MATH FINANCE
LECTURE 3

1/27/16

§1.1 one-period binomial model: mathematical framework:

- Two trading dates ($t=0$, $t=1$)
- Sample space $\Omega = \{H, T\}$

  $\mathbb{P}(H)$, $\mathbb{P}(T) > 0$

  $\mathbb{P}(H) + \mathbb{P}(T) = 1$
- Bank account w/ 1-period interest rate $r \geq 0$.
- Single stock w/ initial price S_0/share; price per share, S_1, at time 1 is a random variable on Ω:

  $S_1(H) = u S_0 \qquad S_1(T) = d S_0$

  where

  $$0 < \underbrace{d}_{\text{"down factor"}} < 1 + r < \underbrace{u}_{\text{"up factor"}}$$

Remarks

- Strategies cannot look into the future.
- A strategy, formally, is characterized by
  - initial capital X_0
  - number of stock shares Δ_0

Rmk For an arbitrary deriv. sec., there is exactly one derivative security:

Suppose this derivative pays V_1 at time $t=1$ (i.e. $V_1(H)$ if H occurs, $V_1(T)$ if T occurs).

Note that to replicate, we need

(1) $$\Delta_0 S_1(H) + (X_0 - \Delta_0 S_0)(1+r) = V_1(H)$$

(2) $$\Delta_0 S_1(T) + (X_0 - \Delta_0 S_0)(1+r) = V_1(T)$$

Subtracting (1)-(2) and solving for Δ_0 yields.

$$\boxed{\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}.}$$

"Delta-Hedging Formula"

Solving for $X_0(1+r)$ in (1) & (2) gives

$$\boxed{X_0(1+r) = \frac{u-(1+r)}{u-d} V_1(T) + \frac{(1+r)-d}{u-d} V_1(H)}$$

We'll use these coefficients a lot so let's give them names:

$$\tilde{p} = \frac{1+r-d}{u-d} \qquad \tilde{q} = \frac{u-1-r}{u-d}$$

* no need to memorize! $\tilde{p}+\tilde{q}=1$ & $\tilde{p}u+\tilde{q}d=1+r$

Note that $\tilde{p}+\tilde{q}=1$ & $1 \geq \tilde{p}, \tilde{q} \geq 0$, so we can think of $\tilde{p}$ and $\tilde{q}$ as probabilities.

Def We'll define this as the risk-neutral measure on Ω: $\tilde{\mathbb{P}}(H) = \tilde{p}$, $\tilde{\mathbb{P}}(T) = \tilde{q}$.

Then we have that the initial capital, X_0, of a replicating strategy is

$$\boxed{X_0 = \frac{1}{1+r}\left[\tilde{p}\,V_1(H) + \tilde{q}\,V_1(T)\right]} = \frac{1}{1+r}\tilde{\mathbb{E}}[V_1]$$

i.e. the interest-adjusted expected value under the risk-neutral measure.

Rmk. The actual probabilities p & q of H & T, respectively, still don't appear in the pricing formula.

Rmk. The risk-neutral probabilities are NOT the real-world probabilities of anything. In fact: changing the currency would change $\tilde{p}$ & $\tilde{q}$.

Examples. $u=1.4$, $d=0.8$, $r=0.05$

$S_0=100$.

(i) Price a (European) call with $K_c=92$

(ii) " " put with $K_p=92$.

(iii) " " call with $K_c=116$.

(iv) " " put with $K_p=116$.

First of all, $\tilde{p} = \frac{1+r-d}{u-d} = \frac{.25}{.6} = \frac{5}{12}$

$\tilde{q} = \frac{u-1-r}{u-d} = 1 - \frac{5}{12} = \frac{7}{12}$

So:

(i) $C_1(H) = 140 - 92 = 48$ $\qquad C_1(T) = 0$

$$X_0 = \frac{1}{1.05}\left[\frac{5}{12}\cdot 48 + \frac{7}{12}\cdot 0\right] = \frac{20}{21}\cdot 20 = \frac{400}{21} \approx \$19.05.$$

(ii) $P_1(H) = 0$ $\qquad P_1(T) = 12$

$$X_0 = \frac{1}{1.05}\left[\frac{5}{12}\cdot 0 + \frac{7}{12}\cdot 12\right] = \frac{20}{21}\cdot 7 = \frac{140}{21} \approx \$6.67$$

(iii) $C_1(H) = 24$ $\qquad C_1(T) = 0$

$$X_0 = \frac{1}{1.05}\left[\frac{5}{12}\cdot 24\right] = \frac{200}{21} \approx \$9.52$$

(iv) $P_1(H) = 0$ $\qquad P_1(T) = 36$

$$X_0 = \frac{1}{1.05}\left[\frac{7}{12}\cdot 36\right] = \$20 \qquad \blacksquare$$

[That was much quicker!]

