Lecture 21: Setting up for Black-Scholes (Merton)

The goal for the next few lectures will be to set up and motivate the Black-Scholes model, which is used on Wall Street to model financial markets with derivative securities. We'll derive the Black-Scholes formula, which is widely used to model (European) derivative securities.

(This is the analog of our $V_0 = \frac{1}{(1+r)^N} \widetilde{\mathbb{E}}[V_N]$ formula, when we add in more of the real world complexity than we've allowed in the N-period binomial models we've been considering.)

The gist: Suppose we have to model T years in our model (because we have a derivative with maturity date T years from now). Divide the interval $[0,T]$ into N equal steps (so each step, or period, is $\frac{T}{N}$ years long) and let $N \to \infty$. (So we'll get to Black-Scholes by taking suitable limits of an N-period binomial.)

For convenience, let's make $\tilde{p} = \tilde{q} = \frac{1}{2}$.

Since $\tilde{p} = \dfrac{1+r-d}{u-d}$ and $\tilde{q} = \dfrac{u-1-r}{u-d}$,

we'll need $1+r-d = u-1-r = \sigma$ for some constant $\sigma > 0$. that is, our up & down factors will depend on r and σ by

$$\boxed{\begin{aligned} u &= 1+r+\sigma \\ d &= 1+r-\sigma \end{aligned}}$$

σ has a meaning $\rightarrow$ it's the volatility of the model. So u and d are determined by interest rate and volatility.

These values of r and σ are for one period. But we're going to let the number of periods $\to \infty$. What we'll be actually interested in are r_* and σ_*, the annual rate and volatility. These are scaled by

$$r = \frac{r_* T}{N}, \quad \sigma = \sigma_* \frac{\sqrt{T}}{\sqrt{N}} \left(\text{since } \sigma^2 = \frac{\sigma_*^2 T}{N}\right)$$

So we can now also scale the up and down factors:

Fact

As N rises, $u \to e^{\sigma \frac{\sqrt{T}}{\sqrt{N}}}$

$d \to e^{-\sigma \frac{\sqrt{T}}{\sqrt{N}}}$

Proof

This requires Taylor's theorem: if f is continuous and has continuous 1^{st} & 2^{nd} derivatives, then

$$f(x) = f(0) + f'(0)x + O(x^2)$$

(where $O(x^2)$, read "Big O of x^2", means that this term grows no faster than Cx^2 for some constant C).

So if $f(x) = e^x$, we have

$$e^x = 1 + x + O(x^2)$$

Then we have

constants

$$e^{\pm \sigma_* \frac{\sqrt{T}}{\sqrt{N}}} = 1 \pm \frac{\sigma_* \sqrt{T}}{\sqrt{N}} + O\left(\frac{\sigma_*^2 T}{N}\right)$$

$$= 1 \pm \sigma_* \frac{\sqrt{T}}{\sqrt{N}} + O\left(\frac{1}{N}\right).$$

So the difference between u_* and $e^{\sigma_* \frac{\sqrt{T}}{\sqrt{N}}}$ is no bigger than $\frac{c}{N}$ for some constant c, which $\to 0$ as $N \to \infty$. (Analogously, d_* is no more than $\frac{k}{N}$ away from $e^{-\sigma \frac{\sqrt{T}}{\sqrt{N}}}$ for some k, and $\frac{k}{N} \to 0$ as $N \to \infty$.) ■