§4.3 This is an example of a "stopping rule": some rule that says when the security matures.

Formally:

<u>Def</u> By a stopping time or stopping rule in an N-period binomial model, we mean a random variable τ on Ω such that

it's a time

(i) $\tau(\omega) \in \{0, 1, \ldots, N\} \quad \forall\, \omega \in \Omega$

(ii) $\forall\, n \in \{0, 1, \ldots, N\}$, $\forall\, \omega_1, \ldots, \omega_N \in \{H, T\}$,

if you stopped at n, doesn't matter what coin flips come after

if $\tau(\omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots, \omega_N) = n$ then $\tau(\omega_1, \ldots, \omega_n, \hat{\omega}_{n+1}, \ldots, \hat{\omega}_N) = n$ for all $\hat{\omega}_{n+1}, \ldots, \hat{\omega}_N \in \{H, T\}$

<u>Def.</u> A derivative security with random maturity ("DSwRM") V is characterized by

(a) a stopping rule τ, and
(b) a payment function $V_*: \Omega \to \mathbb{R}$.

(So we can write $V = (\tau, V_*)$.)

→ Holder receives $V_*(\omega)$ at time $\tau(\omega)$.

→ Note that since payment function cannot look into the future, $\forall\, n \in \{0, 1, \ldots, N\}$ and $\forall\, \omega_1, \ldots, \omega_N \in \{H, T\}$ and $\forall\, 0 \le n \le N$, if $\tau(\omega_1, \ldots, \omega_N) = n$ then

$$V_*(\omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots, \omega_N) = V_*(\omega_1, \ldots, \omega_n, \hat{\omega}_{n+1}, \ldots, \hat{\omega}_N)$$

$\forall\, \hat{\omega}_{n+1}, \ldots, \hat{\omega}_N \in \{H, T\}$.

Back to the rebate option example:
Can summarize in this table:

ω	$\tau(\omega)$	$V_*(\omega)$
H,H,H	1	\$1
H,H,T	1	\$1
H,T,H	1	\$1
H,T,T	1	\$1
T,H,H	2	\$1
T,H,T	2	\$1
T,T,H	3	\$1
T,T,T	3	\$0

For DSwRM, it's convenient to define $V_{\tau(\omega)}(\omega) = V_*(\omega)$ $\forall\, \omega \in \Omega$.

Proposition Let V be a DSwRM, $V = (\tau, V_*)$.
Then there is a replicating strategy:
For $n \le \tau(\omega)$ capital $X_n(\omega)$ is uniquely determined by backward induction:

(i) $X_N(\omega) = V_N(\omega)$ $\forall\, \omega \in \Omega$

(ii) $\forall\, n \in \{0, 1, \dots, N-1\}$,

$$X_n(\omega_1, \dots, \omega_n) = \begin{cases} V_n(\omega_1, \dots, \omega_n) & \text{if } \tau(\omega) = n \\ \frac{1}{1+r}\left[\tilde{p} X_{n+1}(\omega H) + \tilde{q} X_{n+1}(\omega T)\right] & \text{if } \tau(\omega) > n \end{cases}$$

Pricing DSwRM's:

Let $V=(\tau, V_*)$ be a DSwRM.

$\forall \omega \in \Omega$, $V_*(\omega)$ is deposited at time $\tau(\omega)$.

Value of this deposit at time N is

$$V_*(\omega)\,(1+r)^{N-\tau(\omega)}$$

that's how long it stays in bank

So $$\boxed{V_0 = \frac{1}{(1+r)^N}\,\tilde{\mathbb{E}}\left[(1+r)^{N-\tau}\,V_*\right] = \tilde{\mathbb{E}}\left[\frac{V_*}{(1+r)^{\tau}}\right]}$$

Example, cont'd.

$$\mathbb{P}(\omega) = 1/8 \quad \forall\, \omega \in \Omega = \{H,T\}^3$$

$$V_0 = \tilde{\mathbb{E}}\left[\frac{V_*}{(1+r)^{\tau}}\right] = \frac{1}{8}\left[(4)(1)\cdot\frac{1}{1+r} + (2)(1)\,\frac{1}{(1+r)^2} + (1)(1)\,\frac{1}{(1+r)^3}\right]$$

$$= \frac{1}{8}\left[4\left(\frac{8}{11}\right) + 2\left(\frac{8}{11}\right)^2 + \left(\frac{8}{11}\right)^3\right] \approx \boxed{\$0.544}$$

Exercise. Consider a 2-period binomial model with $u=2$, $d=\frac{1}{2}$, $r=\frac{1}{4}$; a stock is trading at $S_0=4$. Let V be an American put w/ $T=2, K=5$.

a) Define $\tau(\omega) = \min\{k \mid V_k(S_k) = (5-S_k)^+\}$.

