Rmk: This is one reason we love logarithmic utility functions. They also display the correct qualitative behavior (convave, flatten out as x gets large), but would actually be riskier than the typical real-life investor.

Q What's a good way to measure how risk-averse an investor with utility function u is?

→ Arrow-Pratt (absolute) risk aversion (ARA) coefficient

$$a(x) = -\frac{u''(x)}{u'(x)}$$

(The larger $a(x)$ is, the more risk averse an investor w/ utility function and wealth x is.)

Some common utility functions

(i) $\ln(x)$, $x>0$

(ii) x^{α}, $x>0$, $0<\alpha<1$

(iii) $-\frac{1}{x^{\beta}}$, $x>0$, $\beta>0$

(iv) $-e^{-\alpha x}$, $x>0$, $\alpha>0$

Exercise

Analyze the ARA coefficient behavior for all 4 types of common utility functions, and explain what happens to the risk aversion of each type of investor as that investor's wealth increases, as quantified by the ARA coefficient.

(i). $u(x) = \ln(x)$

$$\Rightarrow u'(x) = \frac{1}{x}, \quad u''(x) = -\frac{1}{x^2}$$

$$\Rightarrow a(x) = -\left(-\frac{1/x^2}{1/x}\right) = \frac{1}{x}$$

(ii) $u(x) = x^\alpha$

$$\Rightarrow u'(x) = \alpha x^{\alpha-1}, \quad u''(x) = \alpha(\alpha-1)x^{\alpha-2}$$

$$\Rightarrow a(x) = \frac{-\alpha(\alpha-1)x^{\alpha-2}}{\alpha x^{\alpha-1}} = \frac{1-\alpha}{x}$$

(iii) $u(x) = -\frac{1}{x^\beta}$

$$\Rightarrow u'(x) = \beta x^{-\beta-1}, \quad u''(x) = -\beta(\beta+1)x^{-\beta-2}$$

$$\Rightarrow a(x) = \frac{\beta(\beta+1)x^{-\beta-2}}{\beta x^{-\beta-1}} = \frac{\beta+1}{x}$$

(iv) $U(x) = -e^{-\alpha x}$

$$\Rightarrow u'(x) = \alpha e^{-\alpha x}, \quad u''(x) = -\alpha^2 e^{-\alpha x}$$

$$\Rightarrow a(x) = \frac{\alpha^2 e^{-\alpha x}}{\alpha e^{-\alpha x}} = \alpha$$

- So for utility functions of type (i), (ii), and (iii), the risk aversion decreases when wealth increases. For utility functions of type (iv), the risk aversion stays constant with wealth.
- In type (ii), higher α corresponds to lower risk aversion.
- In type (iii), higher β corresponds to higher risk aversion.
- In type (iv), higher α corresponds to higher risk aversion. ■

Example Foggy Nelson has utility function v, and Matt Murdock, aka Daredevil, has utility function u. Suppose $v(x) = f(u(x))$ for all $x > 0$ where f is increasing and concave. Show that Foggy is more risk-averse than Matt.

(Note: If this isn't true, Daredevil should probably change his name.)

OK so first of all, note that f increasing means $f'(x) > 0 \;\forall x$.

And f concave means $f''(x) < 0 \;\forall x$.

Using the ARA coefficients, we have

$$ARA(\text{Matt}) = -\frac{u''(x)}{u'(x)}$$

and

$$\boxed{ARA(\text{Foggy}) = -\frac{v''(x)}{v'(x)}}$$

Note that $v(x) = f(u(x))$ so $v'(x) = u'(x)\, f'(u(x))$ (chain rule! calc 1!). Now we take the derivative again:

$$v''(x) = u''(x)\, f'(u(x)) + u'(x)\left[u'(x)\, f''(u(x))\right]$$

(chain rule AND product rule! BAM)

So

$$\boxed{\text{ARA (Foggy)}} = -\left[\frac{u''(x)\, f'(u(x)) + (u'(x))^2\, f''(u(x))}{u'(x)\, f'(u(x))}\right]$$

$$= \text{ARA (Matt)} - u'(x)\,\frac{f''(u(x))}{f'(u(x))}$$

Remember that $u'(x) > 0$ (since the more money, the happier Matt is) and $\frac{f''}{f'} < 0$. So we're subtracting a negative number, meaning

$$\boxed{\text{ARA (Foggy)}} > \text{ARA (Matt)}.$$

And that means $\boxed{\text{Foggy}}$ is more risk-averse! KAPOW

☆ Summary of What We've Done So Far (Shreve Chapters 1–4) ☆

- **Introductory stuff** (lectures 1-2)
  - Financial markets & models
  - Arbitrage & replication
  - Pricing basics
  - Basic option types
- **Chapter 1: Binomial model** (lectures 3-5)
  - Set up of binomial model (in one period & multiple periods)
  - Backward induction pricing
  - Risk-neutral measure
  - Pricing puts & calls
  - Self-financing strategies
- **Chapter 2: Probability** (lectures 6-10)
  - Coin-toss space
  - Binomial product measures
  - Random variables
  - Expected values, correlation, independence, and all that good stuff.

- conditional expectation
- martingales & how to use 'em for pricing

- **Chapter 4: American Derivatives** (lectures 10–14)
  - American & Bermudan options
  - Early exercise premiums
  - Derivatives with random maturity
  - Stopping rules

- **Chapter 3: State Prices & CAPM** (Lectures 15–20)
  - Radon-Nikodým stuff & change of measure
  - State prices & state price densities
  - CAPM: optimal investment & asset management
    - utility functions
    - risk aversion
    - lagrange multipliers
    - Arrow-Pratt absolute risk aversion coefficient

- **where we'll go from here (after Exam 3):**

Black-Scholes: motivate, set-up, and derive the pricing model!