§3.3: Capital Asset Pricing Model

Two ways of modeling asset prices:

1. No-arbitrage methodology	2. CAPM
★ gives precise quantitative results	★ balance supply with demand (given investors' UTILITY FUNCTIONS)
★ great in idealized markets (but not incomplete markets)	★ useful qualitative insights (but not precise quantitative pricing results)
★ only need risk-neutral measure	★ works in incomplete markets

When do we need the "actual" reference measure?

↳ Asset management & Risk management

Asset management: trade off b/w risk and (actual) expected return

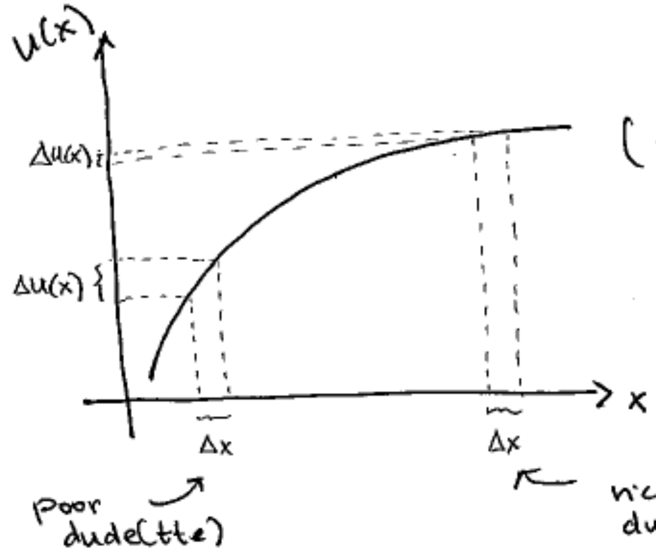
↳ but also need risk-neutral measure (what this section is about)

Risk management: Need probability of catastrophe(s)

but also need risk-neutral measure (for pricing securities in risky portfolio)

Def. A utility function is a nondecreasing, strictly concave function U; may take on value of $-\infty$ but not $+\infty$.

Convention: $\ln(x) = -\infty$ for all $x \le 0$.



(when you have little, each marginal increase in \$ means a lot. when you have a lot, each marginal increase means little.)

Rmk. Utility functions are concave to capture risk aversion of typical investor.

Ex. Gamble: \$1 w/ probability 1/2
\$99 w/ probability 1/2

- expected value \$50
- risk averse gambler would rather take \$50 now than play though!

↳ i.e. $\boxed{\mathbb{E}[u(X)] \le u(\mathbb{E}[X])}$

where X takes on value \$1 or \$99 w/ probability 1/2.

This is Jensen's Inequality for concave functions!

Optimal Investment Problem : N-period model

TAKE!

Given initial wealth X_0, find adapted process $\Delta_0, \Delta_1, \dots, \Delta_{N-1}$ (recall Δ_k = # shares of stock held between time k & $k+1$) that maximizes $\mathbb{E}\left[U(X_N)\right]$ for the investor. Recall

$$(\dagger) \quad X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$$

for all $n = 0, 1, \dots, N-1$.

Rmk. Note we're maximizing utility of actual expected value of terminal capital. Doesn't make sense to maximize risk-neutral expectation because stock and money market have same expected rate of return (under risk-neutral measure) $\rightarrow$ so to maximize $\tilde{\mathbb{E}}[U(X)]$, invest only in money market!

Example $\boxed{\text{THIS IS GONNA GET UGLY}}$

$N=2$, $u=2$, $d=1/2$, $r=1/4$; $p=2/3$, $q=1/3$; $S_0=4$

Suppose investor starts with $X_0 = 4$. How can she choose Δ_0, $\Delta_1(H)$, $\Delta_1(T)$ to maximize $\mathbb{E}\left[U(X_2)\right]$ where $U(\star) = \ln(\star)$?

Let's compute $\tilde{\mathbb{P}}(\omega_1\omega_2)$, $\mathbb{P}(\omega_1\omega_2)$, and $X_2(\omega_1\omega_2)$ for all four $\omega_1\omega_2 \in \{H,T\}^2$:

- $\tilde{\mathbb{P}}(HH) = \frac{1}{4}$ $\quad \tilde{\mathbb{P}}(HT) = \frac{1}{4}$ $\quad \tilde{\mathbb{P}}(TH) = \frac{1}{4}$ $\quad \tilde{\mathbb{P}}(TT) = \frac{1}{4}$
- $\mathbb{P}(HH) = \frac{4}{9}$ $\quad \mathbb{P}(HT) = \frac{2}{9}$ $\quad \mathbb{P}(TH) = \frac{2}{9}$ $\quad \mathbb{P}(TT) = \frac{1}{9}$

From formula (†) we have

$$X_1(H) = 8\Delta_0 + \frac{5}{4}(4 - 4\Delta_0) = 3\Delta_0 + 5$$
$$X_1(T) = 2\Delta_0 + \frac{5}{4}(4 - 4\Delta_0) = -3\Delta_0 + 5$$

So plugging these in to (†) and simplifying, we get:

$$X_2(HH) = 6\Delta_1(H) + \frac{15}{4}\Delta_0 + \frac{25}{4}$$
- $$X_2(HT) = -6\Delta_1(H) + \frac{15}{4}\Delta_0 + \frac{25}{4}$$
$$X_2(TH) = \frac{3}{2}\Delta_1(T) - \frac{15}{4}\Delta_0 + \frac{25}{4}$$
$$X_2(TT) = -\frac{3}{2}\Delta_1(T) - \frac{15}{4}\Delta_0 + \frac{25}{4}$$

(You should verify these algebraic steps.)

