# §3: STATE PRICES

We've been mostly ignoring the reference (or "actual" probability measure, $\mathbb{P}$, since we've been using the risk-neutral (or "pricing") measure $\widetilde{\mathbb{P}}$ for all our pricing needs. Can we price stuff using the reference measure?

⇝ Yes! But we need to do something first.

<u>Def</u>. The Radon-Nikodým derivative of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$ is given by

$$Z(\omega) = \frac{\widetilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)} \quad \text{for all } \omega \in \Omega.$$

<u>Theorem</u> Suppose that $\mathbb{P}(\omega)$ and $\widetilde{\mathbb{P}}(\omega)$ are positive for all $\omega \in \Omega$. Then

1) $\mathbb{P}(Z > 0) = 1$
2) $\mathbb{E}[Z] = 1$
3) For all RV's $Y$, $\widetilde{\mathbb{E}}[Y] = \mathbb{E}[ZY]$.

"Radon-Nikodým theorem"

<u>Proof</u>. 1) $Z(\omega) = \dfrac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)} > 0$ for all $\omega \in \Omega$. //

2) $$\mathbb{E}[\,Z\,] = \sum_{\omega\in\Omega} \frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)} \mathbb{P}(\omega)$$ by definition

$$= \sum_{\omega\in\Omega} \tilde{\mathbb{P}}(\omega)$$ by cancellation, since $\mathbb{P}(\omega) > 0 \ \forall \omega$

$$= 1$$ since $\tilde{\mathbb{P}}$ is a probability measure //

3) $$\tilde{\mathbb{E}}[Y] = \sum_{\omega} Y(\omega)\tilde{\mathbb{P}}(\omega) =$$

$$= \sum_{\omega} Y(\omega) \frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)} \mathbb{P}(\omega)$$

$$= \sum_{\omega} Y(\omega) Z(\omega) \mathbb{P}(\omega)$$

$$= \mathbb{E}[ZY] \quad // \quad \blacksquare$$

Let's use the Radon-Nikodým derivative to price stuff:

<u>Example</u>. $N=3$, $u=2$, $d=1/2$, $r=1/4$
$p=2/3$, $q=1/3$
$S_0 = 4$.

Let $V$ be a lookback option with $V_3 = \max_{0\le n\le 3} S_n - S_3$.

So then $$V_0 = \underbrace{\tilde{\mathbb{E}}\left[\frac{V_3}{(1+r)^3}\right]}_{\text{regular old backwards analysis stuff}} = \underbrace{\mathbb{E}\left[\frac{Z V_3}{(1+r)^3}\right]}_{\text{Radon-Nikodým theorem.}}$$

The probabilities of the different $t=3$ situations are:

$$\mathbb{P}(\omega) = \left(\frac{2}{3}\right)^3 = \frac{8}{27} \quad \text{if } \omega = HHH$$

$$\mathbb{P}(\omega) = \left(\frac{2}{3}\right)^2 \frac{1}{3} = \frac{4}{27} \quad \text{if } \omega \text{ contains exactly 2 H's}$$

(i.e. if $\omega = HHT$, $HTH$, or $THH$)

$$\mathbb{P}(\omega) = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 = \frac{2}{27} \quad \text{if } \omega \text{ contains exactly 1 H}$$

(i.e. if $\omega = HTT$, $THT$, or $TTH$)

$$\mathbb{P}(\omega) = \left(\frac{1}{3}\right)^3 = \frac{1}{27} \quad \text{if } \omega = TTT$$

& $\tilde{\mathbb{P}}(\omega) = 1/8$ for all $\omega \in \{H,T\}^3$. So we get

$$Z(\omega) = \frac{1/8}{8/27} = 27/64 \quad \text{if } \omega = HHH$$

$$Z(\omega) = \frac{1/8}{4/27} = 27/32 \quad \text{if } \omega = HHT, HTH, \text{ or } THH$$

$$Z(\omega) = \frac{1/8}{2/27} = 27/16 \quad \text{if } \omega = HTT, THT, \text{ or } TTH$$

$$Z(\omega) = \frac{1/8}{1/27} = 27/8 \quad \text{if } \omega = TTT$$

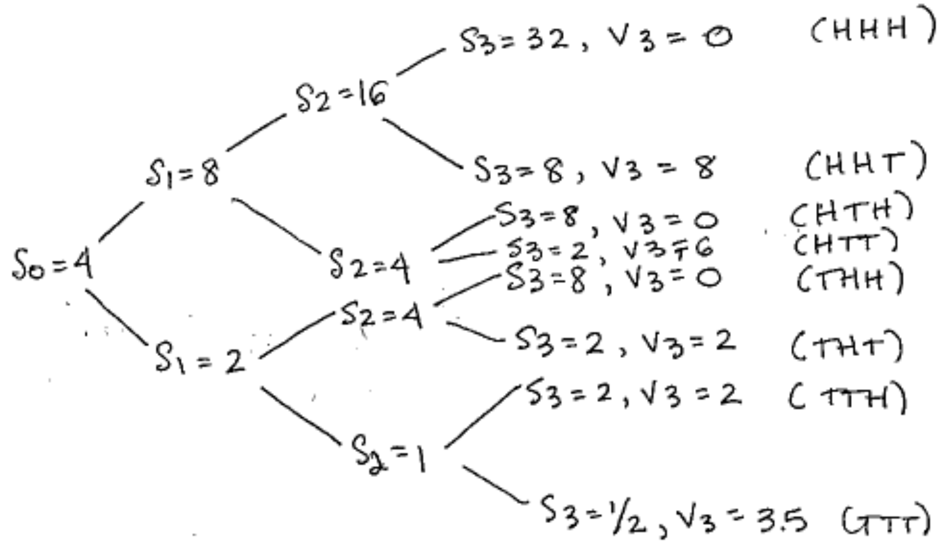
So
$$\mathbb{E}\left[\frac{ZV_3}{(1+r)^3}\right] = \frac{1}{(1+r)^3}\left[\frac{8}{27}\cdot\frac{27}{64}\cdot 0 + \frac{4}{27}\cdot\frac{27}{32}\cdot(8+0+0) + \frac{2}{27}\cdot\frac{27}{16}\cdot(6+2+2) + \frac{1}{27}\cdot\frac{27}{8}\cdot 3.5\right]$$

