Remark re: notation

Note that we aren't calling the price V_0 yet. This is because we reserve "V_0" for the $t=0$ value of the American backward induction algorithm. We have to prove $V_0 = P_0^*$.

Surprising(?) note re: American calls

We'll see in §4.5 that for American calls, $\tau = N$ is always optimal, so there is <u>no</u> early exercise premium. That's why we'll (largely) ignore them.

Prop. Define the adapted process $(V_n)_{0\le n\le N}$ by

$$\text{(i)} \quad G_N(\omega) = V_N(\omega)$$

$$\text{(ii)} \quad \forall\, n<N,\ V_n(\omega) = \max\left\{G_n(\omega),\ \frac{1}{1+r}\tilde{\mathbb{E}}_n[V_{n+1}]\right\}.$$

Then V_0 = (arbitrage-free) price of the option at $t=0$, and the random variable τ_* defined by

$$\tau^*(\omega) = \min\left\{ n\in\{0,1,\dots,N\} \;\middle|\; G_n(\omega) = V_n(\omega)\right\}$$

is an optimal exercise policy.

Rmk

$$\frac{V_n}{(1+r)^n} = \frac{1}{(1+r)^n}\max\left\{G_n,\ \frac{1}{1+r}\tilde{\mathbb{E}}_n[V_{n+1}]\right\} \geq \frac{1}{(1+r)^{n+1}}\tilde{\mathbb{E}}_n[V_{n+1}]$$

so $\left(\frac{V_n}{(1+r)^n}\right)_{0\le n\le N}$ is a supermartingale.

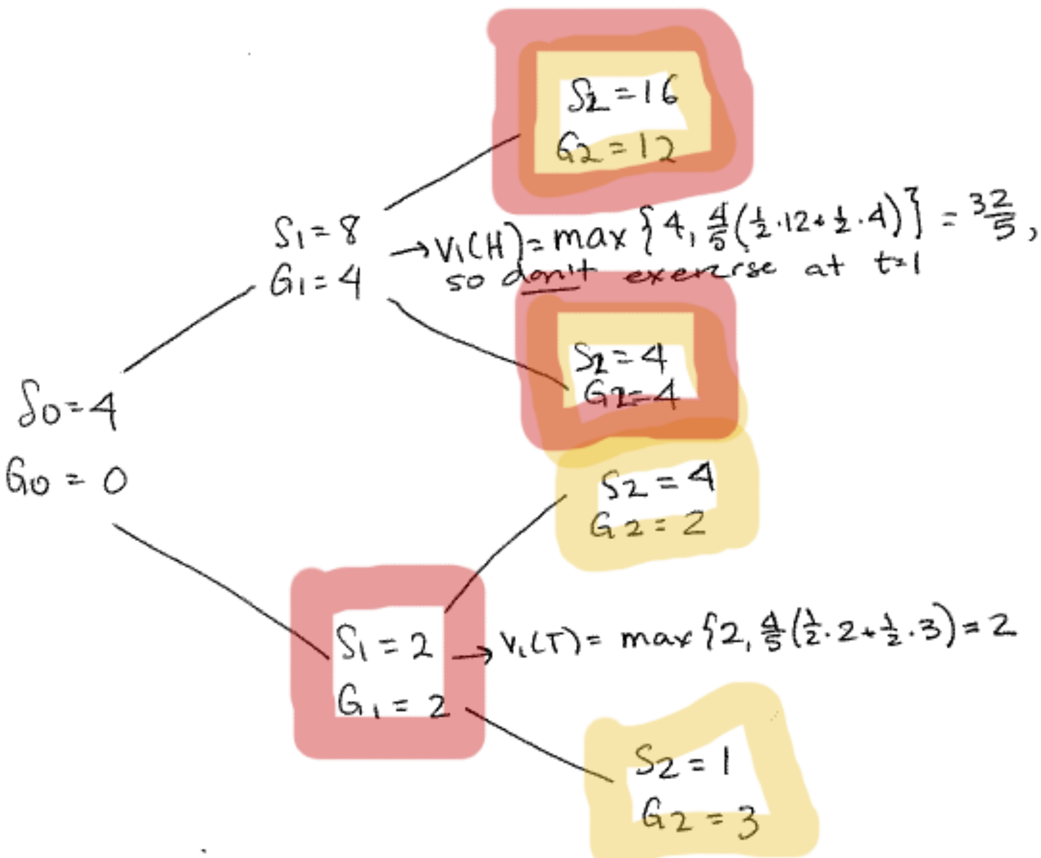
Example: An option with options.

$N=2,\ u=2,\ d=1/2,\ r=1/4,\ S_0=4.$

let $M_n(\omega) = \max_{0\le k\le n} S_n(\omega)$, $L_n(\omega) = \min_{0\le k\le n} S_n(\omega)$.

let $G_n(\omega) = M_n(\omega) - L_n(\omega)$

let V be an American lookback option (w/ intrinsic values G_n). Find V_0.



So by proposition on p.53, τ^* defined by

$$\tau^*(HH) = \tau^*(HT) = 2$$
$$\tau^*(TH) = \tau^*(TT) = 1$$

is an optimal exercise policy.

But $\rho^*(\omega) = 2\ \forall\, \omega \in \{H,T\}^2$ is also an optimal exercise policy. (So they needn't be unique.)

Example Find all the optimal exercise policies for the American derivative security V which has intrinsic values $G_n = S_n$ $\forall n \in \{0, \dots, N\}$.

Note that $V_N = S_N$, and proceed inductively: assume $V_{n+1} = S_{n+1}$, and compute V_n.

$$V_n(\omega_1, \dots, \omega_n) = \max\left\{ G_n, \frac{1}{1+r}\left(\tilde{p}\, V_{n+1}(\omega_1, \dots, \omega_n, H) + \tilde{q}\, V_{n+1}(\omega_1, \dots, \omega_n, T)\right)\right\}$$

$$= \max\left\{ S_n, \frac{1}{1+r}\left(\tilde{p}\, S_{n+1}(\omega_1, \dots, \omega_n, H) + \tilde{q}\, S_{n+1}(\omega_1, \dots, \omega_n, T)\right)\right\}$$

$$= \max\{ S_n, S_n \} = S_n(\omega_1, \dots, \omega_n)$$

So $V_n = S_n$ for all n. This means <u>ANY</u> exercise policy is optimal! In other words,

$$V_0 = S_0 = \tilde{\mathbb{E}}\left[\frac{S_\tau}{(1+r)^\tau}\right] = \tilde{\mathbb{E}}\left[\frac{G_\tau}{(1+r)^\tau}\right]$$

for <u>all</u> stopping rules τ. ■