Goal: prove $V_n = \mathcal{P}_n^*$ where

$$\mathcal{P}_n^* = \max_{\substack{\tau / \tau(\omega) \geq n \\ \forall \omega}} \tilde{\mathbb{E}}_n\left[\frac{G_\tau}{(1+r)^{\tau-n}}\right].$$

(This will prove $\mathcal{P}_0^* = V_0$.)

Lemma 1. $(V_n)_{0 \leq n \leq N}$ satisfies

(i) $V_n \geq G_n \quad \forall n$

(ii) $\left(\frac{V_n}{(1+r)^n}\right)_{0 \leq n \leq N}$ is a supermartingale

Lemma 2. $(\mathcal{P}_n^*)_{0 \leq n \leq N}$ satisfies

(i) $\mathcal{P}_n^* \geq G_n \quad \forall n$

(ii) $\left(\frac{\mathcal{P}_n^*}{(1+r)^n}\right)_{0 \leq n \leq N}$ is a supermartingale

Lemma 3 For all adapted processes $(Y_n)_{0 \leq n \leq N}$ satisfying

(i) $Y_n \geq G_n \quad \forall n$

(ii) $\left(\frac{Y_n}{(1+r)^n}\right)_{0 \leq n \leq N}$ is a supermartingale,

$Y_n \geq \mathcal{P}_n^* \quad \forall n.$

$\rightarrow$ Corollary 3 $V_n \geq \mathcal{P}_n^* \quad \forall n.$

<u>Lemma 4</u>. For all adapted processes $(W_n)_{0 \le n \le N}$ satisfying

(i) $W_n \geq G_n \ \forall n$

(ii) $\left(\frac{W_n}{(1+r)^n}\right)_{0 \le n \le N}$ is a supermartingale,

$V_n \le W_n \ \forall n$.

$\to$ <u>Corollary 4</u> $V_n \le \mathcal{P}_n^* \ \forall n$.

So putting Corollaries 3 & 4 together says

<u>Theorem</u> $V_n = \mathcal{P}_n^* \ \forall n$.

OK, now we prove these thangs.

Lemma 1 was already proven earlier, so — check. ✓

<u>Proof of Lemma 2</u>. Fix $n \in \{0, 1, \ldots, N\}$.

(i) Let $\hat{\tau}$ be the stopping rule $\hat{\tau}(\omega) = n \ \forall \omega \in \Omega$.

Then $\mathcal{P}_n^* \geq \tilde{\mathbb{E}}_n\left[\frac{G_{\hat{\tau}}}{(1+r)^{\hat{\tau}-n}}\right] = G_n$.

(ii) Let τ^* be <u>any</u> stopping rule with $\tau^*(\omega) \geq n+1 \ (\forall \omega)$,

such that $\mathcal{P}_{n+1}^* = \tilde{\mathbb{E}}_{n+1}\left[\frac{G_{\tau^*}}{(1+r)^{\tau^*-(n+1)}}\right]$.

Since $\tau^*(\omega) \geq n \ \forall \omega \in \Omega$, we have

$$\mathcal{P}_n^* = \max_{\substack{\tau \mid \tau(\omega) \geq n \\ \forall \omega \in \Omega}} \tilde{\mathbb{E}}_n\left[\frac{G_\tau}{(1+r)^{\tau-n}}\right] \geq \tilde{\mathbb{E}}_n\left[\frac{G_{\tau^*}}{(1+r)^{\tau^*-n}}\right]$$

$$= \tilde{\mathbb{E}}_n\left[\frac{1}{1+r}\tilde{\mathbb{E}}_{n+1}\left[\frac{G_{\tau^*}}{(1+r)^{\tau^*-(n+1)}}\right]\right] = \tilde{\mathbb{E}}_n\left[\frac{\mathcal{P}_{n+1}^*}{1+r}\right]$$

$$\Rightarrow \frac{P_n^*}{(1+r)^n} \geq \tilde{\mathbb{E}}_n\left[\frac{P_{n+1}^*}{(1+r)^{n+1}}\right]. \blacksquare$$

<u>Proof of Lemma 3.</u> Fix $n \in \{0,1,\dots,N\}$, τ such that $\tau(\omega) \geq n \;\forall\, \omega$.

Let $(Y_n)_{0 \le n \le N}$ be a process satisfying the hypotheses of the lemma. Then

$$\frac{Y_n}{(1+r)^n} \geq \tilde{\mathbb{E}}_n\left[\frac{Y_\tau}{(1+r)^\tau}\right] \geq \tilde{\mathbb{E}}_n\left[\frac{G_\tau}{(1+r)^\tau}\right].$$

Therefore,

$$\frac{Y_n}{(1+r)^n} = \max_{\substack{\tau \mid \tau(\omega) \geq n \\ \forall \omega \in \Omega}} \frac{Y_n}{(1+r)^n} \geq \max_{\substack{\tau \mid \tau(\omega) \geq n \\ \forall \omega \in \Omega}} \tilde{\mathbb{E}}_n\left[\frac{Y_\tau}{(1+r)^\tau}\right]$$

$$= \frac{P_n^*}{(1+r)^n}.$$

So $Y_n \geq P_n^*$ for all n. $\blacksquare$

<u>Proof of Lemma 4.</u> Let $(W_n)_{0 \le n \le N}$ satisfy the hypotheses of the lemma, and let's proceed by induction. (The backward kind.)

<u>Base case:</u> $V_N = G_N \leq W_N$.

<u>Ind. Step:</u> Fix $0 \le n \le N-1$ and suppose $V_{n+1} \leq W_{n+1}$.

(We'll show $V_n \leq W_n$.):

$$\tilde{\mathbb{E}}_n[V_{n+1}] \underset{\substack{\uparrow \\ \text{Ind. hypothesis}}}{\leq} \tilde{\mathbb{E}}_n[W_{n+1}] \underset{\substack{\uparrow \\ \text{by (ii)}}}{\leq} W_n(1+r),$$

and $V_n = \max\left\{G_n, \frac{\tilde{\mathbb{E}}_n[V_{n+1}]}{1+r}\right\} \leq \max\{G_n, W_n\} = W_n$.

So $V_n \leq W_n$, as desired. $\blacksquare$

<u>Theorem</u>. Let τ^* be the random variable defined by

$$\tau^*(\omega) = \min\left\{n \in \{0,1,\ldots,N\} \,\middle|\, V_n(\omega) = G_n(\omega)\right\}$$

$\forall\, \omega \in \Omega$.

τ^* is an optimal exercise policy.

(In other words, exercising as soon as the intrinsic value is at least as high as the discounted risk-neutral expected value of holding the security another period is an optimal policy.)