lecture 16

3/30

From last time:

Radon-Nikodým derivative of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$: $Z(\omega) = \dfrac{\widetilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}$ for all $\omega \in \Omega$

Radon-Nikodým theorem: $\widetilde{\mathbb{E}}[Y] = \mathbb{E}[ZY]$ for all RV's Y

State price density: $\zeta(\omega) = \dfrac{Z(\omega)}{(1+r)^N}$

State price: $\zeta(\omega)\mathbb{P}(\omega)$

(the price of the "state" ω of the world)

Radon-Nikodým process: $(Z_n)_{0 \le n \le N} = (\mathbb{E}_n[Z])_{0 \le n \le N}$

Def $(Z_n)_{0\le n\le N} = (\mathbb{E}_n[Z])_{0\le n\le N}$ is called the Radon-Nikodým process, where here Z refers to the Radon-Nikodým derivative.

Rmk It's a martingale under $\mathbb{P}$! by the previous theorem.

use of the Radon-Nikodým process:

Lemma If $0\le n\le N$, and Y is a RV depending only on the first n coin tosses, then

$$\tilde{\mathbb{E}}[Y] = \mathbb{E}[Z_n Y].$$

This is like the "limited information" version of the Radon-Nikodým theorem.

Example Let $0\le n\le N$.

Let $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \ldots, \bar{\omega}_n)$. Define the RV Y by:

$$Y(\omega) = \begin{cases} 1 & \text{if } \omega_i = \bar{\omega}_i \quad \forall\ i = 0,1,\ldots,n \\ 0 & \text{otherwise} \end{cases}$$

Compute $\tilde{\mathbb{E}}[Y]$, $\mathbb{E}[Z_n Y]$, and $Z_n(\bar{\omega}_1,\ldots,\bar{\omega}_n)$:

(i) $\tilde{\mathbb{E}}[Y] = \sum_{\omega} Y(\omega)\tilde{\mathbb{P}}(\omega) = \tilde{\mathbb{P}}(1^{st}\ n \text{ tosses all match})$

$$= \tilde{p}^{\#H(\bar{\omega}_1,\ldots,\bar{\omega}_n)}\,\tilde{q}^{\#T(\bar{\omega}_1,\ldots,\bar{\omega}_n)}$$

(ii) $\mathbb{E}[Z_n Y] = \sum_{\omega} Z_n(\omega)\, Y(\omega)\, \mathbb{P}(\omega)$

$$= Z_n(\bar{\omega}_1,\ldots,\bar{\omega}_n)\, \mathbb{P}(1^{st}\ n \text{ tosses all match})$$

$$= Z_n(\bar{\omega}_1,\ldots,\bar{\omega}_n)\, p^{\#H(\bar{\omega}_1,\ldots,\bar{\omega}_n)}\, q^{\#T(\bar{\omega}_1,\ldots,\bar{\omega}_n)}$$

(iii) The lemma says $\tilde{\mathbb{E}}[Y] = \mathbb{E}[YZ_n]$, so we have

$$Z_n(\bar{\omega}_1, \dots, \bar{\omega}_n) = \left(\frac{\tilde{p}}{p}\right)^{\#H(\bar{\omega}_1, \dots, \bar{\omega}_n)} \left(\frac{\tilde{q}}{q}\right)^{\#T(\bar{\omega}_1, \dots, \bar{\omega}_n)} \quad \blacksquare$$

Example. Let $N=3$, $u=2$, $d=1/2$, $r=1/4$, $p=2/3$, $q=1/3$

Compute the Radon-Nikodym process $(Z_n)_{0 \le n \le 3}$.

$$\frac{\tilde{p}}{p} = \frac{1/2}{2/3} = \frac{3}{4}$$

$$\frac{\tilde{q}}{q} = \frac{1/2}{1/3} = \frac{3}{2}$$

$Z_0(\omega) = 1$ for all ω

$Z_1(H) = \frac{3}{4}$

$Z_1(T) = \frac{3}{2}$

$Z_2(HH) = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$

$Z_2(HT) = Z_2(TH) = \left(\frac{3}{4}\right)\left(\frac{3}{2}\right) = \frac{9}{8}$

$Z_2(TT) = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

$Z_3(HHH) = \left(\frac{3}{4}\right)^3 = \frac{27}{64}$

$Z_3(HHT) = Z_3(HTH) = Z_3(THH) = \left(\frac{3}{4}\right)^2\left(\frac{3}{2}\right) = \frac{27}{32}$

$Z_3(TTH) = Z_3(THT) = Z_3(HTT) = \left(\frac{3}{4}\right)\left(\frac{3}{2}\right)^2 = \frac{27}{16}$

$Z_3(TTT) = \left(\frac{3}{2}\right)^3 = \frac{9}{8}$.

Ok, so that's a good sanity check, since that fits with the computations we did on page 64. $\blacksquare$

Lemma. Let $1 \le n \le m \le N$. Let Y be a RV depending only on the first m tosses. Then

$$\tilde{\mathbb{E}}_n[Y] = \frac{1}{Z_n} \mathbb{E}_n[Z_m Y].$$

Theorem. Let V be a European option paying V_N at time N. Then the value of V at time n is

$$V_n \underset{⓪}{=} \tilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^{N-n}}\right] \underset{①}{=} \frac{(1+r)^n}{Z_n} \mathbb{E}_n\left[\frac{Z_N V_N}{(1+r)^N}\right] \underset{②}{=} \frac{1}{\zeta_n} \mathbb{E}_n[\zeta_N V_N].$$

Proof. Equality ⓪ just comes from backwards induction, as usual.

For equality ①: says $\tilde{\mathbb{E}}_n[V_N] = \frac{1}{Z_n} \mathbb{E}_n[Z_N V_N]$, which we get from the lemma.

For equality ②, recall that $\xi_n = \frac{Z_n}{(1+r)^n}$ so $\frac{1}{\xi_n} = \frac{(1+r)^n}{Z_n}$.

Plugging that in yields second equality. Done! ■