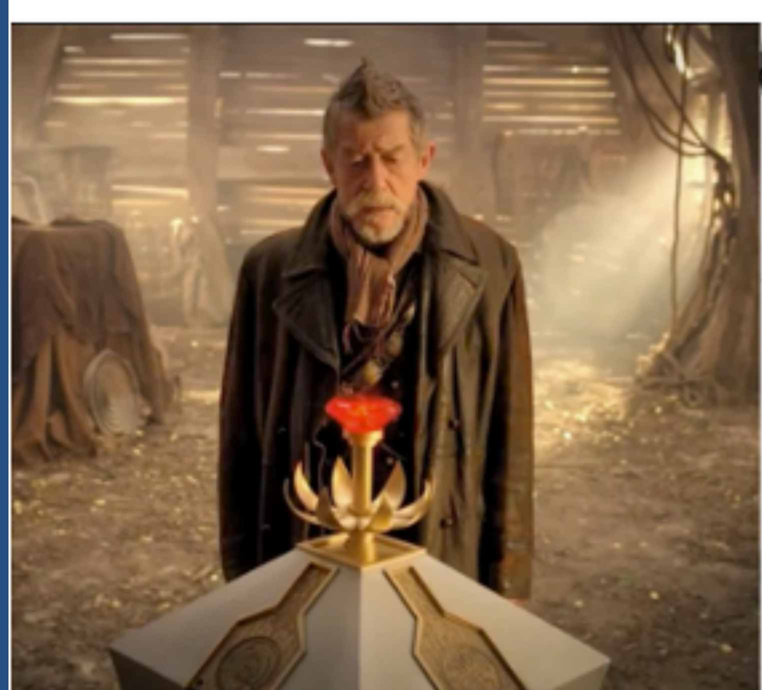
POLICE PUBLIC CALL BOX

MATH 321
LECTURE 5

2/3/16

Welcome to Lecture 5! We're going to talk about... timey-wimey... stuff.

Hey, 10th. You didn't prepare for your lecture, did you. The "timey-wimey" stuff is the N-PERIOD BINOMIAL MODEL.



No More.

Oh, that came out rather dramatic. I mean let's get on with it.

That was OK, for $N=2$ periods. But at time N, there are 2^N outcomes, so there's a practical issue with generalizing this method.

N-period binomial model

(mathematical framework).

Consists of:

- sample space $\Omega = \{ \omega = (\omega_1, \omega_2, \ldots, \omega_N) \mid \omega_i \in \{T, H\} \; \forall i \in \{1, 2, \ldots, N\} \}$.

  (i.e. all strings of H's & T's of length N)

  (so $|\Omega| = 2^N$)

- real #s $u > d > 0$, $r \geq 0$

  (arbitrage-free $\iff d < 1+r < u$)

- stock S w/ initial price S_0 and time $t+1$ price

$$S_{t+1} = (1 + \rho_{t+1}) S_t$$

where
$$\rho_{t+1}(\omega) = \begin{cases} u-1 & \text{if } \omega_t = H \\ d-1 & \text{if } \omega_t = T \end{cases}$$

- Δ_t = # shares of stock held b/w t & $t+1$.

Remark: important value of ρ_t depends <u>only</u> on the outcome of the t^{th} coin toss.

Remark: It follows that

$$\boxed{S_t(\omega) = \prod_{i=1}^{t} (1+\rho_i(\omega_i))\, S_0}$$

price of stock at time t, given that the outcome of the 1st t coin tosses is described by $\omega = (\omega_1, \omega_2, \dots, \omega_t)$

Remark: Random variables pertaining to capitals / prices at time n should only depend on the first n coin tosses, for all n.

Self-financing assumption

(other than in later examples involving dividend-paying stocks) we'll assume strategies are self-financing: i.e. no capital is introduced or removed between time t=0 & t=N.

$$\Rightarrow \quad X_1 = \Delta_0 S_1 + (X_0 - \Delta_0 S_0)(1+r)$$

$$X_2 = \Delta_1 S_2 + (X_1 - \Delta_1 S_1)(1+r)$$

$$\vdots$$

$$X_{t+1} = \Delta_t S_{t+1} + (X_t - \Delta_t S_t)(1+r)$$

Price securities using backward induction:

Proposition. Let V be a derivative security w/ maturity N. There exists a unique replicating strategy, constructed by $(X_t, \Delta_t)_{t\in\{1,2,\dots,N\}}$ where

$$(1)\quad X_N(\omega) = V_N(\omega) \quad \forall\, \omega\in\Omega,$$

and (2) $\forall\, t\in\{0,1,\dots,N-1\}$,

$$X_t(\omega) = X_t(\omega_1,\dots,\omega_t) = \frac{1}{1+r}\left[\tilde{p}\, X_{t+1}(\omega_1,\dots,\omega_t,H) + \tilde{q}\, X_{t+1}(\omega_1,\dots,\omega_t,T)\right]$$

and

$$\Delta_t(\omega) = \Delta_t(\omega_1,\dots,\omega_t) = \frac{X_{t+1}(\omega_1,\dots,\omega_t,H) - X_{t+1}(\omega_1,\dots,\omega_t,T)}{S_{t+1}(\omega_1,\dots,\omega_t,H) - S_{t+1}(\omega_1,\dots,\omega_t,T)}$$

Remark: This algorithm leads to a sum of 2^N terms. Each will have a factor of $\tilde{p}^{h(\omega)}\tilde{q}^{N-h(\omega)}$ where $h(\omega)$ = # heads among ω.
Let's call $\tilde{p}^{h(\omega)}\tilde{q}^{N-h(\omega)} = \tilde{Q}(\omega)$.
Then

