Lecture 22

Continuous probability background: what do we need and why do we need it?

Goal: Compute the Black-Scholes price of a put (European-style): maturity T, strike K.

Let H_N = # heads, T_N = # tails in a sequence of N coin tosses. S_N = stock price at time N = $= S_0 u^{H_N} d^{T_N}$.

The put has price

$$P_0 = \frac{1}{\left(1+\frac{r_* T}{N}\right)^N} \tilde{\mathbb{E}}\left[(K-S_N)^+\right]$$

$$= \frac{1}{\left(1+\frac{r_* T}{N}\right)^N} \tilde{\mathbb{E}}\left[\left(K-S_0 u^{H_N} d^{T_N}\right)^+\right]$$

in the N-period model, so the actual Black-Scholes price, P^*, of the put is

$$\lim_{N\to\infty} P_0 = \lim_{N\to\infty}\left(1+\frac{r_* T}{N}\right)^{-N} \lim_{N\to\infty} \tilde{\mathbb{E}}\left[\left(K-S_0 u^{H_N} d^{T_N}\right)^+\right]$$

provided both limits exist (they do).

For the first limit, we only need calc I and some cleverness:

let's look at the natural log of this quantity first: $\ln\left[\left(1+\frac{r_* T}{N}\right)^{-N}\right] = -N \ln\left(1+\frac{r_* T}{N}\right) =$

$$= -\frac{\ln\left(1+\frac{r_* T}{N}\right)}{1/N}.$$

So $\lim_{N\to\infty} \ln\left[\left(1+\frac{r_* T}{N}\right)^{-N}\right] = -\lim_{N\to\infty} \frac{\ln\left(1+\frac{r_* T}{N}\right)}{1/N} =$

[Both the numerator and denominator go to 0 as $N \to \infty$, so we can use L'Hôpital's rule:]

$$= -\lim_{N\to\infty} \frac{\frac{1}{\left(1+\frac{r_* T}{N}\right)}\left(-\frac{r_* T}{N^2}\right)}{-1/N^2} =$$

$$= \lim_{N\to\infty} \frac{-r_* T}{\left(1+\frac{r_* T}{N}\right)} = -r_* T$$

Raise both sides to be exponents of e and we get:

$e^{-r_* T}$ (this clears the ln away).

So we have

$$\boxed{P^* = e^{-r_* T} \lim_{N\to\infty} \tilde{\mathbb{E}}\left[\left(K - S_0 u^{H_N} d^{T_N}\right)^+\right]}.$$

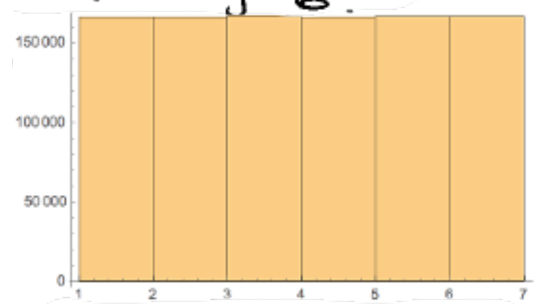
The background we need now is to help evaluate this remaining limit.

Setting up for the Central Limit Theorem

<u>Idea</u>: Take sums of independent random variables and plot them in a histogram. The result is a bell curve-ish looking thing.

<u>Example</u>: Roll a die. You get 1, 2, 3, 4, 5, or 6 uniformly — each with probability $\frac{1}{6}$.

Histogram might look like:

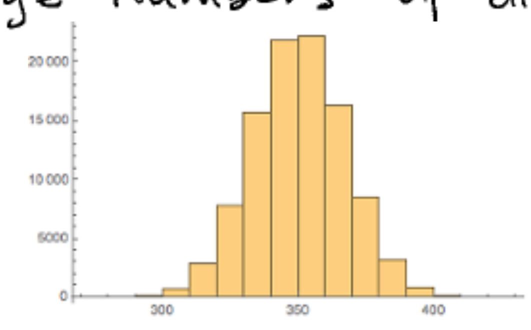


Now roll two dice & sum the results. You get numbers between 2 and 12. Probabilities are:

#	$\mathbb{P}(\#)$
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

So a histogram might look like:

Continue like this for large numbers of dice. Here's an example of a histogram where we sum up 100 dice:



The histogram looks more and more Bell curvy.

<u>Note</u>: Another good way to visualize what's happening is with a BEAN MACHINE. We can watch a video that simulates this for us (we used

http://vis.supstat.com/2013/04/bean-machine

in class).

OK, so that's a good way to visualize what the Central limit theorem says. But what is a Bell curve, exactly?

For continuous R.V.'s X, X has distribution f iff for all $a \le b$, $P(a \le X \le b) = \int_a^b f(x)\,dx$.

A "normal distribution" with mean 0, variance 1, is the function $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

<u>Notation</u>: $X \sim \mathcal{N}(0,1)$.

So we can state:

<u>Central Limit Theorem</u>: If $X_1, X_2, \ldots, X_N$ are independent RV's with $\mathbb{E}[X_i] = 0$ and $Var(X_i) = 1$ $\forall i$, and $X = \frac{1}{N}\sum_{i=1}^{N} X_i$, then Y is approximately $Y \sim \mathcal{N}(0,1)$. (i.e. sum of indep. RVs is roughly normal).

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Central limit theorem : [THE REAL THING]

Let $X_1, X_2, \ldots, X_N$ be independent RVs (under $\tilde{\mathbb{P}}$) with $\tilde{\mathbb{E}}[X_i] = 0$, $Var(X_i) = 1$, $\forall i$.

