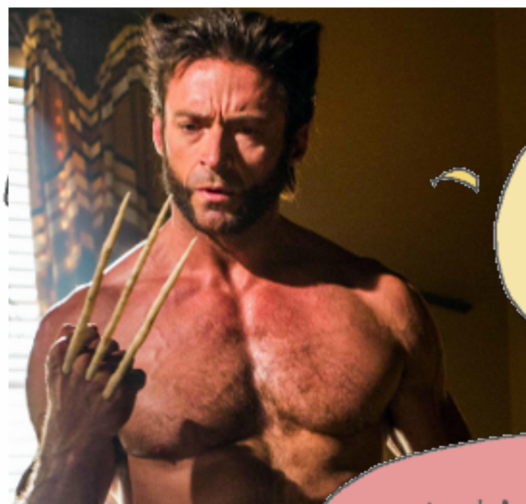
LECTURE 11

3/2

First there were European options. Puts, calls, whatever.



Then came the Americans. You still could go long or short on these.

And then there are Bermuda shorts.

Damn it, Deadpool!

Remark 1 Let $(V_n)_{0 \le n \le N}$ be the values of an American option at times $0, 1, \ldots, N$. These are NOT the capitals of a self-financing strategy!

Remark 2 $\left(\frac{V_n}{(1+r)^n}\right)_{0 \le n \le N}$ is a SUPERMARTINGALE for American securities V (because holder may get unlucky & fail to exercise at optimal date).

But during any period of time when it's not advantageous to exercise, it behaves like a MARTINGALE.

Example from page 42, revisited:

Consider a broker who sells one such American put V, and hedges the short position. She needs to be able 2/5 at $t=1$ if $\omega_1 = H$ and 3 if $\omega_1 = T$.

So she wants

$$\Delta_0 = \frac{2/5 - 3}{8 - 2} = -\frac{13}{30}$$

shares of stock in her portfolio at $t=0$, and $\frac{34}{25} + \frac{13}{30}(4) = 232/75$ in the bank at $t=0$.

Case 1: If $\omega_1 = H$, the portfolio is worth

$$(232/75)(5/4) - (13/30)(8) = 2/5.$$

Need to readjust portfolio so it's worth 0 if $\omega_2 = H$ and 1 if $\omega_2 = T$:

$$\Delta_1 = \frac{0-1}{16-4} = -1/12$$

So amount in the bank is

$$2/5 + (1/12)(8) = 16/15.$$

at $t=1$.

<u>Case 2</u>. If $\omega_1 = T$, portfolio is worth

$$(232/75)(5/4) - (13/30)(2) = 3$$

If holder exercises at $t=1$, broker pays 3 — done. If he doesn't, then she adjusts her portfolio so it's worth 1 at $t=2$ if $\omega_2 = H$ & 4 if $\omega_2 = T$.

→ We need $\frac{2}{5}\left(V_2(4) + V_2(1)\right) = 2$ at $t=1$.

But she has \$3! So she can safely consume \$1 at $t=1$.

"Possibility of consumption" is what makes it a supermartingale. ∎

Challenges of analyzing American securities:

- Time at which payoff is made is initially not known.
- Also depends on holder's preference

(e.g. if intrinsic value at some time n, is equal to the discounted risk-neutral value, risk-averse holder may want to exercise while risk-seeking holder may prefer to wait).

⇝ We'll temporarily eliminate this consideration to get a better handle on American options. This brings us to...

Def. · **Derivatives w/ random maturity** are securities w/ a random payment date (but same payment date for all holders).