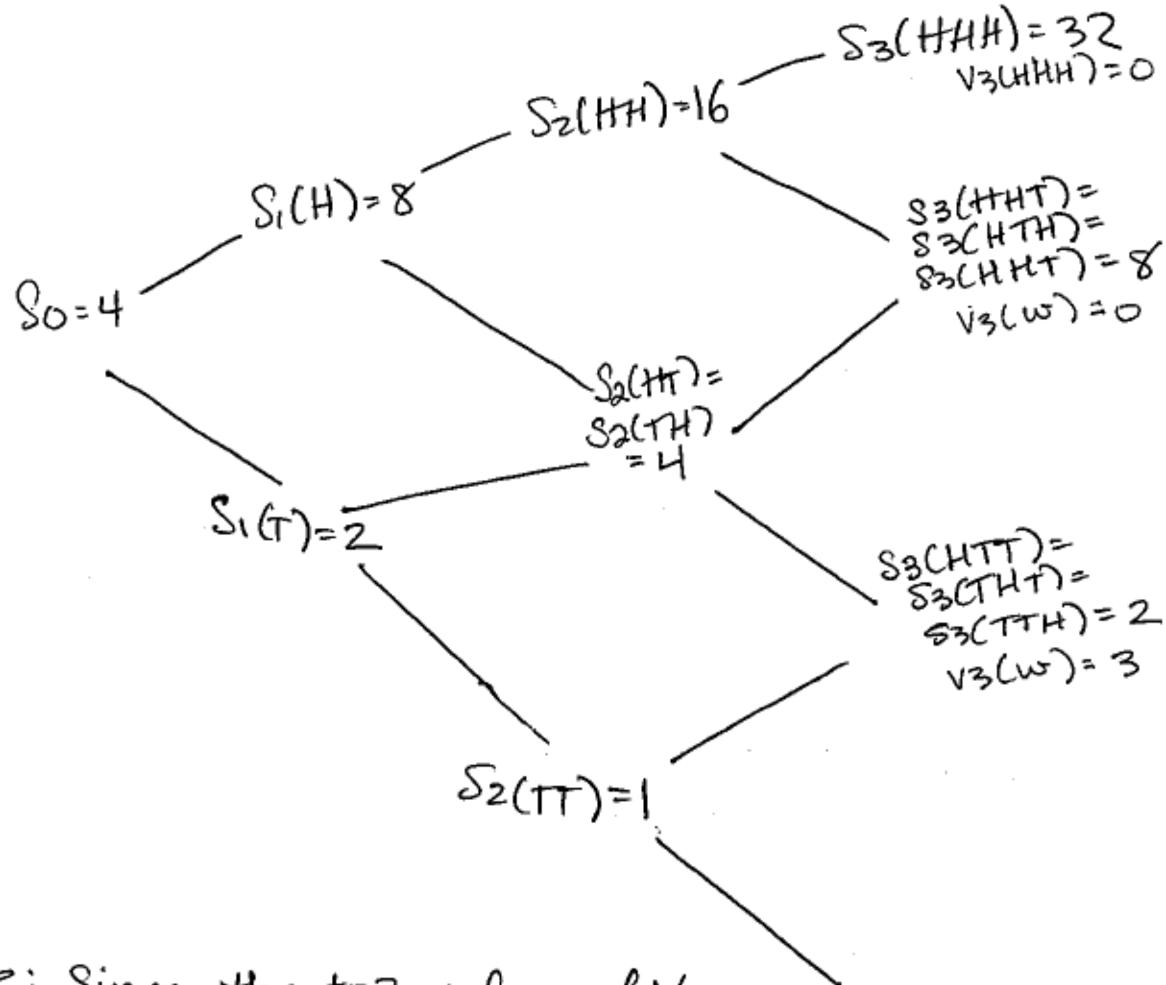
$$V_0 = \frac{1}{(1+r)^N}\sum_{\omega\in\Omega}\tilde{Q}(\omega)\, V_N(\omega)$$

(i.e. V_0 is the discounted risk-neutral expected value of V_N).

Example:

$N=3,\ S_0=4,\ u=2,\ d=1/2,\ r=1/4$

European put V w/ $K_p=5,\ T=3$:



Note: Since the $t=3$ value of V only depends on the value of stock at $t=3$ (not on the coin tosses), we get down to 4 outcomes:

$$V_3(32)=0,\ V_3(8)=0,\ V_3(2)=3,\ V_3(\tfrac{1}{2})=\tfrac{9}{2}.$$

Now the algorithm goes:

$$V_2:\begin{cases} V_2(16)=\frac{4}{5}\left[\frac{1}{2}\cdot 0+\frac{1}{2}\cdot 0\right]=0 \\ V_2(4)=\frac{4}{5}\left[\frac{1}{2}\cdot 0+\frac{1}{2}\cdot 3\right]=6/5 \\ V_2(1)=\frac{4}{5}\left[\frac{1}{2}\cdot 3+\frac{1}{2}\cdot\frac{9}{2}\right]=3 \end{cases}$$

$$V_1:\begin{cases} V_1(8)=\frac{4}{5}\left[\frac{1}{2}\cdot 0+\frac{1}{2}\cdot\frac{6}{5}\right]=12/25 \\ V_1(2)=\frac{4}{5}\left[\frac{1}{2}\cdot\frac{6}{5}+\frac{1}{2}\cdot 3\right]=42/25 \end{cases}$$

$$V_0=\frac{4}{5}\left[\frac{1}{2}\cdot\frac{12}{25}+\frac{1}{2}\cdot\frac{42}{25}\right]=\$0.864$$