Let $\gamma \in \mathbb{R}$. Let $y \in \mathbb{R}$.
Let $Y_1, Y_2, \ldots$ be RVs such that

$$\lim_{N \to \infty} Y_N = y.$$

Let $f(x)$ be a bounded, continuous function.

Then:

$$\lim_{N \to \infty} \tilde{\mathbb{E}}\left[f\left(\frac{\gamma}{\sqrt{N}} \sum_{i=1}^{N} X_i + Y_N \right) \right] = \int_{-\infty}^{\infty} f(\gamma x + y) \varphi(x)\, dx.$$

How we'll use this: we'll want $f(x) = (K - e^x)^+$
(this is so that $f(\ln(S_N)) = (K - S_N)^+$.)

Let's set this up:

$u = 1 + \frac{r_* T}{N} + \frac{\sigma_* \sqrt{T}}{\sqrt{N}}$, $d = 1 + \frac{r_* T}{N} - \frac{\sigma_* \sqrt{T}}{\sqrt{N}}$

so taking logs gives:

$$\boxed{\begin{aligned}\ln u &= \ln\left(1+\frac{r_* T}{N}+\frac{\sigma_*\sqrt{T}}{\sqrt{N}}\right)\\ \ln d &= \ln\left(1+\frac{r_* T}{N}-\frac{\sigma_*\sqrt{T}}{\sqrt{N}}\right)\\ \ln S_N &= \ln\left(S_0\, u^{H_N} d^{T_N}\right)\\ &= \ln S_0 + H_N \ln u + T_N \ln d.\end{aligned}}$$

Remember Taylor's theorem! Applied here, we get

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + O(x^3)$$

So letting $x = \frac{r_* T}{N} \pm \frac{\sigma_*\sqrt{T}}{\sqrt{N}}$ gives

$$\ln u = \frac{r_* T}{N} + \frac{\sigma_*\sqrt{T}}{\sqrt{N}} - \frac{1}{2}\left(\frac{r_* T}{N}+\frac{\sigma_*\sqrt{T}}{\sqrt{N}}\right)^2 + O\left(\frac{1}{N\sqrt{N}}\right)$$

Note: $O(x^3) = O\left(\frac{r_* T}{N}+\frac{\sigma_*\sqrt{T}}{\sqrt{N}}\right)^3 = O\left(\left(\frac{1}{N}+\frac{1}{\sqrt{N}}\right)^3\right)$

$$= O\left(\frac{1}{N^3}+\frac{3}{N^2\sqrt{N}}+\frac{3}{N^2}+\frac{1}{N^{3/2}}\right)$$

$$= O\left(\frac{1}{N^{3/2}}\right)$$ since the other terms are all smaller

Note also: $-\frac{1}{2}\left(\frac{r_* T}{N}+\frac{\sigma_*\sqrt{T}}{\sqrt{N}}\right)^2 = -\frac{1}{2}\left(\frac{r_*^2 T^2}{N^2}+\frac{\sigma_*^2 T}{N}+\frac{2r_*\sigma_* T\sqrt{T}}{N\sqrt{N}}\right)$

$$= \frac{\sigma_*^2 T}{2N} + O\left(\frac{1}{N\sqrt{N}}\right).$$

So $\boxed{\ln u = \frac{\sigma_*\sqrt{T}}{\sqrt{N}} + \frac{1}{N}\left(r_* T + \frac{\sigma_*^2 T}{2}\right) + O\left(\frac{1}{N\sqrt{N}}\right)}$

Redoing these computations for d similarly yields

$$\boxed{\ln d = -\frac{\sigma_* \sqrt{T}}{\sqrt{N}} + \frac{1}{N}\left(r_* T + \frac{\sigma_*^2 T}{2}\right) + O\left(\frac{1}{N\sqrt{N}}\right)}$$

Putting this together gives

$$\ln S_N = \ln S_0 + H_N\left[\frac{\sigma_*\sqrt{T}}{\sqrt{N}} + \frac{r_* T + \sigma_*^2 T/2}{N} + O\left(\frac{1}{N\sqrt{N}}\right)\right]$$

$$+ T_N\left[-\frac{\sigma_*\sqrt{T}}{\sqrt{N}} + \frac{r_* T + \sigma_*^2 T/2}{N} + O\left(\frac{1}{N\sqrt{N}}\right)\right]$$

since $H_N + T_N = N$ and $H_N, T_N \le N$ so $H_N O\left(\frac{1}{N\sqrt{N}}\right) = O\left(\frac{1}{\sqrt{N}}\right)$

$$= \ln S_0 + \frac{\sigma_*\sqrt{T}}{\sqrt{N}}(H_N - T_N) + \frac{\sigma_*^2 T}{2} + r_* T + O\left(\frac{1}{\sqrt{N}}\right)$$

$$= \boxed{\ln S_0 + \frac{\sigma_*^2 T}{2} + r_* T + O\left(\frac{1}{\sqrt{N}}\right).}$$

Let's call this $\boxed{Y_N.}$

(Note $\lim_{N\to\infty} Y_N = \ln S_0 + \frac{\sigma_*^2 T}{2} + r_* T$, so this is our $\boxed{\text{"}y\text{"}}$.)

Note that the RV $H_N - T_N = \boxed{\sum_{i=1}^{N} X_i}$ where

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ toss is } H \\ -1 & \text{if } i^{th} \text{ toss is } T \end{cases}$$

. So we have $\boxed{\gamma = \sigma_*\sqrt{T}.}$ Now we have everything we need to plug into the central limit theorem.