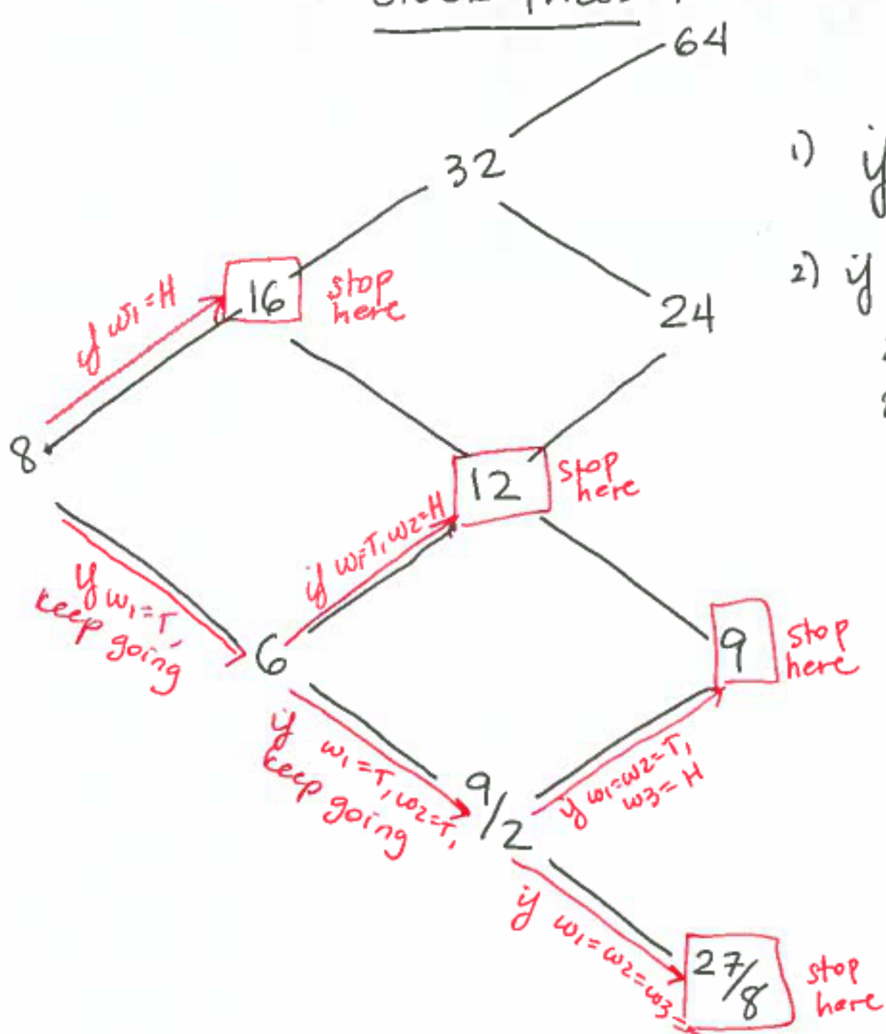
· **Maturity**: Smallest time such that it's known that holder receives no future payment. (We'll see more in our discussion of "stopping times.")

Example $N=3,\ u=2,\ d=3/4,\ r=3/8,\ S_0=8$

let V be a "rebate option": it pays \$1 the first time stock crosses upper barrier $K=9$, or expires at $t=3$ if this never happens.

(So here, maturity $= \min\{n \mid S_n \geq 9 \text{ or } n=3\}$.)

Stock Prices:



We have:

1) if $\omega_1=H \rightsquigarrow$ get \$1, done

2) if $\omega_1=T \rightarrow$

2a) if $\omega_2=H \rightsquigarrow$ get \$1, done

2b) if $\omega_2=T \rightarrow$

2bi) if $\omega_3=H \rightarrow$ get \$1

if $\omega_3=T \rightarrow$ get \$0

So $V_1(H)=1$

$V_2(TH)=1$

$V_2(TTH)=1$

$V_2(TTT)=0$

Note $V_2(TT) = \frac{1}{1+r}\left[1\cdot\tilde{p} + 0\cdot\tilde{q}\right] = \frac{8}{11}\cdot\frac{1}{2} = \frac{4}{11}$

$V_2(TH) = 1$

$V_1(T) = \frac{8}{11}\left[1\cdot\frac{1}{2} + \frac{4}{11}\cdot\frac{1}{2}\right] = \frac{60}{121}$

$V_1(H) = 1$

So $V_0 = \frac{8}{11}\left[\frac{60}{121}\cdot\frac{1}{2} + 1\cdot\frac{1}{2}\right] \approx \boxed{\$0.544}$

§4.3 This is an example of a "stopping rule": some rule that says when the security matures.