Put-Call Parity

Rmk

Note that for any put P and call C that have the same strike price, $P_1(\omega)$ and $C_1(\omega)$ cannot both be positive (for any outcome $\omega \in \Omega$).

Def

Let $(x)^+$ denote the "positive part" of x. That is, $(x)^+ = \max(x, 0)$.

so $C_1 = (S_1 - K)^+$

$P_1 = (K - S_1)^+$

<u>Fact</u> For any x, $x^+ - (-x)^+ = x$.

<u>Pf</u>. Try proving this yourself! Hint: consider the cases $x=0$, $x<0$, $x>0$ separately. ∎

Put-Call Parity

<u>Portfolio 1</u>: Buy one put, P, & sell short one call, C.
The cost is $P_0 - C_0$ and the payout at the maturity date T is

$$V_1 = (K - S_1)^+ - (S_1 - K)^+ = K - S_1.$$

<u>Porfolio 2</u>: Invest \$$K/(1+r)$ in the bank and sell short one share of stock.
This costs $\frac{1}{1+r}K - S_0$ at $t=0$ and the payout at time 1 is $K - S_1$.

Since the two portfolios have the same payout (in any situation) they must require the same initial capital, so we see

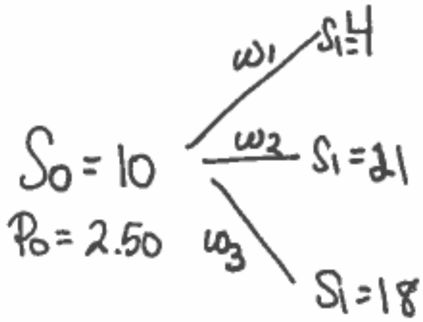
$$\boxed{P_0 - C_0 = \frac{K}{1+r} - S_0} \; *$$

*This is independent of the exact choice of model!

Example

Recall Example 1.6(i) from the supplementary notes.

We had the situation $r = 0.2$



where P is a put w/ $K = 10$, $T = 1$.

We wanted to find C_0 for a call C with $K = 10$ & $T = 1$.

Put-call parity now quickly gives

$$C_0 = P_0 + S_0 - \frac{K}{1+r}$$

$$= 2.50 + 10 - \frac{10}{1.2}$$

$$\approx \$4.17$$

(which matches what we got by solving for α, β, γ). ∎

More on the risk-neutral measure...

Proposition. In the one-period model,

there's no arbitrage $\Leftrightarrow$ $d < 1+r < u$.

Proof (overview):

($\Rightarrow$) Proof by contrapositive:

Suppose $d \geq 1+r$. Then if we borrow \$$S_0$ from bank to buy a share of stock ($X_0 = 0$) we'll have
$$X_1 = S_1 - S_0(1+r) = \begin{cases} (u-(1+r))S_0 \text{ if } H \\ \text{or} \\ (d-(1+r))S_0 \text{ if } T \end{cases}$$

