Rmk. In using the more clever approach on page 73, we effectively rewrote the optimal investment problem to say:

Optimal Investment Problem : N-period model
TAKE 2

Given an investor with initial wealth X_0 and utility function U, find a random variable X_N that maximizes $\mathbb{E}[U(X_N)]$ subject to the constraint $\tilde{\mathbb{E}}\left[\frac{X_N}{(1+r)^N}\right] = X_0$.

(We can then use X_N to construct the portfolio process using delta hedging.)

Obvious question: how do we find a random variable that maximizes (an expression) subject to (another expression)?

↳ Lagrange multipliers!
(Also friends from Calc 3 — let's review.)

We'll set the problem up using state prices:

The Radon-Nikodým derivative $Z = \frac{\widetilde{\mathbb{P}}}{\mathbb{P}}$ is given by

- $Z(HH) = \frac{9}{16}$ $\quad$ $Z(HT) = \frac{9}{8}$

  $Z(TH) = \frac{9}{8}$ $\quad$ $Z(TT) = \frac{9}{4}$

So the state price densities $\xi = \frac{Z}{(1+r)^2}$ are

- $\xi(HH) = \frac{9}{25}$ $\quad$ $\xi(HT) = \frac{18}{25}$

  $\xi(TH) = \frac{18}{25}$ $\quad$ $\xi(TT) = \frac{36}{25}$

For any problem like:

"Find x_1, x_2, x_3, x_4 that maximize $p_1 U(x_1) + p_2 U(x_2) + p_3 U(x_3) + p_4 U(x_4)$ subject to $p_1 \xi_1 x_1 + p_2 \xi_2 x_2 + p_3 \xi_3 x_3 + p_4 \xi_4 x_4 = X_0$."

the Lagrangian is

$$\boxed{\begin{aligned} L = {} & p_1 U(x_1) + p_2 U(x_2) + p_3 U(x_3) + p_4 U(x_4) \\ & - \lambda \left(p_1 \xi_1 x_1 + p_2 \xi_2 x_2 + p_3 \xi_3 x_3 + p_4 \xi_4 x_4 - X_0 \right) \end{aligned}}$$

for some value of λ. Set up the equations

$$\frac{\partial L}{\partial x_1} = 0 \qquad \frac{\partial L}{\partial x_2} = 0 \qquad \frac{\partial L}{\partial x_3} = 0 \qquad \frac{\partial L}{\partial x_4} = 0$$

and solve for λ. Finally, plug the value of λ into the equations to solve for $x_1, x_2, x_3,$ and x_4.

let's try it

Here, our x_i's are $x_1 = X_2(HH)$, $x_2 = X_2(HT)$, $x_3 = X_2(TH)$, $x_4 = X_2(TT)$

our ξ_i's are $\xi_1 = \xi(HH)$, $\xi_2 = \xi(HT)$, $\xi_3 = \xi(TH)$, $\xi_4 = \xi(TT)$

and our p_i's are $p_1 = \mathbb{P}(HH)$, $p_2 = \mathbb{P}(HT)$, $p_3 = \mathbb{P}(TH)$, $p_4 = \mathbb{P}(TT)$

so the Lagrangian is

$$L = \frac{4}{9}\ln x_1 + \frac{2}{9}\ln x_2 + \frac{2}{9}\ln x_3 + \frac{1}{9}\ln x_4$$
$$-\lambda\left(\frac{4}{9}\cdot\frac{9}{25}x_1 + \frac{2}{9}\cdot\frac{18}{25}x_2 + \frac{2}{9}\cdot\frac{18}{25}x_3 + \frac{1}{9}\cdot\frac{36}{25}x_4 - 4\right)$$

so taking partials and setting them equal to zero gives

$$\frac{\partial L}{\partial x_1} = \frac{4}{9x_1} - \frac{4}{25}\lambda = 0 \Rightarrow x_1 = \frac{25}{9\lambda}$$

$$\frac{\partial L}{\partial x_2} = \frac{2}{9x_2} - \frac{4}{25}\lambda = 0 \Rightarrow x_2 = \frac{25}{18\lambda}$$

$$\frac{\partial L}{\partial x_3} = \frac{2}{9x_3} - \frac{4}{25}\lambda = 0 \Rightarrow x_3 = \frac{25}{18\lambda}$$

$$\frac{\partial L}{\partial x_4} = \frac{1}{9x_4} - \frac{4}{25}\lambda = 0 \Rightarrow x_4 = \frac{25}{36\lambda}$$

plug these values into the constraint

$$\frac{4}{9}\cdot\frac{9}{25}\cdot x_1+\frac{2}{9}\cdot\frac{18}{25}x_2+\frac{2}{9}\cdot\frac{18}{25}x_3+\frac{1}{9}\cdot\frac{36}{25}x_4=4$$

and solve for λ:

$$\frac{4}{25}\cdot\frac{25}{9\lambda}+\frac{4}{25}\cdot\frac{25}{18\lambda}+\frac{4}{25}\cdot\frac{25}{18\lambda}+\frac{4}{25}\cdot\frac{25}{36\lambda}=4$$

$$\Rightarrow\left(\frac{4}{9}+\frac{2}{9}+\frac{2}{9}+\frac{1}{9}\right)\frac{1}{\lambda}=4$$

$$\Rightarrow \boxed{\frac{1}{\lambda}=4.}$$

Plugging this back in to the equations at the bottom of the previous page gives

$$\boxed{\begin{array}{ll} x_1=\frac{100}{9} & x_2=\frac{50}{9} \\ x_3=\frac{50}{9} & x_4=\frac{25}{9}\end{array}}$$

which matches what we got the first time. $\square$

<u>Rmk</u> We can rewrite the constraint $X_0=\tilde{\mathbb{E}}\left[\frac{X_N}{(1+r)^N}\right]$ as $X_0=\mathbb{E}\left[\frac{Z_N X_N}{(1+r)^N}\right]=\mathbb{E}\left[\zeta X_N\right]$ so that we don't have both actual and risk-neutral expectations to deal with.

OK, so let's restate the optimal investment problem a third and final way, using Lagrange multipliers.

We'll call the 2^N outcomes in $\{H,T\}^N$: $\omega^1, \omega^2, \ldots, \omega^N$

and let

$$\left.\begin{aligned} \zeta_k &= \zeta(\omega^k) \\ p_k &= \mathbb{P}(\omega^k) \\ x_m &= X_N(\omega^k) \end{aligned}\right\} \text{ for each } k=1,2,\ldots,N.$$

Then we can say

The Optimal Investment Problem - take 3

Given an investor with initial wealth X_0 and utility function U, find $x_1, x_2, \ldots, x_N$ that maximize

$$\sum_{k=1}^{N} p_k U(x_k)$$

Subject to

$$X_0 = \sum_{k=1}^{N} p_k x_k \zeta_k$$

Method

Lagrangian: $L = \sum_{k=1}^{N} p_k U(x_k) - \lambda \sum_{k=1}^{N} p_k x_k \zeta_k + \lambda X_0.$

Lagrange multiplier equations: $\frac{\partial L}{\partial x_k} = p_k U'(x_k) - \lambda p_k \zeta_k = 0$

which reduce to $U'(x_k) = \lambda \zeta_k$ for each k.

(i.e. $U'(X_N) = \frac{\lambda Z}{(1+r)^N}$).