Is τ a stopping time? If not, why not? If so, describe τ in words, and find $\tau(\omega)$ for each $\omega \in \Omega = \{H,T\}^2$.

b) Define $\rho(\omega) = \min\{k \mid S_k(\omega) = \min_{0\le j\le 2} S_j(\omega)\}$.
Is ρ a stopping time? If not, why not? If so, describe τ in words, and find $\rho(\omega)$ for each $\omega \in \Omega = \{H,T\}^2$.

(For solutions, try Questions 111-113 on OHP.)

Example

A non-example of a stopping rule

$\rho(HH)=1, \ \rho(HT)=1, \ \rho(TH)=1, \ \rho(TT)=2.$

Why isn't it a stopping rule?

→ if $\omega_1 = H$, stop at $t=1$. This is ok.
But if $\omega_1 = T$, you don't know if you should stop at $t=1$ or $t=2$ without looking into the future, which a stopping rule can't do! ■

§4.4: General American Derivative Securities

OK, so derivatives with random maturity have the same maturity for each holder. How do we deal with American derivatives, where we (as holders) can choose exercise time?

Def. An optimal exercise policy/rule for an American option is a stopping rule τ^* such that

(intrinsic values, discounted)

$$\mathcal{P}_0(\tau^*) \geq \mathcal{P}_0(\tau)$$

$\forall$ stopping rules τ, where $\mathcal{P}_0(\tau) = \tilde{\mathbb{E}}\left[\frac{G_\tau}{(1+r)^\tau}\right]$

(arbitrage free price of the security w/ stopping time τ)

Rmk The arbitrage-free price of the American option is thus

$$\mathcal{P}_0^* = \mathcal{P}_0(\tau^*)$$

for any optimal exercise policy τ^*.

i.e. $$\mathcal{P}_0^* = \tilde{\mathbb{E}}\left[\frac{G_{\tau^*}}{(1+r)^{\tau^*}}\right] = \max_{\tau \in \mathcal{S}_0} \tilde{\mathbb{E}}\left[\frac{G_\tau}{(1+r)^\tau}\right]$$

where $\mathcal{S}_0$ = collection of all stopping rules.

Remark re: notation

Note that we aren't calling the price V_0 yet. This is because we reserve "V_0" for the $t=0$ value of the American backward induction algorithm. We have to prove $V_0 = P_0^*$.

Surprising(?) note re: American calls

We'll see in §4.5 that for American calls, $\tau = N$ is always optimal, so there is no early exercise premium. That's why we'll (largely) ignore them.

Prop. Define the adapted process $(V_n)_{0 \le n \le N}$ by

(i) $G_N(\omega) = V_N(\omega)$

(ii) $\forall\, n < N$, $V_n(\omega) = \max\left\{ G_n(\omega), \frac{1}{1+r} \tilde{\mathbb{E}}_n[V_{n+1}] \right\}$.

Then V_0 = (arbitrage-free) price of the option at $t=0$, and the random variable τ_* defined by

$$\tau^*(\omega) = \min \left\{ n \in \{0,1,\ldots,N\} \;\middle|\; G_n(\omega) = V_n(\omega) \right\}$$

is an optimal exercise policy.

Rmk

$$\frac{V_n}{(1+r)^n} = \frac{1}{(1+r)^n} \max\left\{ G_n, \frac{1}{1+r} \tilde{\mathbb{E}}_n[V_{n+1}] \right\} \geqslant \frac{1}{(1+r)^{n+1}} \tilde{\mathbb{E}}_n[V_{n+1}]$$

so $\left(\frac{V_n}{(1+r)^n}\right)_{0 \le n \le N}$ is a supermartingale.