Formally:

<u>Def</u> By a stopping time or stopping rule in an N-period binomial model, we mean a random variable τ on Ω such that

it's a time

(i) $\tau(\omega) \in \{0, 1, \dots, N\} \quad \forall\, \omega \in \Omega$

(ii) $\forall\, n \in \{0, 1, \dots, N\}$, $\forall\, \omega_1, \dots, \omega_N \in \{H, T\}$,

if you stopped at n, doesn't matter what coin flips come after

if $\tau(\omega_1, \dots, \omega_n, \omega_{n+1}, \dots, \omega_N) = n$ then $\tau(\omega_1, \dots, \omega_n, \hat{\omega}_{n+1}, \dots, \hat{\omega}_N) = n$ for all $\hat{\omega}_{n+1}, \dots, \hat{\omega}_N \in \{H, T\}$

<u>Def</u>. A derivative security with random maturity ("DSwRM") V is characterized by

(a) a stopping rule τ, and
(b) a payment function $V_* : \Omega \to \mathbb{R}$.

(So we can write $V = (\tau, V_*)$.)

→ Holder receives $V_*(\omega)$ at time $\tau(\omega)$.

→ Note that since payment function cannot look into the future, $\forall\, n \in \{0, 1, \dots, N\}$ and $\forall\, \omega_1, \dots, \omega_N \in \{H, T\}$ and $\forall\, 0 \le n \le N$, if $\tau(\omega_1, \dots, \omega_N) = n$ then

$$V_*(\omega_1, \dots, \omega_n, \omega_{n+1}, \dots, \omega_N) = V_*(\omega_1, \dots, \omega_n, \hat{\omega}_{n+1}, \dots, \hat{\omega}_N)$$

$\forall\, \hat{\omega}_{n+1}, \dots, \hat{\omega}_N \in \{H, T\}$.

Back to the rebate option example:
Can summarize in this table:

ω	$\tau(\omega)$	$V_*(\omega)$
H,H,H	1	\$1
H,H,T	1	\$1
H,T,H	1	\$1
H,T,T	1	\$1
T,H,H	2	\$1
T,H,T	2	\$1
T,T,H	3	\$1
T,T,T	3	\$0

For DSwRM, it's convenient to define $V_{\tau(\omega)}(\omega) = V_*(\omega)$ $\forall\, \omega \in \Omega$.

Proposition Let V be a DSwRM, $V = (\tau, V_*)$.
Then there is a replicating strategy:
For $n \le \tau(\omega)$ capital $X_n(\omega)$ is uniquely determined by backward induction:

(i) $X_N(\omega) = V_N(\omega)$ $\forall\, \omega \in \Omega$

(ii) $\forall\, n \in \{0, 1, \ldots, N-1\}$,

$$X_n(\omega_1, \ldots, \omega_n) = \begin{cases} V_n(\omega_1, \ldots, \omega_n) & \text{if } \tau(\omega) = n \\ \frac{1}{1+r}\left[\tilde{p}\, X_{n+1}(\omega H) + \tilde{q}\, X_{n+1}(\omega T)\right] & \text{if } \tau(\omega) > n \end{cases}$$

Pricing DSwRM's:

Let $V = (\tau, V_*)$ be a DSwRM.

$\forall \omega \in \Omega$, $V_*(\omega)$ is deposited at time $\tau(\omega)$.

Value of this deposit at time N is

$$V_*(\omega)\,(1+r)^{N-\tau(\omega)}$$

that's how long it stays in bank

So $$V_0 = \frac{1}{(1+r)^N}\,\tilde{\mathbb{E}}\left[(1+r)^{N-\tau}\,V_*\right]$$

$$= \tilde{\mathbb{E}}\left[\frac{V_*}{(1+r)^{\tau}}\right]$$

Example, cont'd.

$$\mathbb{P}(\omega) = 1/8 \quad \forall\, \omega \in \Omega = \{H,T\}^3$$

$$V_0 = \tilde{\mathbb{E}}\left[\frac{V_*}{(1+r)^{\tau}}\right] = \frac{1}{8}\left[(4)(1)\cdot\frac{1}{1+r} + (2)(1)\,\frac{1}{(1+r)^2} + (1)\frac{1}{(1+r)^3}\right]$$

$$= \frac{1}{8}\left[4\left(\frac{8}{11}\right) + 2\left(\frac{8}{11}\right)^2 + \left(\frac{8}{11}\right)^3\right] \approx \$0.544$$