So, everything we've been doing in optimal investment is about maximizing $\mathbb{E}[U(X_N)]$ subject to the self-financing constraint: $\tilde{\mathbb{E}}[X_N] = X_0(1+r)^N$.

All this Lagrangian stuff can be summarized in the following theorem, which for particularly nice utility functions (mostly, in our case, logarithmic utility) yields much simpler calculations:

<u>Theorem</u>. Assume U is a twice-differentiable utility function on $(0,\infty)$ such that for all $x>0$, $U'(x)>0$ and $U''(x)<0$. Then $\hat{X}_N : \Omega \to \mathbb{R}$ is the random variable which maximizes $\mathbb{E}[U(\hat{X}_N)]$ over all choices of X_N if and only if $\exists$ $\mu > 0$ such that

$$U'(\hat{X}_N(\omega)) = \mu Z(\omega)$$

$\forall \omega \in \Omega$.

★ Proof: see supplementaryMaterial.pdf!

<u>Remark</u>

Ok, so practically speaking, we need to solve the pair of equations:

$$\boxed{\begin{array}{l}(i)\quad U'(\hat{X}_N) = \mu Z \\ (ii)\quad \tilde{\mathbb{E}}[\hat{X}_N] = X_0(1+r)^N\end{array}}$$

for μ and $\hat{X}_N$.

Let's see how this works out in the case of...

Logarithmic Utility

$$U(x) = \ln(x) \Rightarrow U'(x) = \frac{1}{x} \Rightarrow U''(x) = -\frac{1}{x^2}.$$

So we are solving

$$(i)\quad \frac{1}{\hat{X}_N} = \mu Z$$

$$(ii)\quad \tilde{\mathbb{E}}[\hat{X}_N] = X_0(1+r)^N$$

From (i), $\hat{X}_N = \frac{1}{\mu Z}$, so from (ii), $\tilde{\mathbb{E}}[\hat{X}_N] = \tilde{\mathbb{E}}[\frac{1}{\mu Z}] = X_0(1+r)^N$.

Recall that $\tilde{\mathbb{E}}[\frac{1}{Z}] = 1$ so (ii) becomes:

$$\frac{1}{\mu} = \tilde{\mathbb{E}}\left[\frac{1}{\mu Z}\right] = X_0(1+r)^N$$

$$\Rightarrow \boxed{\mu = \frac{1}{X_0(1+r)^N}}$$

Plugging into (i) gives

$$\boxed{\hat{X}_N = \frac{X_0(1+r)^N}{Z}}$$

Recall that $Z(\omega) = Z_N(\omega) = \left(\frac{\tilde{p}}{p}\right)^{\#H(\omega)}\left(\frac{\tilde{q}}{q}\right)^{\#T(\omega)}$ so we can go a little further and get

$$\hat{X}_N(\omega) = X_0 (1+r)^N \left(\frac{p}{\tilde{p}}\right)^{\#H(\omega)} \left(\frac{q}{\tilde{q}}\right)^{\#T(\omega)} \quad \forall \omega \in \Omega.$$

So what percentage of capital at time $t=n$ should be invested in stock?

Well, let's apply the self-financing condition again: for all $n = 0, \dots, N$,

$$\hat{X}_n(\omega_1 \dots \omega_n) = \frac{\tilde{\mathbb{E}}_n[\hat{X}_N]}{(1+r)^{N-n}}$$

$$= X_0 (1+r)^n \left(\frac{p}{\tilde{p}}\right)^{\#H(\omega_1 \dots \omega_n)} \left(\frac{q}{\tilde{q}}\right)^{\#T(\omega_1 \dots \omega_n)}$$

