliptic $\wp_\Lambda$ map by looking at g_3-space as g_3 ranges over the nonzero complex numbers. This was done by the first author and Koss [2004]. In Figs. 8 and 9 we color points black if there is an attracting periodic orbit. Each point in a central cardioid corresponds to lattice Λ such that the corresponding elliptic $\wp$-function has a Sierpinski Julia set. Throughout the interior of the black regions we have quadratic-like maps.

6.2. *Other examples*

We can show that there are other lattices satisfying the conditions of Proposition 4.3. Any hyperbolic map with three distinct attracting fixed points has a locally Sierpinski Julia set using the same proof as was used in the triangular case.

In order to show that square lattices with attracting fixed points have locally Sierpinski Julia

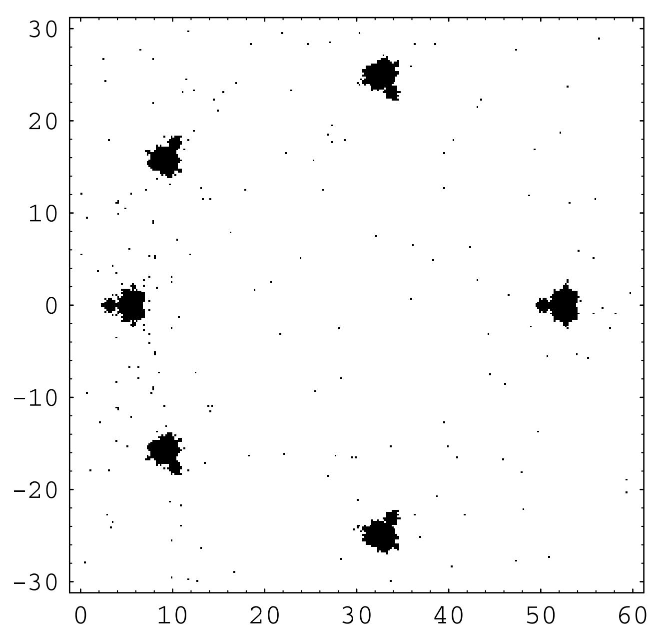

Fig. 8. g_3-space with attracting periodic orbits.

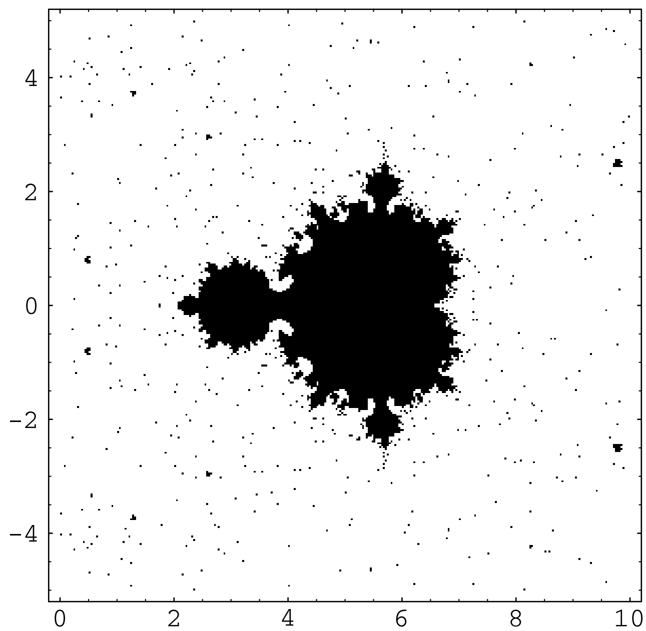

Fig. 9. A cardioid of locally Sierpinski Julia sets in g_3 space.

