MATH FINANCE - LECTURE 4

Alternate derivation of put-call parity:

Recall the pricing equations we found on pgs 11-12:

$$(1)\quad C_0 = \frac{1}{1+r}\left[\tilde{p}\, C_1(H) + \tilde{q}\, C_1(T)\right]$$

$$(2)\quad P_0 = \frac{1}{1+r}\left[\tilde{p}\, P_1(H) + \tilde{q}\, P_1(T)\right]$$

Using the "positive part" notation (see pgs 13-14) we have $C_1 = (S_1 - K)^+$ & $P_1 = (K - S_1)^+$, so

$$(1^*)\quad C_0 = \frac{1}{1+r}\left[\tilde{p}\,(S_1(H)-K)^+ + \tilde{q}\,(S_1(T)-K)^+\right]$$

$$(2^*)\quad P_0 = \frac{1}{1+r}\left[\tilde{p}\,(K-S_1(H))^+ + \tilde{q}\,(K-S_1(T))^+\right]$$

and therefore $[(2^*)-(1^*)]$:

$$\begin{aligned}
P_0 - C_0 &= \frac{1}{1+r}\left[\tilde{p}\,(K-S_1(H)) + \tilde{q}\,(K-S_1(T))\right]\\
&= \frac{1}{1+r}\left[K - \tilde{p}\,S_1(H) - \tilde{q}\,S_1(T)\right]\\
&= \frac{K}{1+r} - \frac{1}{1+r}\left[\tilde{p}\,S_1(H) + \tilde{q}\,S_1(T)\right]\\
&= \frac{K}{1+r} - S_0
\end{aligned}$$

with the last equality comming from the fact that, since the model is arbitrage-free, S_0 is the price of a security paying S_1 at time 1.

Def. A model is **complete** if it is arbitrage-free and every derivative security is replicable.

The Fundamental Theorems of Asset Pricing

FIRST FUNDAMENTAL THEOREM OF ASSET PRICING: The following statements are equivalent, in a general 1-period binomial model.

(i) The model is arbitrage-free.

(ii) There is at least one risk-neutral measure $\tilde{\mathbb{P}}$.

SECOND FUNDAMENTAL THEOREM OF ASSET PRICING:

The following statements are equivalent, in a general 1-period binomial model.

(i) The model is complete.

(ii) There is exactly one risk-neutral measure $\tilde{\mathbb{P}}$.

Let's try using this...

Example Suppose $\Omega = \{\omega_1, \omega_2, \omega_3\}$

$r = 0.1$

Stock 1: S^1

Stock 2: S^2

$$S^1_0 = 20 \begin{cases} \nearrow S^1_1(\omega_1) = 24 \\ \rightarrow S^1_1(\omega_2) = 18 \\ \searrow S^1_1(\omega_3) = 16 \end{cases}$$

$$S^2_1 = 20 \begin{cases} \nearrow S^2_1(\omega_1) = 24 \\ \rightarrow S^2_1(\omega_2) = 24 \\ \searrow S^2_1(\omega_3) = 8 \end{cases}$$

Let's see if we can find a risk-neutral measure $\tilde{\mathbb{P}}$.

Must have:

(i) $p_1 + p_2 + p_3 = 1$.

(ii) $0 \le p_i \le 1 \quad \forall \; i = 1, 2, 3$.

(iii) $24\tilde{p}_1 + 18\tilde{p}_2 + 16\tilde{p}_3 = (1.1)20 = 22$

(iv) $24\tilde{p}_1 + 24\tilde{p}_2 + 8\tilde{p}_3 = (1.1)20 = 22$

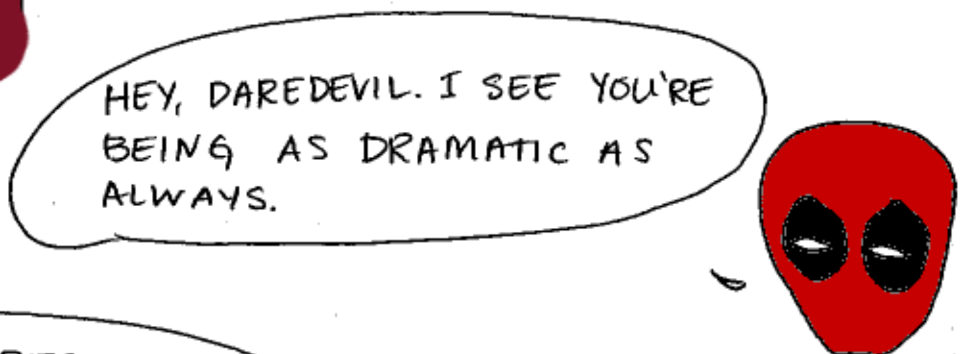
By doing out the algebra, we get

$$\tilde{p}_1 = \frac{17}{24}$$

$$\tilde{p}_2 = \frac{1}{6}$$

$$\tilde{p}_3 = \frac{1}{8}.$$

By the 2nd fundamental theorem of asset pricing, this is a complete model. $\blacksquare$



ONE DIRECTION:

more choices for outcomes in period 1 (e.g. trinomial model – this is what we were using in the example on pg. 18)

ANOTHER DIRECTION:

more periods $\longrightarrow$ MULTIPERIOD MODEL (§1.2 & §1.3 in Shreve)

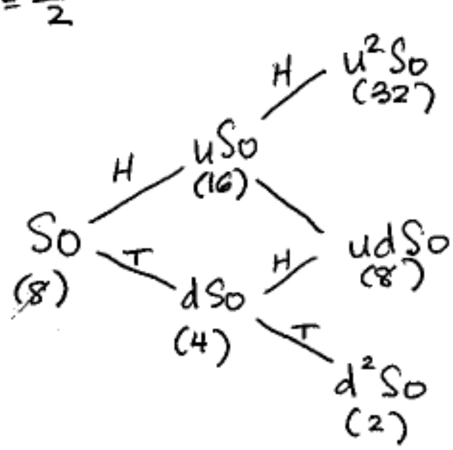
2-period example:

Trading times $t=0, t=1, t=2$.

European call: $T=2$, $K=20$.

$u=2$, $d=1/2$, $r=1/4$, $S_0=8$ (in each period)

$\rightarrow \tilde{p}=\tilde{q}=\frac{1}{2}$



$C_2=(S_2-20)^+$, so

$C_2(HH)=12$, $C_2(HT)=C_2(TH)=0$,

$C_2(TT)=0$.

How much is the call worth @ $t=1$?

$$C_1(H)=\frac{1}{1+r}\left[\tilde{p}\,C_2(HH)+\tilde{q}\,C_2(HT)\right]$$

$$=\frac{4}{5}\left[\frac{1}{2}\cdot 12+\frac{1}{2}\cdot 0\right]=\frac{24}{5}$$

$$C_1(T)=\frac{4}{5}\left[\frac{1}{2}\cdot 0+\frac{1}{2}\cdot 0\right]=0$$

So $C_0=\frac{4}{5}\cdot\left[\frac{1}{2}\cdot\frac{24}{5}+\frac{1}{2}\cdot 0\right]=\frac{48}{25}=\underline{\underline{\$1.92}}$ ■