So we get

$$\hat{X}_{n+1}(\omega_1 \dots \omega_n H) = (1+r)\left(\frac{p}{\tilde{p}}\right) \hat{X}_n(\omega_1 \dots \omega_n)$$
$$\hat{X}_{n+1}(\omega_1 \dots \omega_n T) = (1+r)\left(\frac{q}{\tilde{q}}\right) \hat{X}_n(\omega_1 \dots \omega_n)$$

and therefore the # of shares of stock held between $t=n$ and $t=n+1$ is

$$\hat{\Delta}_n = \frac{(1+r)\hat{X}_n \left(\frac{p}{\tilde{p}} - \frac{q}{\tilde{q}}\right)}{uS_n - dS_n}$$

and the percentage of time n capital is

$$\alpha_n = \frac{\hat{\Delta}_n S_n}{\hat{X}_n} = \frac{1+r}{u-d}\left(\frac{p}{\tilde{p}} - \frac{q}{\tilde{q}}\right).$$

Note: α_n is the same for each n! (But $\hat{\Delta}_n$ needn't be.)

Example N-period model with $u=2, d=\frac{1}{2}, r=\frac{1}{4}$, $p=\frac{2}{3}, q=\frac{1}{3}, S_0=4, X_0=4, U(x)=\ln(x)$.

To maximize expected utility at time N, need

$$\alpha_n = \frac{1+r}{u-d}\left(\frac{p}{\tilde{p}} - \frac{q}{\tilde{q}}\right)$$

$$= \frac{5/4}{3/2}\left(\frac{2/3}{1/2} - \frac{1/3}{1/2}\right) = \frac{5}{9}$$

of capital in stock at each time $t=n$.

So initially invest $\frac{5}{9}X_0$ in stock.

$$\Rightarrow \Delta_0 = \frac{5}{9}\cdot\frac{4}{4} = \frac{5}{9}$$

Then invest $\frac{5}{9}X_1$ at time $t=1$:

$$X_1(H) = \frac{5}{9}\cdot 8 + \frac{5}{4}\left(4 - \frac{5}{9}\cdot 4\right) = \frac{60}{9}$$

$$X_1(T) = \frac{5}{9}\cdot 2 + \frac{5}{4}\left(4 - \frac{5}{9}\cdot 4\right) = \frac{30}{9}$$

So $\Delta_1(H) = \frac{5}{9}\,\frac{60/9}{8} = \frac{25}{54}$

$\Delta_1(T) = \frac{5}{9}\,\frac{30/9}{2} = \frac{25}{27}$ ■

(Note that these match our computations from the example on pages 73–74 & prior.)

↳ This computation was way quicker than any of those though!

$\boxed{Q}$ why don't we always just use this method?

↳ For one thing, we'd have a different formula to memorize for α_n for each different type of utility function.

↳ For another, it was highly convenient (and important) that we know $\tilde{\mathbb{E}}[\frac{1}{Z}] = 1$. Let's look at another type of utility function & see what happens:

<u>Power-type utility functions</u>

$$u(x) = x^{\alpha} \Rightarrow u'(x) = \alpha x^{\alpha - 1} \qquad (0 < \alpha < 1)$$

So we are now solving

$$\text{(i) } \alpha \hat{X}_N^{\alpha-1} = \mu Z \quad \& \quad \text{(ii) } \tilde{\mathbb{E}}[\hat{X}_N] = X_0(1+r)^N$$

Need to use (i) to solve for $\hat{X}_N$, so we get

$$\hat{X}_N = \left(\frac{\mu Z}{\alpha}\right)^{\frac{1}{\alpha-1}} = \left(\frac{\mu}{\alpha}\right)^{\frac{1}{\alpha-1}} Z^{\frac{1}{\alpha-1}}$$

So $\tilde{\mathbb{E}}[\hat{X}_N] = \tilde{\mathbb{E}}\left[\left(\frac{\mu}{\alpha}\right)^{\frac{1}{\alpha-1}} Z^{\frac{1}{\alpha-1}}\right] =$

$$= \left(\frac{\mu}{\alpha}\right)^{\frac{1}{\alpha-1}} \tilde{\mathbb{E}}\left[Z^{\frac{1}{\alpha-1}}\right].$$

We could spend some time computing this expected value, but it's tedious enough / messy enough to memorize that the Lagrangian becomes a better option.