So we want to maximize

$$\mathbb{E}[\ln(X_2)] = \frac{4}{9}\ln(X_2(HH)) + \frac{2}{9}\ln(X_2(HT)) + \frac{2}{9}\ln(X_2(TH)) + \frac{1}{9}\ln(X_2(TT))$$

To do this, need to take partial derivatives (remember those from Calc 3??) with respect to the variables we care about (here: $\Delta_0, \Delta_1(T), \Delta_1(H)$).

We get:

$$\frac{\partial}{\partial \Delta_0} \mathbb{E}[\ln(X_2)] = \frac{4}{9}\cdot\frac{15}{4}\cdot\frac{1}{X_2(HH)} + \frac{2}{9}\cdot\frac{15}{4}\cdot\frac{1}{X_2(HT)} + \frac{2}{9}\left(-\frac{15}{4}\right)\frac{1}{X_2(TH)} + \frac{1}{9}\left(-\frac{15}{4}\right)\frac{1}{X_2(TT)}$$

$$= \frac{5}{12}\left(\frac{4}{X_2(HH)} + \frac{2}{X_2(HT)} - \frac{2}{X_2(TH)} - \frac{1}{X_2(TT)}\right)$$

$$\frac{\partial}{\partial \Delta_1(H)} \mathbb{E}[\ln(X_2)] = \frac{4}{9}\cdot\frac{6}{X_2(HH)} + \frac{2}{9}\frac{(-6)}{X_2(HT)}$$

$$= \frac{4}{3}\left(\frac{2}{X_2(HH)} - \frac{1}{X_2(HT)}\right)$$

$$\frac{\partial}{\partial \Delta_1(T)} \mathbb{E}[\ln(X_2)] = \frac{2}{9}\cdot\frac{3}{2}\cdot\frac{1}{X_2(TH)} + \frac{1}{9}\left(-\frac{3}{2}\right)\frac{1}{X_2(TT)}$$

$$= \frac{1}{6}\left(\frac{2}{X_2(TH)} - \frac{1}{X_2(TT)}\right)$$

To find the critical point of $\mathbb{E}[\ln(X_2)]$, set these equal to zero and solve; after some more algebraic simplification (sense a pattern?) we get

$$X_2(HH) = 2X_2(HT)$$
$$X_2(TH) = 2X_2(TT)$$
$$X_2(HT) = 2X_2(TT)$$

one method from here is to rewrite these 3 equations using the variables Δ_0, $\Delta_1(H)$, $\Delta_1(T)$ and solve. this is a huge pain, but involves the least heavy machinery. (This method = butts.)

A more efficient method involves using the fact that

$$X_0 = \tilde{\mathbb{E}}\left[\frac{X_N}{(1+r)^N}\right]$$

to get a fourth equation:

$$4 = \frac{16}{25}\left[\frac{1}{4}X_2(HH) + \frac{1}{4}X_2(HT) + \frac{1}{4}X_2(TH) + \frac{1}{4}X_2(TT)\right].$$

using the first three equations and writing everything in terms of, say, $X_2(HT)$, we get

$$X_2(HH) = 2X_2(HT) = 2X_2(TH) = 4X_2(TT)$$

So

$$25 = 2X_2(HT) + X_2(HT) + X_2(TH) + \frac{1}{2}X_2(HT)$$

$$= \frac{9}{2}X_2(HT)$$

$$\Rightarrow X_2(HT) = \frac{50}{9}$$

Thus

$$X_2(HH) = \frac{100}{9} \qquad X_2(TH) = \frac{50}{9} \qquad X_2(TT) = \frac{25}{9}.$$

Instead of solving the equations on page 71 now, we can use the delta-hedging formulas:

- $$\Delta_1(H) = \frac{X_2(HH) - X_2(HT)}{S_2(HH) - S_2(HT)} = \frac{50/9}{12} = \frac{25}{54}$$
- $$\Delta_1(T) = \frac{X_2(TH) - X_2(TT)}{S_2(TH) - S_2(TT)} = \frac{25/9}{3} = \frac{25}{27}$$

$$\Delta_0 = \frac{X_1(H) - X_1(T)}{S_1(H) - S_1(T)} = \frac{X_1(H) - X_1(T)}{6}$$

$$X_1(H) = \frac{4}{5}\left[\frac{1}{2}\cdot\frac{100}{9} + \frac{1}{2}\cdot\frac{50}{9}\right] = \frac{20}{3}$$

$$X_1(T) = \frac{4}{5}\left[\frac{1}{2}\,\frac{50}{9} + \frac{1}{2}\cdot\frac{25}{9}\right] = \frac{10}{3}$$

$$\Rightarrow \Delta_0 = \frac{10/3}{6} = \frac{5}{9}$$

Rmk. In using the more clever approach on page 73, we effectively rewrote the optimal investment problem to say:

Given an investor with initial wealth X_0 and utility function U, find a random variable X_N that maximizes $\mathbb{E}[U(X_N)]$ subject to the constraint $\tilde{\mathbb{E}}\left[\frac{X_N}{(1+r)^N}\right] = X_0$.

(We can then use X_N to construct the portfolio process using delta hedging.)

Obvious question: how do we find a random variable that maximizes (an expression) subject to (another expression)?

↳ Lagrange multipliers!
(Also friends from Calc 3 — let's review.)