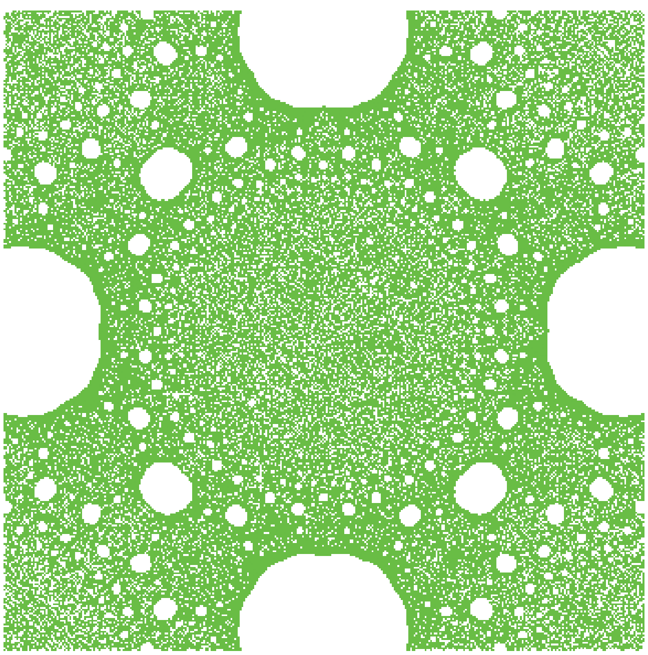

Fig. 10. A Sierpinski carpet tile for a square lattice?

sets, it remains to show that given any $\epsilon > 0$, there are at most finitely many Fatou components of radius $> \epsilon$. While we believe this to be the case (see Fig. 10), $\wp_\Lambda$ is not hyperbolic for a square lattice, so the proof given here does not apply; we leave it as a conjecture.

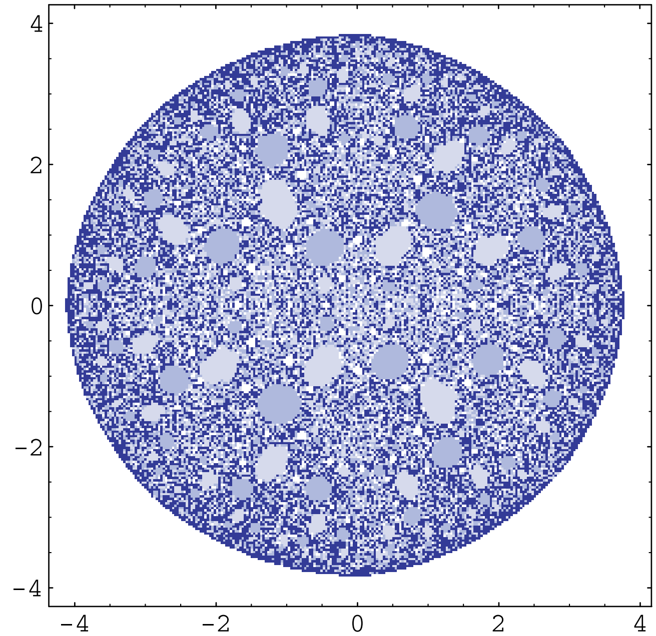

Fig. 11. A conjugate copy $J(\wp_\Lambda)$ for a triangular lattice.

6.3. *Sierpinski curves on the torus and planar $\mathcal{S}$-curves*

Suppose we have a lattice Λ for which we have a locally Sierpinski Julia set resulting from an attracting fixed point as above. We let F_o denote a forward invariant component of $F(\wp_\Lambda)$ corresponding to an attracting fixed point p_o. By mapping the fixed point p_o to ∞ via a Mobius transformation, we send the region F_o to a neighborhood of ∞. The boundary of F_o is a simple closed curve as shown above. We can then apply Remark 1.1 to obtain a planar $\mathcal{S}$-curve. This is illustrated in Fig. 11. In Fig. 11 the Julia set is bright blue, while the complementary regions are colored light blue and white according to which of the three attracting fixed points each Fatou component is attracted. One of the fixed points has been mapped to infinity and its basin of attraction is white.

Furthermore, if we identify fundamental regions for $\wp_\Lambda$, we can view $J(\wp_\Lambda)$ on $\mathbb{C}/\Lambda$, a torus, and for each example above we obtain a Sierpinski curve on T^2.

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