LECTURE 23

Last time we set up all the stuff we need to apply the Central Limit Theorem:

- $Y_N = \ln S_0 + \frac{\sigma_*^2 T}{2} + r_* T + O\left(\frac{1}{N}\right)$
- $y = \ln S_0 + \frac{\sigma_*^2 T}{2} + r_* T$
- $\sum_{i=1}^{N} X_i = H_N - T_N$
- $\gamma = \sigma_* \sqrt{T}$
- $f(x) = \left(K - e^x\right)^+$

Note that $\frac{\sigma_* \sqrt{T}}{\sqrt{N}} \sum_{i=1}^{N} X_i + Y_N = \ln S_N$, so we get

$$\lim_{N\to\infty} \tilde{\mathbb{E}}\left[f(\ln S_N)\right] = \lim_{N\to\infty} \tilde{\mathbb{E}}\left[\left(K - \exp\left\{\frac{\sigma_*\sqrt{T}}{\sqrt{N}}(H_N - T_N) + \ln S_0 + \frac{\sigma_*^2 T}{2} + r_* T + O\left(\tfrac{1}{\sqrt{N}}\right)\right\}\right)^+\right]$$

$$= \int_{-\infty}^{\infty} \left(K - \exp\left\{\sigma_* \sqrt{T}\, x + \ln S_0 + \frac{\sigma_*^2 T}{2} + r_* T\right\}\right)^+ \varphi(x)\, dx$$

Recall that $f(\ln S_N) = (K - S_N)^+$, so this is exactly what we wanted! Now let's try to evaluate the integral.

$$\lim_{N\to\infty} \tilde{\mathbb{E}}[f(\ln S_N)] = \int_{-\infty}^{\infty} f\left(\sigma_* \sqrt{T}\, x + \ln S_0 + \left(r_* - \frac{\sigma_*^2}{2}\right)T\right) \varphi(x)\,dx.$$

Taking $f(x) = (K - e^x)^+$ gives $f(\ln S_N) = (K - S_N)^+$.

Therefore

this means we're putting all the $[\cdot]$ stuff in the exponent of $e^{[\cdot]}$.

$$P^* = e^{-r_* T} \int_{-\infty}^{\infty} \left(K - \exp\left[\sigma_* \sqrt{T}\, x + \ln S_0 + \left(r_* - \frac{\sigma_*^2}{2}\right)T\right]\right)^+ \varphi(x)\,dx$$

$$= \frac{e^{-r_* T}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(K - S_0 \exp\left[\sigma_* \sqrt{T} x + \left(r_* - \frac{\sigma_*^2}{2}\right)T\right]\right)^+ e^{-x^2/2}\,dx$$

When is the stuff in the $(\cdot)^+$ positive?

$$\iff K > S_0 \exp\left[\sigma_* \sqrt{T}\, x + \left(r_* - \frac{\sigma_*^2}{2}\right)T\right]$$

$$\iff \ln\left(\frac{K}{S_0}\right) > \sigma_* \sqrt{T}\, x + \left(r_* - \frac{\sigma_*^2}{2} T\right)$$

$$\iff \underbrace{- \frac{\ln\left(\frac{S_0}{K}\right) + \left(r_* - \frac{\sigma_*^2}{2}\right)T}{\sigma_* \sqrt{T}}}_{\text{Let's call this } d.} > x$$

So integrand $> 0 \iff x < -d$. So

$$P^* = \frac{e^{-r_* T}}{\sqrt{2\pi}} \int_{-\infty}^{-d} \left(K - S_0 \exp\left[\sigma_* \sqrt{T}\, x + \left(r_* - \frac{\sigma_*^2}{2}\right)T\right]\right) e^{-x^2/2}\,dx$$

$$= \frac{Ke^{-r_* T}}{\sqrt{2\pi}} \int_{-\infty}^{-d} e^{-x^2/2}\, dx$$

$$- \frac{S_0}{\sqrt{2\pi}} \int_{-\infty}^{-d} e^{-r_* T} e^{\sigma_* \sqrt{T} x} e^{(r_* - \frac{\sigma_*^2}{2})T} e^{-x^2/2}\, dx$$

$$= Ke^{-r_* T} \varphi(-d) - \frac{S_0}{\sqrt{2\pi}} \int_{-\infty}^{-d} e^{-\frac{(x - \sigma_* \sqrt{T})^2}{2}}\, dx$$

Do w-substitution to evaluate this integral.

$$w = x - \sigma_* \sqrt{T} \Rightarrow dw = dx$$

New Bounds: when $x = -d$, $w = -d - \sigma_* \sqrt{T}$

This gives

$$p^* = Ke^{-r_* T} \varphi(-d) - \frac{S_0}{\sqrt{2\pi}} \int_{-\infty}^{-d-\sigma_* \sqrt{T}} e^{-w^2/2}\, dw$$

$$= Ke^{-r_* T} \varphi(-d) - S_0 \varphi\left(-d - \sigma_* \sqrt{T}\right)$$

That was a lot of calculus and algebra, but notice that all we really did was take limits of the stuff we derived in the binomial model!!

Note also that we do not need to do all this again for the call price! We use put-call parity:

$$C_0 - P_0 = \frac{1}{\left(1+\frac{r_* T}{N}\right)^N} \tilde{E}\left[S_N - K\right] = S_0 - \frac{K}{\left(1+\frac{r_* T}{N}\right)^N}.$$

Thus

$$\boxed{\begin{aligned} C^* &= \lim_{N\to\infty} C_0 = P^* + S_0 - Ke^{-r_* T} \\ &= Ke^{-r_* T}\left(\varphi(-d) - 1\right) + S_0\left(1 - \varphi(-d - \sigma_*\sqrt{T})\right) \end{aligned}}$$

The Greeks

Among many handy things that we can do with these formulas are THE GREEKS (so called because they are quantities represented by Greek letters). These quantities are partial derivatives of the Black-Scholes prices, and measure the sensitivity of the price to changes in underlying parameters. They're VERY important in the real world because they measure risk.