$$= \frac{1}{(1+r)^3}\left[\frac{1}{8}\cdot 0 + \frac{1}{8}\cdot(8+0+0) + \frac{1}{8}(6+2+2) + \frac{1}{8}(3.5)\right]$$

$$= \tilde{\mathbb{E}}\left[\frac{V_3}{(1+r)^3}\right]$$

(see tree on next page if you're unsure where these numbers come from)

(Numerically, both equal $\left(\frac{4}{5}\right)^3\left(\frac{1}{8}\right)\left(\frac{43}{2}\right) = 1.376$.)




How are these things used in finance?

<u>Def</u> The state price density is defined as $\xi(\omega)=\dfrac{Z(\omega)}{(1+r)^N}$ (the Greek letter "xi")

(price per unit of probability).

The state price is $\xi(\omega)\,\mathbb{P}(\omega)$.

↳ what does a state price tell us?

Idea: this is the price of a certain state ($\omega$) of the world.

<u>Example</u>.

Let $V$ be a security which pays \$1 if a particular outcome $\bar{\omega}$ (of coin flips) occurs, and \$0 otherwise.

$$V_0=\frac{1}{(1+r)^N}\tilde{\mathbb{E}}[V_N]=\frac{\tilde{\mathbb{P}}(\bar{\omega})}{(1+r)^N}=\frac{Z(\bar{\omega})\,.\,\mathbb{P}(\bar{\omega})}{(1+r)^N}=\xi(\bar{\omega})\,\mathbb{P}(\bar{\omega}).$$

↳ So the initial price of the security which pays \$1 if (and only if) $\bar{\omega}$ occurs, is the <u>state price</u> of that state of the world.

(In general, $V_0=\tilde{\mathbb{E}}\left[\frac{V_N}{(1+r)^N}\right]=\mathbb{E}[\xi V_N]$.)

**Theorem** $(Z_n)_{0 \le n \le N}$ is a martingale under $\mathbb{P}$, where for all $n = 0, 1, \ldots, N$, $Z_n = \mathbb{E}_n[Z]$. (for <u>any</u> RV $Z$!)

**Proof**: Let $n \in \{0, 1, \ldots, N-1\}$.

$$\mathbb{E}_n[Z_{n+1}] = \mathbb{E}_n\big[\mathbb{E}_{n+1}[Z]\big]$$

$$= \mathbb{E}_n[Z] \quad \text{(by iterated expectation property)}$$

$$= Z_n \quad \blacksquare$$

Back to Example from pg 61: Let's compute $\mathbb{E}[Z]$:

$$Z_3 = \mathbb{E}_3[Z] = Z$$

$$Z_2(HH) = \mathbb{E}_2[Z](HH) = \tfrac{2}{3} Z(HHH) + \tfrac{1}{3} Z(HHT)$$

$$= \tfrac{2}{3}\left(\tfrac{27}{64}\right) + \tfrac{1}{3}\left(\tfrac{27}{32}\right) = \tfrac{9}{16}$$

Proceeding in similar fashion, we get (you should check this)

$$Z_2(HT) = \tfrac{9}{8}, \quad Z_2(TH) = 9/8, \quad Z_2(TT) = \tfrac{9}{4}$$

So then

$$Z_1(H) = \mathbb{E}_1[Z](H) = \mathbb{E}_1\big[\mathbb{E}_2[Z]\big](H)$$

$$= \mathbb{E}_1[Z_2](H) = \tfrac{2}{3}\left(\tfrac{9}{16}\right) + \tfrac{1}{3}\left(\tfrac{9}{8}\right) = \tfrac{3}{4}$$

Similar computation for $\omega_1 = T$ yields

$$Z_1(T) = 3/2 \quad \text{(again, check this)}$$

and so $Z_0 = \mathbb{E}[Z] = \mathbb{E}\big[\mathbb{E}_1[Z]\big] = \mathbb{E}[Z_1]$

$$= \tfrac{2}{3}\left(\tfrac{3}{4}\right) + \tfrac{1}{3}\left(\tfrac{3}{2}\right) = 1.$$

Note: This shouldn't be a surprise, as it follows from the Radon-Nikodym theorem! $\blacksquare$

**Def** $(Z_n)_{0\le n\le N} = \left(\mathbb{E}_n[Z]\right)_{0\le n\le N}$ is called the Radon-Nikodým process, where here $Z$ refers to the Radon-Nikodým derivative.

**Rmk** It's a martingale under $\mathbb{P}$! by the previous theorem.

use of the Radon-Nikodým process:

**Lemma** If $0\le n\le N$, and $Y$ is a RV depending only on the first $n$ coin tosses, then

$$\tilde{\mathbb{E}}[Y] = \mathbb{E}[Z_n Y].$$

This is like the "limited information" version of the Radon-Nikodým theorem.