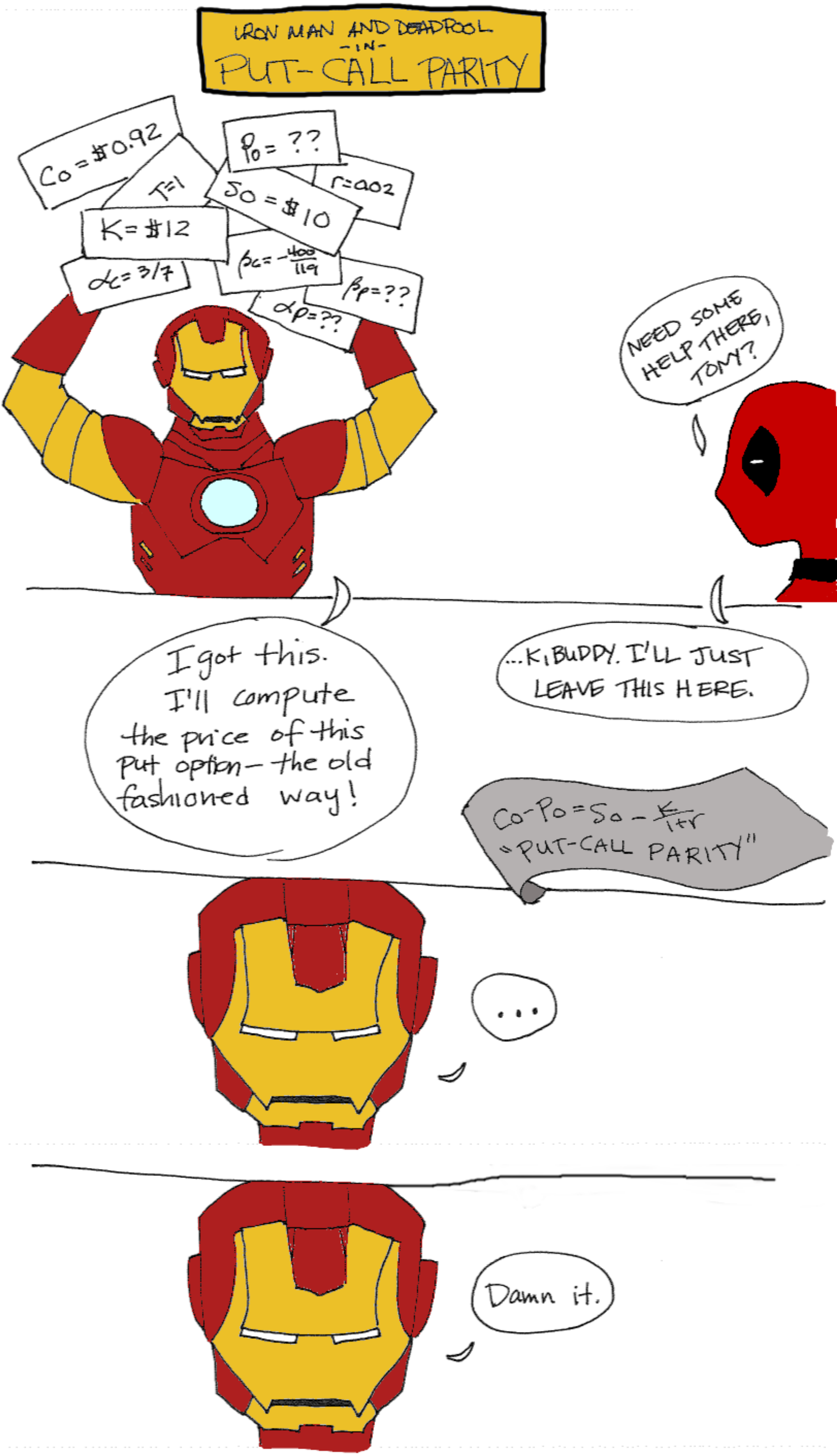
Since $u > d \geq 1+r$, we have no risk of loss. So this is an arbitrage!! Same idea if $u \leq 1+r$ (can you fill out the corresponding proof for that case?).

($\Leftarrow$) Suppose $d < 1+r < u$. Let X be some strategy w/ $X_0 = 0$. It has
$$X_T = \Delta_0 S_1 + (X_0 - \Delta_0 S_0)(1+r)$$
$$= \Delta_0 S_1 + (-\Delta_0 S_0)(1+r).$$

This is $\Delta_0 u S_0 - S_0 \Delta_0(1+r) = \Delta_0 S_0 (u-1-r)$ if H

& $\Delta_0 d S_0 - S_0 \Delta_0 (1+r) = \Delta_0 S_0 (d-1-r)$ if T.

Note: if $\Delta_0 = 0$ then the payout $X_1 = 0$. If $\Delta_0 > 0$, $X_1(H) > 0$ & $X_1(T) < 0$. If $\Delta_0 < 0$, $X_1(H) < 0$ & $X_1(T) > 0$. So no arbitrage is possible (always chance of loss). ■



IRON MAN AND DEADPOOL -IN- PUT-CALL PARITY
$C_0 = \$0.92$
$T=1$
$P_0 = ??$
$r=0.02$
$S_0 = \$10$
$K = \$12$
$\beta_C = -\frac{400}{119}$
$\alpha_C = 3/7$
$\beta_P = ??$
$\alpha_P = ??$
NEED SOME HELP THERE, TONY?
I got this. I'll compute the price of this put option — the old fashioned way!
...K, BUDDY. I'LL JUST LEAVE THIS HERE.
$C_0 - P_0 = S_0 - \frac{K}{1+r}$
"PUT-CALL PARITY"
...
Damn it.