MATH 321: Lecture 10

When are things martingales & when can we turn non-martingales into martingales?

Theorem Let $(\Delta_n)_{0\le n\le N}$ be an adapted process, let $X_0 \in \mathbb{R}$, and $\forall\, n \in \{0,1,\ldots,N-1\}$ let

$$X_{n+1} = \Delta_n S_{n+1} + (X_n - \Delta_n S_n)(1+r).$$

Then $\left(\frac{X_n}{(1+r)^n}\right)_{0\le n\le N}$ is a martingale (under $\tilde{\mathbb{P}}$).

Proof. Fix $n \in \{0,1,\ldots,N-1\}$. We'll show $\tilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] = \frac{X_n}{(1+r)^n}$:

$$\tilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] \underset{\substack{\text{by definition}\\ \text{of } X_{n+1}}}{=} \tilde{\mathbb{E}}\left[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}} + \underbrace{\frac{X_n - \Delta_n S_n}{(1+r)^n}}_{\text{(this is a constant)}}\right]$$

by pulling out constants, what's known @ time t, & resolving $\tilde{E}_n[S_{n+1}]$

$$= \frac{\Delta_n}{(1+r)^{n+1}}\left(\tilde{p}\,u S_n + \tilde{q}\,d S_n\right) + \frac{X_n - \Delta_n S_n}{(1+r)^n}$$

$$= \Delta_n S_n \frac{\tilde{p}u + \tilde{q}d}{(1+r)^{n+1}} + \frac{X_n - \Delta_n S_n}{(1+r)^n}$$

since $\tilde{p}u + \tilde{q}d = 1+r$

$$= \frac{\Delta_n S_n + X_n - \Delta_n S_n}{(1+r)^n} = \frac{X_n}{(1+r)^n} \quad \blacksquare$$

implications of & remarks about this theorem:

<u>Corollary 1</u>. $X_0 = \tilde{\mathbb{E}}\left[\frac{X_N}{(1+r)^N}\right] = \tilde{\mathbb{E}}\left[\frac{X_n}{(1+r)^n}\right] \; \forall\, n \in \{0, 1, \ldots, N\}$.

<u>Corollary 2</u>. There is no arbitrage in the N-period binomial model.

<u>Proof</u>. Consider a strategy with $X_N(\omega) \geq 0 \; \forall \omega \in \Omega$ (so no possibility of losing money), and s.t. $\exists\; \bar{\omega} \in \Omega$ with $X_N(\bar{\omega}) > 0$ (some possibility of gaining money). To be an arbitrage strategy, this would have to have $X_0 = 0$. But:

$$X_0 = \frac{1}{(1+r)^N} \tilde{\mathbb{E}}[X_N] \geq \frac{1}{(1+r)^N}\left(X_N(\bar{\omega})\,\mathbb{P}(\bar{\omega})\right) > 0. \quad \blacksquare$$

<u>Remark</u> Consider a derivative security with payout $V_N = g(S_N)$ (for some function g). Generally, $(g(S_n))_{0 \leq n \leq N}$ and $\left(\frac{g(S_n)}{(1+r)^n}\right)_{0 \leq n \leq N}$ are <u>not</u> martingales under $\tilde{\mathbb{P}}$. Need constants $\alpha_0, \ldots, \alpha_N \neq 0$ such that $(\alpha_n\, g(S_n))_{0 \leq n \leq N}$ <u>is</u> a martingale under $\tilde{\mathbb{P}}$.

When we do, we get:

$$V_0 = \frac{1}{(1+r)^N} \tilde{\mathbb{E}}[g(S_N)] = \frac{1}{(1+r)^N} \tilde{\mathbb{E}}\left[\frac{1}{\alpha_N} \cdot \alpha_N \cdot g(S_N)\right]$$

$$= \frac{1}{\alpha_N (1+r)^N} \tilde{\mathbb{E}}[\alpha_N g(S_N)]$$

because it's a martingale $\rightarrow$ $= \dfrac{1}{\alpha_N (1+r)^N} \alpha_0 g(S_n)$

It's convenient to let $\alpha_0 = 1$ here.

Remark. $\tilde{\mathbb{P}}$ is the only measure s.t. $\forall$ wealth processes $(W_n)_{0 \le n \le N}$, $\exists$ constants $\alpha_0, \ldots, \alpha_N \neq 0$ s.t. $(\alpha_n X_n)_{0 \le n \le N}$ is a martingale.

American Deriv. Secs

- American put/call w/ exercise/expiration/maturity date N can be exercised at any time $n \in \{0,1,\dots,N\}$.

  Note: Price of Am $\geq$ Price of Euro w/ same strike & maturity, or else arbitrage.

Def If V is American, W European, securities w/ same payment functions, then $V_0 - W_0$ is called the early exercise premium.

- Bermudan option: can be exercised at any time $n \in \mathcal{E} \subseteq \{0,1,\dots,N\}$ (some "black out dates").

Example. $N=2$, $u=2$, $d=1/2$, $r=1/4$, $S_0=4$.

U = Euro. put w/ $T=2$, $K=5$

V = Amer. put " "

$$\left.\begin{aligned} U_2(HH) &= (5-16)^+ = 0 \\ U_2(HT) = U_2(TH) &= (5-4)^+ = 1 \\ U_2(TT) &= (5-1)^+ = 4. \end{aligned}\right\} \begin{aligned} U_1(H) &= \tfrac{2}{5}(1) = \tfrac{2}{5} \\ U_1(T) &= \tfrac{2}{5}(5) = 2 \end{aligned}$$

$$\Rightarrow U_0 = \frac{2}{5}\left(\frac{2}{5}+2\right) = \frac{2}{5}\left(\frac{12}{5}\right) = \frac{24}{25}.$$

Note since can make $(5-S_0)^+ = 1$ buy selling V immediately, we already have $V_0 > U_0$.

Since can exercise any time, Amer. puts have intrinsic value. Let's denote intrinsic value of V at time n by G_n. So here $G_n(\omega) = (5 - S_n(\omega))^+ \; \forall \omega$.

$$G_0 = 1$$

$$G_1(H) = 0 \qquad G_1(T) = 3$$

$$G_2(HH) = 0 \quad G_2(HT) = 1 \quad G_2(TH) = 1 \quad G_2(TT) = 4.$$

If we arrive at $t=2$ w/o having exercised V yet, we have $V_2(\omega) = G_2(\omega)$.

What's at time 1?

① Suppose $\omega_1 = H$. $G_1(H) = 0$ so won't exercise now.

$$V_1(H) = \frac{1}{1+r}\left[\tilde{p} V_2(HH) + \tilde{q} V_2(HT)\right] = \frac{2}{5}(1) = \frac{2}{5}.$$

② Suppose $\omega_1 = T$. If exercise now, get $G_1(T) = 3$.

Value of waiting is

$$\frac{1}{1+r}\left(\tilde{p} V_2(TH) + \tilde{q} V_2(TT)\right) = \frac{2}{5}(5) = 2.$$

$\Rightarrow$ should exercise at $t=1$ if $\omega_1 = T$.

$\Rightarrow V_1(T) = 3.$

$$\Rightarrow V_0 = \frac{2}{5}\left(\frac{2}{5} + 3\right) = \frac{34}{25}$$

$G_0 = 1$ so should NOT exercise at $t=0$.

$\Rightarrow$ $\boxed{\text{price is } \frac{34}{25}}$ & early exercise premium is

$$\frac{34}{25} - \frac{24}{25} = \frac{10}{25}$$